Transfer operators and estimating characteristic values
(or Computing numbers illustrated by a couple of examples)

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I. The Gauss Problem

Continued Fractions

It is a classical fact that every irrational $x \in (0,1)$ has a unique **continued fraction** expansion

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} =: [a_1, a_2, a_3, \ldots], \qquad a_n \in \mathbb{N}.$$

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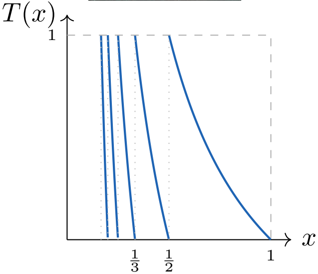
A convenient way to generate the series (a_n) is to use the orbit of x under the *Gauss map*:

$$T(x) = \frac{1}{x} - \left[\frac{1}{x}\right]$$



In particular, $a_n = k$ if $\underbrace{T \circ \cdots \circ T}_{T^n}(x) \in \left(\frac{1}{k+1}, \frac{1}{k}\right)$, $n \geq 0$.

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In a famous letter sent on 30 January, 1812, from Gauss to Laplace, it was claimed (in modern terminology) that for any fixed $0 < t < 1$ (and Lebesgue measure ℓ):

$$\boxed{\ell(\{x \in [0,1] : T^n(x) \leq t\}) \to \frac{1}{\log 2} \int_0^t \frac{du}{1+u}, \text{ as } n \to +\infty.}$$

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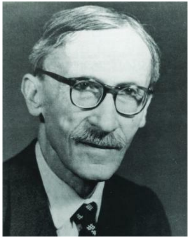
Gauss-Laplace correspondence

Kuzmin–Lévy-Wirsing Theorem (1928 -1974)

$$\exists 0 < q < 1, C > 0 \text{ such that } |\lambda(\{x : T^n(x) \leq t\}) - \frac{1}{\log 2}\int_0^t \frac{du}{1+u}| = Cq^n + o(q^n)$$



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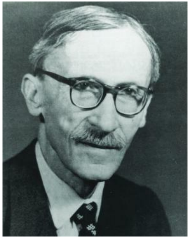

Gauss Problem

What is the value of the Kuzmin–Lévy-Wirsing constant q?

The Transfer Operator (Deus ex machina)

The **transfer operator** for the Gauss map is the linear operator $\mathcal{L} : \mathcal{B} \to \mathcal{B}$ defined by

$$(\mathcal{L}f)(x) = \sum_{n=1}^{\infty} \frac{1}{(x+n)^2} f\left(\frac{1}{x+n}\right),$$

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Reformulated Gauss Problem

What is the value of the second eigenvalue λ_2 (and thus q)?

Validated Numerics for $q = |\lambda_2|$: Result and idea of the approach

- We can replace $\mathcal{L}$ by the conjugate operator

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Lemma (Relating the operator $\mathcal{M}$ to q)

Then for (strictly increasing) polynomials $p_-, p_+ : [0,1] \to \mathbb{R}^+$ *we have bounds*

$$\inf_{0 \le x \le 1} \frac{(\mathcal{M}p_-)'(x)}{p_-(x)} \le q = |\lambda_2| \le \sup_{0 \le x \le 1} \frac{(\mathcal{M}p_+)'(x)}{p_+(x)}$$

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- **Example**. Choosing appropriate polynomials $p_\pm$ of degree 50 gives a (validated) estimate

$$q = |\lambda_2| = 0.30366300289873265859744812190155623311087735225365\ldots$$

accurate to 50 decimal places.

