# Lectures on Equilibrium states, mixing and dimension 

Mark Pollicott


#### Abstract

These notes correspond to two separate mini-lecture courses presented at the Banach Centre, Warsaw, in the Spring of 2023. However, thery are united by a common theme was the study of hyperbolic dynamical systems and ideas from Thermodynamic Formalism and its applications.


## Contents

I Equilibrium measures, pressure and resonances ..... 2
1 Introduction ..... 2
1.1 Overview of topics ..... 2
1.2 A little statistical mechanics ..... 2
2 Hyperbolic diffeomorphisms and equilibrium states ..... 4
2.1 Hyperbolic attractors and Anosov diffeomorphisms ..... 4
2.2 Invariant measures and Equilbrium states ..... 5
2.3 The SRB measure and its construction ..... 6
2.4 Constructing other equilibrium measures ..... 7
2.5 Results for attracting flows ..... 9
3 Subshifts of finite type ..... 10
3.1 Equilibrium measures for subshifts ..... 11
3.2 Changing Equilibrium measures for subshifts ..... 11
3.3 Rigidity and flexability of pressure ..... 14
4 Resonances ..... 16
4.1 Speed of mixing ..... 16
4.2 Examples ..... 17
5 Geodesic flows ..... 18
5.1 Definition of the geodesic flow ..... 18
5.2 Topological entropy ..... 19
5.3 Entropies of measures ..... 19
5.4 Smoothness of entropy ..... 21
5.5 The Anosov property and Lyapunov exponents ..... 21
5.6 Ricci flow and entropy ..... 22
II Estimating Dimension and Lyapunov exponents ..... 23
6 Introduction ..... 23
6.1 Classical Examples of sets and their dimensions ..... 24
6.2 General setting ..... 24
7 Different approachs to estimation dimension ..... 26
7.1 Approach I : Approximation by similarities ..... 26
7.2 The Bowen-Ruelle pressure formula ..... 28
7.3 Approach II : Determinants ..... 29
7.4 Approach III : "min-max" ..... 31
8 Estimating Lyapunov exponents ..... 34
8.1 Lyapunov exponents for interval maps ..... 34
8.2 Lyapunov exponents for random matrix products ..... 38
8.3 Positive matrices ..... 41
8.4 Variations on themes ..... 42
Part I
Equilibrium measures, pressure and resonances

## 1 Introduction

We begin with some classical background and motivation on theomodynamic formalism, before developing these ideas in the particular context of the speed of mixing of hyperbolic systems and applications to geodesic flows.

### 1.1 Overview of topics

In the first part of these notes we want to consider a selection of the following topics related to Equilibrium states (or Gibbs measures) for Anosov systems.

1. Gibbs measures for Anosov diffeomorphisms (and how to construct them).
2. The pressure for functions (and what information it gives).
3. Examples of the speed of mixing (for toral auromorphisms).
4. Entropy and geodesic flows.

As a prelude to this we begin with some historical context.

### 1.2 A little statistical mechanics

The historical origins of Equilibrium measures (or Gibbs measures), pressure, transfer operators give some insight into the dynamical applications, but let us begin with a dash of motivating statistical mechanics. In particular the Ising model (from the PhD thesis of E. Ising (1925) which was supervised and influenced by W. Lenz).

This was originally proposed as a model for "ferromagnetism". Unfortuntately, it isn't very successful physically in this respect in one dimension (where it is easier to analyze) but instead leads to a very successful dynamical application.

- Assume that we have $N$ "sites" corresponding to $\mathbb{Z} / N \mathbb{Z}=\{0,1, \cdots, N-1\}$.
- Each site $i$ is occupied with particles with one of two possible states ("spins") denoted by $\sigma_{i} \in$ $\{-1,1\}$, for $i \in \mathbb{Z} / N \mathbb{Z}$.

Assume that only neighbouring particles (at sites $i$ and $i+1$ ), say can interact and that their contribution to the energy is $-J \sigma_{i} \sigma_{i+1} \in\{-J, J\}$, for some fixed value $J>0$ ("the interactions").