Ingredients in the proof (for enthusiasts)

- The inequalites

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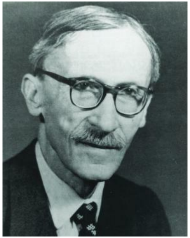

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 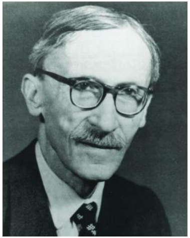 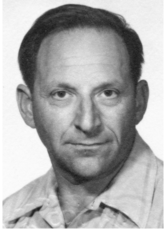

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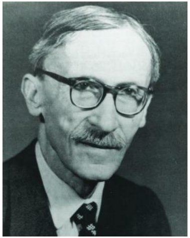
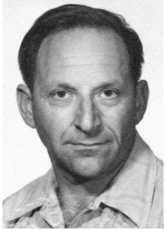


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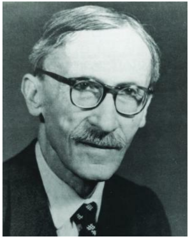
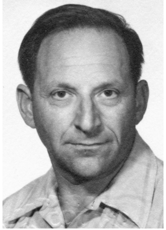

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 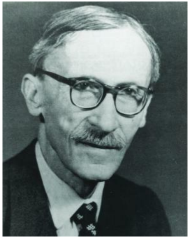 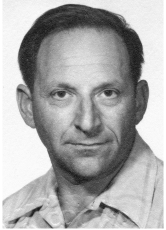  

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There are more estimates on q (due to Flajolet-Vallée, Briggs, Nisoli, Mayer-Roepstorff) but I wanted a validated value and a simple approach ... and ran out of space for pictures.

II. The Minkowski Question Mark Function

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The Minkowski **question mark function** $? : [0,1] \to [0,1]$ is defined for $x = [a_1, a_2, a_3, \ldots]$ by

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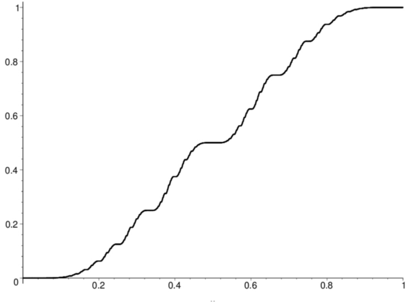

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- Derivative $?'(x) = 0$ for a.e.(ℓ) x.

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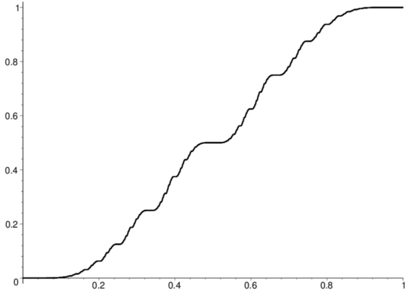

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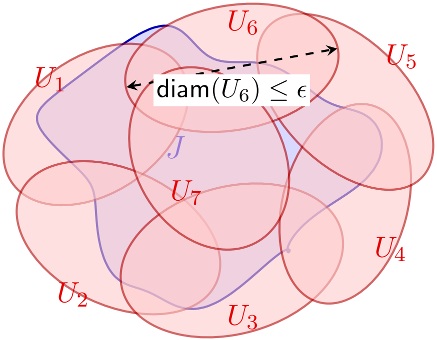
Problem

What is the size of the zero measure set $\{x : ?'(x) \neq 0\}$ (e.g., the Hausdorff dimension).

Recall the definition of Hausdorff dimension of a set $X \subset \mathbb{R}^d$.

$$\boxed{\dim_H(J) = \inf\{s \geq 0 : \text{ for all } \epsilon > 0 \text{ there exists cover } \{U_i\} \text{ of } X \text{ with } \sum_i \mathrm{diam}(U_i)^s \leq \epsilon\}}$$



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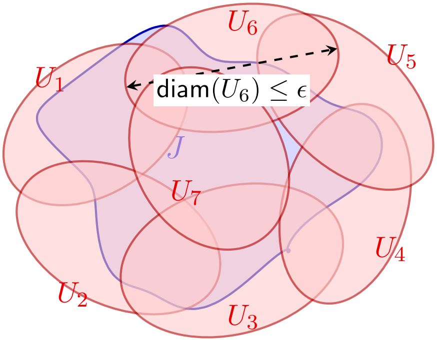

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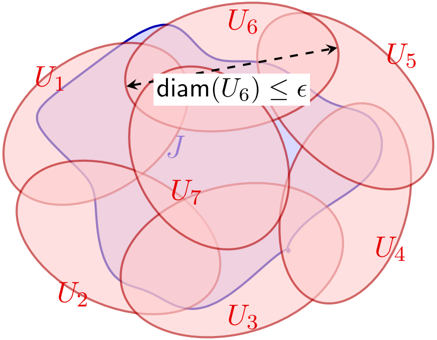

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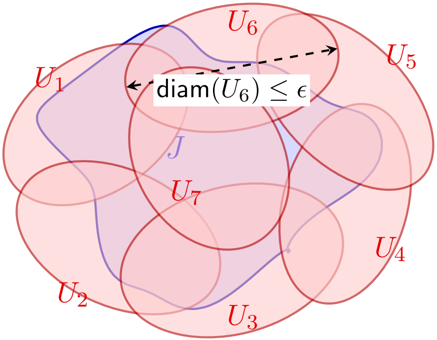

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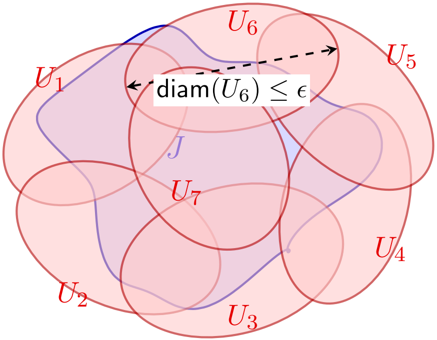

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A dynamical viewpoint

By a result of Kinney (1962) and Kesseboemer-Stratmann (2008):

$$\boxed{\dim_H(\{x : ?'(x) \neq 0\}) = \frac{\log 2}{2\int_0^1 \log(1+x)d\mu(x)}}$$

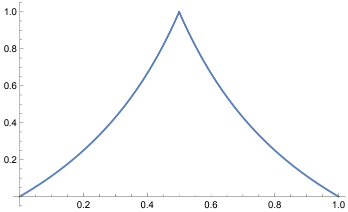
A dynamical viewpoint

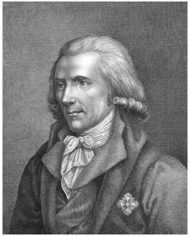
By a result of Kinney (1962) and Kesseboemer-Stratmann (2008):

$$\boxed{\dim_H(\{x : ?'(x) \neq 0\}) = \frac{\log 2}{2\int_0^1 \log(1+x)d\mu(x)}}$$

where μ is the measure of maximal entropy (i.e., "Bernoulli measure") for the Farey map $F : [0,1] \to [0,1]$ defined by

$$F(x) = \begin{cases} \frac{x}{1-x} & \text{if } 0 \leq x \leq \frac{1}{2} \\ \frac{1-x}{x} & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$





Enter a second transfer operator

For each $t \in \mathbb{R}$ we define $\mathcal{L}_t : C([0,1]) \to C([0,1])$ by

$$\boxed{\mathcal{L}_t w(x) = \frac{1}{2(1+x)^{2t}} \left(w\left(\frac{x}{1+x}\right) + w\left(\frac{1}{1+x}\right)\right) \text{ where } w \in C([0,1]).}$$

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Lemma (Relating transfer operators to the dimension)

Let $\epsilon > 0$. Let $p_-, p_+ : [0,1] \to \mathbb{R}^+$ be polynomials and denote

$$r_+ = \inf_{0 \le x \le 1} \left\{ \frac{\mathcal{L}_\epsilon p_+(x)}{p_+(x)} \right\} \text{ and } r_- = \sup_{0 \le x \le 1} \left\{ \frac{\mathcal{L}_{-\epsilon} p_-(x)}{p_-(x)} \right\}$$

then

$$\frac{\epsilon \log 2}{|\log r_-|} \le \dim_H(\{x : ?'(x) = \infty\}) \le \frac{\epsilon \log 2}{|\log r_+|}$$

Making smart choices (of $\epsilon > 0$ and polynomials)

Example (By hand)

With $\epsilon = 10^{-3}$ and polynomials of degree 5 gives a validated estimate to 2 decimal places

$$\mathbf{0.87}4169\ldots \leq \dim_H(\{x : ?'(x) \neq 0\}) \leq \mathbf{0.87}5016\ldots$$