For each of the $2^{N}$ possible configurations $\sigma=\left(\sigma_{1}, \cdots, \sigma_{n}\right) \in\{-1,1\}^{\mathbb{Z} / N \mathbb{Z}}$ that can occur we can associate their total contribution to the energy in the form

$$
H(\sigma)=-J \sum_{i=0}^{N-1} \sigma_{i} \sigma_{i+1}
$$

The Boltzmann distribution is a probability distribution on the configurations $\sigma \in\{-1,1\}^{\mathbb{Z} / N \mathbb{Z}}$ of the form

$$
\frac{\exp (-\beta H(\sigma))}{Z_{N}}
$$

where $\beta>0$ (related to the "inverse temperature") and

$$
Z_{N}=\sum_{\sigma \in\{-1,1\}^{Z / N \mathbb{Z}}} \exp (-\beta H(\sigma))
$$

is the normalizing constant (the "partition function") coming from summing the exponential of this quantity over the $2^{N}$ possible states. In order to evaluate $Z_{N}$ explicitly, we can introduce a $2 \times 2$ real matrix (a "transfer matrix") given by

$$
T=\left(\begin{array}{cc}
e^{J \beta} & e^{-J \beta} \\
e^{-J \beta} & e^{J \beta}
\end{array}\right) .
$$

Finally, we can rewrite the partition function as

$$
Z_{N}=\sum_{i_{1}, \cdots, i_{N-1}} \exp \left(-J \sum_{i=0}^{N-1} \sigma_{i} \sigma_{i+1}\right)=\operatorname{trace}\left(T^{N}\right) .
$$

A simple calculation gives that the eigenvalues $\lambda_{1}>\lambda_{2}$ are:

$$
\lambda_{1}=2 \cosh (\beta J) \text { and } \lambda_{2}=2 \sinh (\beta J) .
$$

We can get rid of the value $N$ by taking a limit ("thermodynamic limit" ):

$$
\lim _{N \rightarrow+\infty} \frac{1}{N} \log Z_{N}=\log \lambda_{1} .
$$

This quantity is called the free energy. In the present one dimensional setting it is disappointing from the point of view of physicists that small changes in $\beta$ never lead to abrupt changes in the system, e.g., the value $\lambda_{1}$ (which would have been a "phase transition"). Although this phenomenon can occur in higher dimensions, this is not the dynamical setting we want to consider.

Of course many of these concepts have evolved into familiar objects in the ergodic theory of so-called thermodynamic formalism: Shift spaces ("the configuration spaces"); function spaces ("interactions"); the pressure function ("free energy"); the transfer operator ("transfer matrix"); Gibbs measures (from the "Boltzmann distribution"). Whereas this original source is interesting, it probably doesn't help directly in what now follows - and can now be safely forgotten.

## 2 Hyperbolic diffeomorphisms and equilibrium states

We need a suitable family of transformations and measures which, inspired by the previous physical motivation, lead to interesting results. The basic class are hyperbolic diffeomorphisms (particularly Anosov diffeomorphisms).

### 2.1 Hyperbolic attractors and Anosov diffeomorphisms

We recall the general definition of a hyperbolic attractor. Let $f: M \rightarrow M$ be a $C^{1+\alpha}$ diffeomorphism on a compact Riemannian manifold, and let $X \subset M$ be a closed $f$-invariant set.

Definition 2.1. The $C^{\infty}$ map $f: X \rightarrow X$ is called a mixing hyperbolic attracting diffeomorphism if:

1. there exists a continuous splitting $T_{X} M=E^{s} \oplus E^{u}$ and $C>0$ and $0<\lambda<1$ such that

$$
\left\|D f^{n} \mid E^{s}\right\| \leq C \lambda^{n} \text { and }\left\|D f^{-n} \mid E^{u}\right\| \leq C \lambda^{n}
$$

for $n \geq 0$ (hyperbolicity);
2. there exists an open set $X \subset U \subset M$ such that $X=\cap_{n=0}^{\infty} f^{n} U$ (attractor);
3. For non-empty open sets $U, V \subset M$ there exists $n \in \mathbb{Z}$ with $f^{-n} U \cap V \neq \emptyset$ (topologically mixing); and
4. the periodic orbits for $f: X \rightarrow X$ are dense in $X$.

In the particular case $X=M$ the diffeomorphism $f$ is a mixing Anosov diffeomorphism. In particular, we have that:

Definition 2.2. A diffeomorphism $f: M \rightarrow M$ is called a mixing Anosov diffeomorphism if:

1. there exists a continuous splitting $T M=E^{s} \oplus E^{u}$ and $C>0$ and $0<\lambda<1$ such that

$$
\left\|D f^{n} \mid E^{s}\right\| \leq C \lambda^{n} \text { and }\left\|D f^{-n} \mid E^{u}\right\| \leq C \lambda^{n}
$$

for $n \geq 0$; and
2. $f: M \rightarrow M$ is topologically mixing.

We will recall two simple examples. The first is the familiar example of the Arnol'd CAT map.
Example 2.3 (A CAT map). Let $M=\mathbb{T}^{1} \times \mathbb{T}^{1}$, writing $\mathbb{T}^{1}=\{z \in \mathbb{C}:|z|=1\}$. We can then define an Anosov diffeomorphism $f: \mathbb{T}^{1} \times \mathbb{T}^{1} \rightarrow \mathbb{T}^{1} \times \mathbb{T}^{1}$ by

$$
f(z, w)=\left(z^{2} w, z w\right)
$$

This is one of many possible examples of linear hyperbolic toral automorphisms. Furthermore, any nearby diffeomorphism will also be Anosov (by structural stability [45]). However, we can also consider an explicit class of such (non-linear) maps which arise from replacing some of the terms by Blaschke products. Thus a slightly more exotic example than the CAT map is the following.