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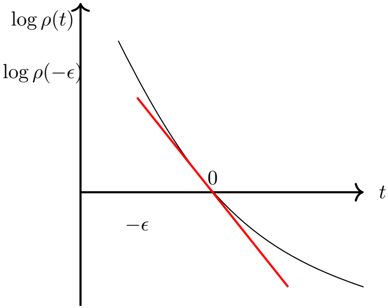
Example (By Computer)

With $\epsilon = 10^{-120}$ and polynomials of degree 300 gives a validated estimate to 52 decimal places

$$\mathbf{0.8747163051082111422151529042191597577579272897515}3223751895331660433\ldots$$
$$\leq \dim_H(\{x : ?'(x) \neq 0\}) \leq$$
$$\mathbf{0.8747163051082111422151529042191597577579272897515}31895976574426163 97\ldots$$

Elements of the proof for enthusiasts



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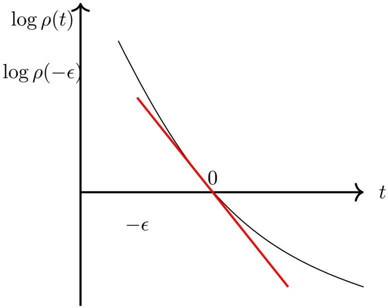

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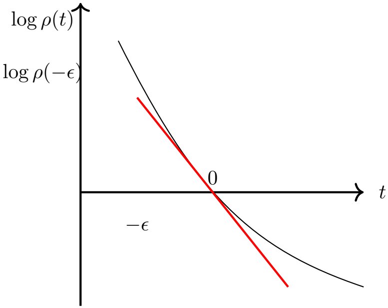

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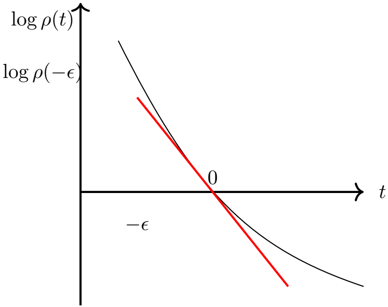

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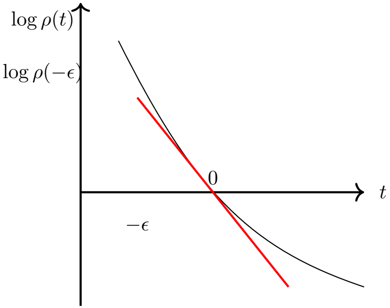

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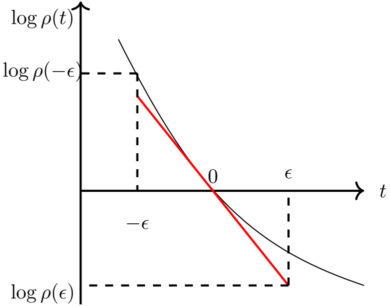

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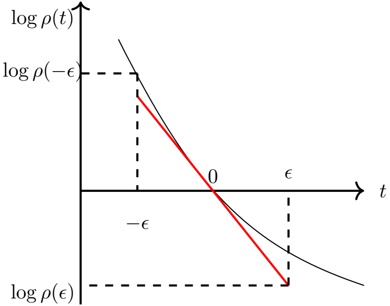

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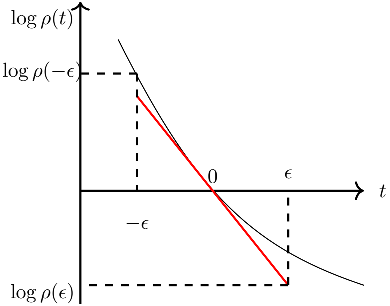

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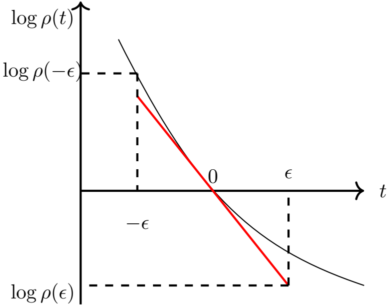

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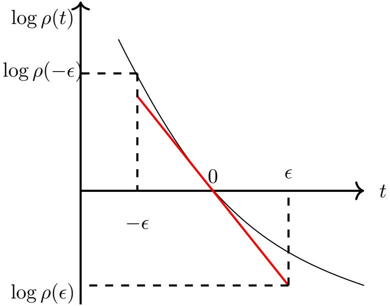

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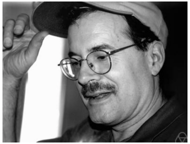

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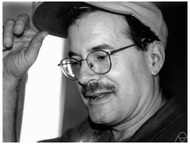  

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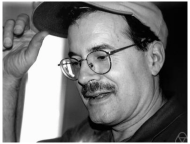    

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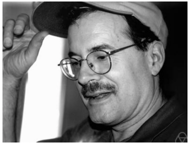




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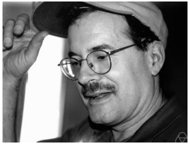      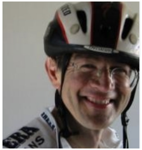

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Go to End

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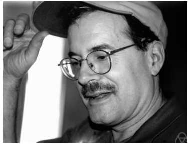      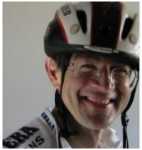

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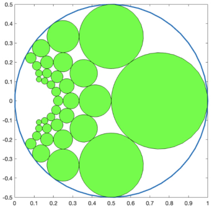
III. “Complex continued fraction set”

Bonus: Complex Continued fraction set

Consider $T_{mn} : D \to D$ on $D = B(\frac{1}{2}, \frac{1}{2})$ given by $T_{nm}(z) := \frac{1}{z+m+in}$ for $(m, n) \in \mathbb{N} \times \mathbb{Z}$.



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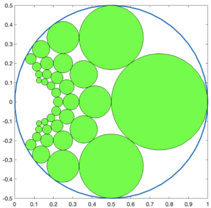

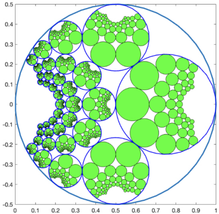

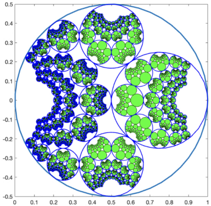

The limit set $J \subset D$ is defined by iteration

$$J := \left\{ \lim_{n \to +\infty} T_{m_1 n_1} \circ \cdots \circ T_{m_k n_k}(D) \text{ where } (m_i, n_i) \in \mathbb{N} \times \mathbb{Z} \right\}.$$

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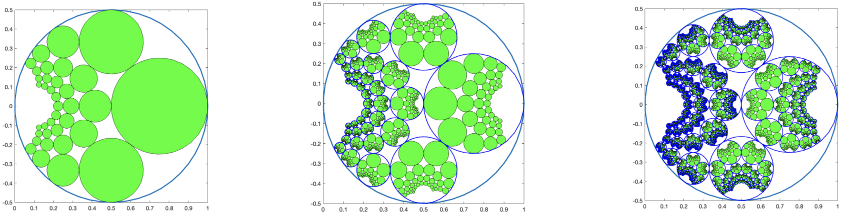

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Dimension Problem

What is the size (Hausdorff dimension) $\dim_H(J)$ of the set J?

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$$1.855519932766765 < \dim_H(J) < 1.855519932767202$$

(i.e., accurate to 11 decimal places) using methods similar to those described before.

Go to End

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$$\sup_{(x,y)\in B} \frac{(\mathcal{L}_a p_a)(x,y)}{p_a(x,y)} < 1 < \inf_{(x,y)\in B} \frac{(\mathcal{L}_b p_b)(x,y)}{p_b(x,y)} \implies a \leq \dim_H(X) \leq b. \quad (*)$$

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  Finally (*) guarantees bounds are completely rigorous.

Thank you for your time

Merci de m'avoir consacré du temps