Example 2.4 (Blacshke products). Let $B_{\lambda}: \mathbb{T}^{1} \times \mathbb{T}^{1} \rightarrow \mathbb{T} \times \mathbb{T}^{1}$ be defined by

$$
B_{\lambda}(z, w)=\left(\left(\frac{z+\lambda}{1-\bar{\lambda} z}\right) z w,\left(\frac{z+\lambda}{1-\bar{\lambda} z}\right) w\right)
$$

where $|\lambda|<1$. Fortunately, Blaschke products are well known for preserving $\mathbb{T}^{1}$ making these maps well defined. These maps $B_{\lambda}$ can be shown to be Anosov and area preserving [44], [37]. In the special case $\lambda=0$ this reduces to Example 2.3.

A classical example of an attracting hyperbolic diffeomorphism (which is not Anosov) is the Solenoid. Let $\mathbb{D}^{2}=\left\{(x, y) \in \mathbb{R}^{2}: x^{2}+y^{2} \leq 1\right\}$.

Example 2.5 (Solenoid). We can define $M=\mathbb{D}^{2} \times \mathbb{T}^{1}$ and $f: X \rightarrow X$ is the (natural) extension of the doubling map on the circle. This can be constructed by mapping the solid torus $M$ inside itself (interior of $M$ plays the role of the neighbourhood $U$ ) in Definition 2.1, see [47] and [17], §7.1. More concretely, let $f: M \rightarrow M$ be defined by

$$
f(z, x, y)=\left(z^{2}, \frac{x}{10}+\frac{\cos \theta}{2}, \frac{x}{10}+\frac{\sin \theta}{2}\right) .
$$

It is easily seen that $f(M) \subset \operatorname{int}(M)$ and the $f$-invariant set $\Lambda=\cap_{n=0}^{\infty} f^{n} \operatorname{int}(M)$ is an attractor (see [17], Proposition 17.1.2). In this example the unstable bundle $E^{u}$ is one dimensional and the stable manifold is two dimensional. The SRB induces the usual Lebesgue measure on the unstable manifolds, which are locally parameterized by the $\theta$-coordinate and projects down to the Haar measure on the circle.

### 2.2 Invariant measures and Equilbrium states

We briefly recall some background material. Let $\mathcal{M}_{f}(X)$ denote the space of $f$-invariant probability measures, i.e., $\mu$ such that $\mu\left(f^{-1} B\right)=\mu(B)$ for any Borel set $B \subset X$.

Lemma 2.6 (Alaoglu's Theorem). $\mathcal{M}_{f}$ is compact with respect to the usual weak star topology (i.e., a sequence of measures $\mu_{n} \rightarrow \mu$ in the weak star topology as $n \rightarrow+\infty$ if for any continuous function $F \in C(X, \mathbb{R})$ we have that $\int F d \mu_{n} \rightarrow \int F d \mu$ as $\left.n \rightarrow+\infty\right)$.

We will be interested in a particular (sub)class of invariant probability measures, which will be the equilibrium measures.

Given a (Hölder) continuous function $G: X \rightarrow \mathbb{R}$ there are various ways to describe the associated equilibrium measures. One standard approach (following Ruelle and Walters) is to use the following variational principle.

Definition 2.7. Given a continuous function $G: X \rightarrow \mathbb{R}$ we say that an $f$-invariant probability measure $\mu_{G} \in \mathcal{M}_{f}$ is an equilibrium measure for $G$ if

$$
\begin{equation*}
h\left(\mu_{G}\right)+\int G d \mu_{G}=\sup \left\{h(\mu)+\int G d \mu: \mu \in \mathcal{M}_{f}(X)\right\} \tag{2.1}
\end{equation*}
$$

where $h(\mu)$ is the entropy (or Komogorov-Sinai invariant) for $\mu \in \mathcal{M}_{f}(X)$, i.e., $\mu_{G}$ is a measure which maximizes the sum of the entropy and the integral of $G$, over all $f$-invariant measures. [38], [46].

The value attained by the supremum in (2.1) above is called the pressure and denoted $P(G)$.
Remark 2.8. For hyperbolic maps the pressure is also characterized by the formula:

$$
P(G):=\lim _{n \rightarrow+\infty} \frac{1}{n} \log \left(\sum_{T^{n} x=x} \exp \left(\sum_{k=0}^{n-1} G\left(T^{k} x\right)\right)\right)
$$

which looks reminiscent of the free energy for statistical physics described in §2.1.
There always exists at least one equilibrium state, and under an additional Hölder regularity assumptions on $G$ there will be a unique equilibrium state. We formally state this in the following proposition.

Proposition 2.9. If $G: X \rightarrow \mathbb{R}$ is continuous then there is at least one measure $\mu=\mu_{G}$ realizing the supremum above. Furthermore, if $G: X \rightarrow \mathbb{R}$ is Hölder continuous (e.g., the restriction of a $C^{1}$ function on $M$ ) then there is a unique measure $\mu=\mu_{G}$ realizing the supremum above.

The existence is a consequence of the map $\mathcal{M}_{f}(X) \ni \mu \mapsto h(\mu)+\int G d \mu$ being upper semicontinuous in the weak star topology and the space $\mathcal{M}_{f}(X)$ being weak star compact (by Lemma 2.6). (The uniqueness proof for Hölder functions can be found in [6].)

Remark 2.10. There are various ways other to construct $\mu_{G}$ when $G$ is Hölder continuous. For example we can consider the family of probability measures

$$
\mu_{G}^{(n)}=\frac{\sum_{T^{n} x=x} \exp \left(\sum_{k=0}^{n-1} G\left(T^{k} x\right)\right) \delta_{x}}{\sum_{T^{n} x=x} \exp \left(\sum_{k=0}^{n-1} G\left(T^{k} x\right)\right)} \text { for } n \geq 1
$$

where $\delta_{x}$ is a Dirac measure supported on the periodic point $x$. Then $\mu_{G}^{(n)}$ converges to $\mu_{G}$ in the weak star topology. This is reminiscent of the statistical mechanics background in §2.1.

In the next subsection we consider the best known example of an equilibrium state.

### 2.3 The SRB measure and its construction

There is a particular example of a natural invariant measures, namely the SRB measures (named after Sinai-Ruelle-Bowen) which is a measure absolutely continuous on the unstable manifolds.

In the special case that $f: M \rightarrow M$ is an Anosov diffeomorphism which preserves the Riemannian volume, say, then the SRB measure will be precisely the (normalized) volume. More generally, if $f$ does not preserve the volume then the SRB measure will be a $f$-invariant probability measure which induces a measure equivalent to the volume on pieces of unstable More precisely, let $W_{l o c}^{u}(x)$ be a small piece of unstable manifold defined, for example, by

$$
W_{l o c}^{u}(x)=\left\{y \in M: d\left(f^{n} x, f^{n} y\right) \leq \epsilon, \forall n \geq 0\right\}
$$

(with $\epsilon>0$ chosen sufficiently small). This is an embedded disk of dimension $\operatorname{dim} E^{u}$.
The SRB measure is an equilibrium state for an appropriate Hölder continuous function, which we now recall.

Theorem 2.11. Let $G(x)=\Phi(x):=-\log \left|\operatorname{det}\left(D f \mid E_{x}^{u}\right)\right|$ then the associated (unique) equilibrium state $\mu_{G}$ is the SRB measure.

Remark 2.12. The potential $\Phi$ is sometimes called the expansion coefficient.
Let $\lambda$ denote the (normalized) volume on $W_{l o c}^{u}(x)$. It follows from the following very classical result there always exists at least one invariant measure (which can be constructed using $\lambda$, or any other [non-invariant] probability measures).

Theorem 2.13 (Krylov-Bogolyubov). For any homeomorphism $f$ of a compact space, the weak star limit points of

$$
\frac{1}{n} \sum_{k=0}^{n-1} f_{*}^{k} \lambda, \quad n \geq 1
$$

are $f$-invariant. (Here we denote by $f_{*}^{k} \lambda(A)=\lambda\left(f^{-k} A\right)$ the push forward measure supported on $\left.f^{k} W_{\delta}^{u}(x).\right)$

However, in the context of hyperbolic attractors much more is true. There is a famous construction due to Sinai (for the particular case of Anosov systems) and Ruelle (in the general setting of hyperbolic attractors). This says that the Sinai-Ruelle-Bowen measure $\mu_{S R B}$ arises using the push forward of the normalized volume $\lambda$ on any piece of local unstable manifold $W_{\delta}^{u}(x)$.

Theorem 2.14 (Sinai [42], Ruelle [38]). Let $f: X \rightarrow X$ be a $C^{1+\alpha}$ topologically mixing hyperbolic attractor. Given $x \in X$ and $\delta>0$ consider a (normalized) volume measure $\lambda=\lambda_{W_{\delta}^{u}(x)}$ on a piece of local unstable manifold $W_{\delta}^{u}(x)$, say. Then the averages

$$
\mu_{n}^{S R B}=\frac{1}{n} \sum_{k=0}^{n-1} f_{*}^{k} \lambda, \quad n \geq 1
$$

converge in the weak star topology to $\mu_{S R B}$ as $n \rightarrow+\infty^{1}$ [45].

### 2.4 Constructing other equilibrium measures

It is natural to ask about modifying the Sinai construction to construct other Equilibrium states.
Question. Let $G: M \rightarrow \mathbb{R}$ be a Hölder continuous function. Can one construct the unique equilibrium state $\mu_{G}$ by modifying the Sinai construction for the SRB measure?

One solution to this problem was presented by Climenhaga, Pesin and Zelerowicz who replaced the volume $\lambda$ on the unstable manifold by a new reference measure on $W_{\delta}^{u}(x)$ defined in terms of $G[?]$.

However, we want to consider an alternative approach where $\lambda$ is instead replaced by a family of absolutely continuous measures $\lambda_{n} \ll \lambda$ (where the density $\frac{d \lambda_{n}}{d \lambda}$ is defined in terms of $G$ and changes with $n$ ) [35]:

Theorem 2.15. Let $f: X \rightarrow X$ be a $C^{1+\alpha}$ topologically mixing hyperbolic attracting diffeomorphism and let $G: X \rightarrow \mathbb{R}$ be a Hölder continuous function. Given $x \in X$ and $\delta>0$ consider the sequence of probability measures $\left(\lambda_{n}\right)_{n=1}^{\infty}$ supported on $W_{\delta}^{u}(x)$ and absolutely continuous with respect to the induced volume $\lambda=\lambda_{W_{\delta}^{u}(x)}$ with densities

$$
\begin{equation*}
\frac{d \lambda_{n}}{d \lambda}(y):=\frac{\left.\exp \left(\sum_{i=0}^{n-1}(G-\Phi)\left(f^{i} y\right)\right)\right)}{\int_{W_{\delta}^{u}(x)} \exp \left(\sum_{i=0}^{n-1}(G-\Phi)\left(f^{i} z\right)\right) d \lambda(z)} \quad \text { for } y \in W_{\delta}^{u}(x) . \tag{2.2}
\end{equation*}
$$

Then the averages

$$
\begin{equation*}
\mu_{n}:=\frac{1}{n} \sum_{k=0}^{n-1} f_{*}^{k} \lambda_{n}, \quad n \geq 1, \tag{2.3}
\end{equation*}
$$

converge in the weak star topology to $\mu_{G}$.
In some vague sense we are compensating for changing from the SRB measure (with potential $\Phi$ ) to another measure (with potential $G$ ) by changing the weighting by the difference.

Example 2.16 (SRB-measure). If we let $G=\Phi$ then $\lambda_{n}=\lambda(n \geq 1)$ and this just reduces to Sinai's theorem.

[^0]Example 2.17 (Bowen-Margulis measure). If we let $G=0$ be identically zero then

$$
\frac{d \lambda_{n}}{d \lambda}(y):=\frac{\left.\exp \left(-\sum_{i=0}^{n-1} \Phi\right)\left(f^{i} y\right)\right)}{\int_{W_{\delta}^{u}(x)} \exp \left(-\sum_{i=0}^{n-1} \Phi\left(f^{i} z\right)\right) d \lambda(z)} \quad \text { for } y \in W_{\delta}^{u}(x)
$$

Then the averages

$$
\mu_{n}:=\frac{1}{n} \sum_{k=0}^{n-1} f_{*}^{k} \lambda_{n}, \quad n \geq 1
$$

converge in the weak star topology to a measure $\mu_{0}$ which is precisely the measure which maximizes the entropy (i.e., it is the Bowen-Margulis measure of maximal entropy).

It is clear from the statement that the construction is independent of the choice of $x$ and $\delta>0$.


Figure 1: A representation of the push forward of the measure $\lambda_{n}$ on $W_{\delta}^{u}(x)$ by $f^{k}$.
The existence of limit points follows from the weak star compactness of $\mathcal{M}_{f}(X)$. One of the key ingredients in the proof is:
Proposition 2.18. Let $f: X \rightarrow X$ be a mixing hyperbolic attracting diffeomorphism. For any continuous function $G: X \rightarrow \mathbb{R}$ we have

$$
\begin{equation*}
P(G)=\lim _{n \rightarrow+\infty} \frac{1}{n} \log \int_{W_{\delta}^{u}(x)} \exp \left(\sum_{k=0}^{n-1}(G-\Phi)\left(f^{k} y\right)\right) d \lambda_{W_{\delta}^{u}(x)}(y) . \tag{2.4}
\end{equation*}
$$

The essence of the proof of the theorem is to show that if $\mu$ is a limit point of $\left\{\mu_{n}\right\}$ then it satisfies $h(\mu)+\int G d \mu \geq P(G)$ (and thus there must be equality by the definition of pressure, therefore implying that $\left.\mu=\mu_{G}\right)$. To achieve this the idea is the following.

1. We recall a useful inequality for the entropy: Given a finite measurable partition $\mathcal{P}=\left\{P_{1}, \cdots, P_{k}\right\}$ and a probablity measure $\nu$ we can write

$$
\begin{equation*}
q H_{\lambda_{n}}\left(\bigvee_{h=0}^{n-1} f^{-h} \mathcal{P}\right) \leq n H_{\mu_{n}}\left(\bigvee_{i=0}^{q-1} f^{-i} \mathcal{P}\right)+2 q^{2} \log \underbrace{\operatorname{Card}(\mathcal{P})}_{=: k} . \tag{2.5}
\end{equation*}
$$

where $H_{\nu}(\mathcal{P})=-\sum_{i=1}^{k} \nu\left(P_{i}\right) \log \nu\left(P_{i}\right)$. and $0<q<n .{ }^{2}$
2. One then has that if $\mu_{n_{k}} \rightarrow \mu$ then combining (2.4) and (2.5) leads to

$$
P(G) \leq \frac{1}{q} H_{\mu_{n}}\left(\bigvee_{i=0}^{q-1} f^{-i} \mathcal{P}\right)+\int G d \mu
$$

[^1]3. Finally, letting $q \rightarrow+\infty$ we have
\[

$$
\begin{aligned}
P(G) & \leq \frac{1}{q} H_{\mu}\left(\bigvee_{i=0}^{q-1} f^{-i} \mathcal{P}\right)+\int G d \mu \\
& \rightarrow h_{\mu}(\mathcal{P})+\int G d \mu \leq h_{\mu}(f)+\int G d \mu
\end{aligned}
$$
\]

since $h_{\mu}(f)=\sup _{\mathcal{P}}\left\{h_{\mu}(\mathcal{P})\right\}$.
Moreover, one sees even from this very short sketch that there is a slightly stronger result when we assume only that $G$ is continuous (and thus we know that equilibrium states exist, but we don't know that they are unique):

Theorem 2.19. If $G: X \rightarrow \mathbb{R}$ is continuous then the weak star accumulation points for the measures $\lambda_{n}(n \geq 1)$ are equilibrium states for $G$.

These constructions could also be done by using Markov partitions and proving a corresponding result subshifts of finite type (although that is not how it was originally proved).

### 2.5 Results for attracting flows

The corresponding results are true for attracting hyperbolic flows. Moreover, the definitions and statements are what one might expect. Let $\phi_{t}: M \rightarrow M(t \in \mathbb{R})$ be a $C^{1+\alpha}$ flow on a compact Riemannian manifold, and let $X \subset M$ be a closed $\phi$-invariant set $\left(\phi_{t}(X)=X\right.$ for all $\left.t \in \mathbb{R}\right)$.

Definition 2.20. The flow $\phi_{t}: X \rightarrow X$ is called a mixing attracting hyperbolic flow if:

1. there exists a continuous splitting $T_{X} M=E^{0} \oplus E^{s} \oplus E^{u}$ where $E^{0}$ is a one dimensional subbundle tangent to the flow orbits and there exist $C>0$ and $0<\lambda<1$ such that

$$
\left\|D \phi_{t} \mid E^{s}\right\| \leq C \lambda^{t} \text { and }\left\|D \phi^{-t} \mid E^{u}\right\| \leq C \lambda^{t}
$$

for $t \geq 0$ (Hyperbolic);
2. there exists an open set $X \subset U \subset M$ such that $X=\cap_{t \in \mathbb{R}} \phi_{t} U$ (Attractor);
3. for non-empty open sets $U, V \subset X$ there exists $T>0$ such that $\phi_{t} U \cap V \neq \emptyset$ (Mixing)
4. the periodic orbits for $\phi_{t}: X \rightarrow X$ are dense in $X$; and
5. $X$ contains no fixed points and $X$ is not a single closed orbit.

We can define equilibrium states and states for flows by analogy with those for diffeomorphisms.
Definition 2.21. Given a continuous function $G: X \rightarrow \mathbb{R}$ we say that an $f$-invariant probability measure $\mu_{G} \in \mathcal{M}_{\phi}(X)$ is an equilibrium measure for $G$ if

$$
\begin{equation*}
h\left(\mu_{G}\right)+\int G d \mu_{G}=\sup \left\{h(\mu)+\int G d \mu: \mu \in \mathcal{M}_{\phi}(X)\right\} \tag{2.6}
\end{equation*}
$$

where $h(\mu)$ is the entropy (or Komogorov-Sinai invariant) for $\mu \in \mathcal{M}_{\phi}(X)$, i.e., $\mu_{G}$ is a measure which maximizes the sum of the entropy and the integral of $G$, over all $\phi$-invariant measures [38], [46].

The value attained by the supremum in (2.1) above is called the pressure and denoted $P(G)$. We can associate a Hölder continuous function $\Phi: X \rightarrow \mathbb{R}$ defined by

$$
\Phi(x)=\lim _{t \rightarrow 0} \frac{1}{t} \log \operatorname{Jac}\left(\phi_{t} \mid E_{x}^{u}\right)
$$

The associated equilibrium state for $\Phi: X \rightarrow \mathbb{R}$ is the Sinai-Ruelle-Bowen measure $\mu_{\Phi}$ and we denote by $\mu_{\Phi}^{u}$ the induced measure on unstable leaves $W_{\delta}^{u}(x)$.

The corresponding result to Theorem 2.15 for constructing equilibrium states for hyperbolic flows is the following:

Theorem 2.22. Let $\phi_{t}: X \rightarrow X$ be a hyperbolic attracting flow and $G: X \rightarrow \mathbb{R}$ a continuous potential. For some $\delta>0$ sufficiently small we can define the measures supported on $W_{\delta}^{u}(x)$ by

$$
\lambda_{T}(A)=\frac{\int_{W_{\delta}^{u}(x) \cap A} e^{\int_{0}^{T}(G-\Phi)\left(\phi_{v} y\right) d v} d \mu_{\Phi}^{u}(y)}{\int_{W_{\delta}^{u}(x)} e^{\int_{0}^{T}(G-\Phi)\left(\phi_{v} y\right) d v} d \mu_{\phi_{1}}^{u}(y)}
$$

for Borel sets $A \subset M$. Taking the average of the push forwards,

$$
\mu_{T}=\frac{1}{T} \int_{0}^{T}\left(\phi_{t}\right)^{*} \lambda_{T} d t, \quad T>0
$$

the weak star limit point of this sequence at $T \rightarrow+\infty$ are equilibrium states for $G$.
If $G$ is only assumed to be continuous then we have the analogue of Theorem 2.19.
Theorem 2.23. If $G: X \rightarrow \mathbb{R}$ is continuous then the weak star accumulation points for the measures $\lambda_{T}$ as $T \rightarrow+\infty$ are equilibrium states for $G$.

Remark 2.24. There are some similar results for partially hyperbolic maps and flows in the particular case that the map(s) act as isometries on the neutral submanifolds.

Question What is the correct formulation of these results for general partially hyperbolic diffeomorphisms?

We should also address the more general hyperbolic setting. This leads to the following questions.
Question What happens if we take some other reference measure $\lambda$ (e.g. Perhaps $\lambda$ could be related to the Hausdorff measure)? What happens if $f$ is not an attractor?

In this spirit we can consider below the more general setting of subshifts of finite type and other Gibbs measures. These then translate into results for more general hyperbolic sets (via Markov partitions, for example).

## 3 Subshifts of finite type

The symbolic analogue of hyperbolic diffeomorphisms are subshifts of finite type. We briefly recall the definition. Let $A$ be a $k \times k$ matrix $(k \geq 2)$ with entries either 0 or 1 .

Definition 3.1. We define the two sided shift space $\Sigma_{A}$ by

$$
\Sigma_{A}=\left\{\underline{x}=\left(x_{n}\right)_{-\infty}^{\infty} \in\{1, \ldots, k\}^{\mathbb{Z}}: A\left(x_{n}, x_{n+1}\right)=1, n \in \mathbb{Z}\right\},
$$

where $A$ is an aperiodic $k \times k$ matrix, and the shift map $\sigma: \Sigma_{A} \rightarrow \Sigma_{A}$ is defined by $(\sigma \underline{x})_{n}=x_{n+1}$ (for $n \geq 0$ ).

In particular, the shift map is topologically mixing if $A$ is aperiodic (i.e., $A^{N}>0$, for $N$ sufficiently large). We will also need a metric on $\Sigma_{A}$ (in order to describe the Hölder properties of potentials):

Definition 3.2. For $\theta \in(0,1)$, we will use the metric on $\Sigma_{A}$ defined by

$$
d(\underline{x}, \underline{y})=\left\{\begin{array}{l}
\theta^{k} \text { when } \underline{x} \neq \underline{y} \text { and } k=\inf \left\{|n|: x_{n} \neq y_{n}\right\} \\
d(\underline{x}, \underline{x})=0 .
\end{array}\right.
$$

In many cases we can model hyperbolic diffeomorphisms by subshifts of finite type and translate results from this symbolic setting (via Markov partitions).

### 3.1 Equilibrium measures for subshifts

We recall that equilibrium states for (Hölder) continuous probabilities on subshifts can be defined in much the same way as for hyperbolic diffeomorphisms.

Definition 3.3. Given a continuous function $G: \Sigma_{A} \rightarrow \mathbb{R}$ we say that an $f$-invariant probability measure $\mu_{G}$ is an equilibrium state for $G$ if

$$
\begin{equation*}
h\left(\mu_{G}\right)+\int G d \mu_{G}=\sup \left\{h(\mu)+\int G d \mu: \mu \in \mathcal{M}_{\sigma}\left(\Sigma_{A}\right)\right\} \tag{3.1}
\end{equation*}
$$

where we recall that $\mathcal{M}_{\sigma}\left(\Sigma_{A}\right)$ denotes the space of $f$-invariant probability measures on $\Sigma_{A}$, i.e., $\mu_{G}$ is a measure which maximizes the sum of the entropy and the integral of $G$, over all $\sigma$-invariant measures. [38], [46].

The value attained by the supremum in (3.1) is (again) called the pressure and denoted by $P(G)$. We can consider sets of a special form:

Definition 3.4. For each $n \in \mathbb{N}$, we denote by

$$
\left[i_{-n}, \ldots, i_{n}\right]=\left\{\underline{x}=\left(x_{n}\right)_{-\infty}^{\infty} \in \Sigma_{A}: x_{-n}=i_{-n}, \ldots, x_{n}=i_{n}\right\}
$$

$a$ cylinder in $\Sigma_{A}$ (where $i_{-n}, \cdots, i_{n} \in\{1, \cdots, k\}$ and $A\left(i_{j}, i_{j+1}\right)=1$ for $-n \leq j \leq n-1$ ).
There is an alternative characterization of the Equilibrium measures as Gibbs measures (which is more reminiscent of the statistical mechanics origins) which we won't need in the sequel:

Remark 3.5 (Gibbsian property of Hölder functions). Given a Hölder continuous potential $\phi: \Sigma_{A} \rightarrow \mathbb{R}$ we can deduce that the associated unique equilibrium state $\mu_{\phi}$ has a Gibbsian property of the following form. There exists $C \geq 1$ such that for any $\underline{x}=\left(x_{n}\right)_{-\infty}^{\infty} \in \Sigma_{A}$ and $n>0$ :

$$
\frac{1}{C} \leq \frac{\mu_{G}\left(\left[i_{-n}, \ldots, i_{n}\right]\right)}{\exp \left(-(2 n+1) P(G)+\sum_{j=-n}^{n} G\left(\sigma^{j} x\right)\right)} \leq C
$$

### 3.2 Changing Equilibrium measures for subshifts

We can mimic Theorem 2.15 (by "generalizing" Sinai's theorem from SRB measures to Equilibrium states). However, here there may not be a canonical reference measure (on small pieces of unstable manifolds).

The analogue of the (small) pieces of unstable manifold in the symbolic setting are the following.

Definition 3.6. For any $\delta>0$ sufficiently small there exists $N=N(\delta)$ such that $\underline{x}=\left(x_{n}\right)_{n=-\infty}^{\infty}$ we have

$$
W_{\delta}^{u}(\underline{x})=\left\{\underline{y}=\left(y_{n}\right)_{n=-\infty}^{\infty} \in \Sigma_{A}: y_{i}=x_{i} \text { for } i \leq N\right\}
$$

is a local unstable manifold (and $\operatorname{diam}\left(W_{\delta}^{u}(\underline{x})\right)<\delta$ ).
Remark 3.7. In the case that $N=0$ then $W_{\delta}^{u}(\underline{x})$ can be identified with sequences $\underline{y}=\left(y_{n}\right)_{n=-\infty}^{\infty}$ for which $x_{i}=y_{i}$ for $i \leq 0$.

In the absence of a natural reference measure it is more natural to take two different equilibrium states (for different Hölder continuous functions) and then try to convert one equilibrium state to another.

We next need to define the analogue of the density of the sequence of measures. To this end we introduce the following notation.

Definition 3.8. Given continuous potentials $G_{1}, G_{2}: \Sigma_{A} \rightarrow \mathbb{R}$ we can consider the weights

$$
S_{n}\left(G_{2}-G_{1}\right)(\underline{x}):=\sum_{i=0}^{n-1}\left(G_{2}-G_{1}\right)\left(\sigma^{i} \underline{x}\right) \text { for } x \in \Sigma_{A}
$$

We can proceed as in the case of hyperbolic attractors. Assume that $G_{1}$ is Hölder continuolus function and $G_{2}$ is continuous. Let $\mu_{G_{1}}$ be the unique equilibrium state for a function $G_{1}$. Then
(i) We associate to $\mu_{G_{1}}$ the (normalized) induced measure $\lambda_{G_{1}}$ supported on the unstable manifold $W_{\delta}^{u}(x) \subset \Sigma_{A}$.
(ii) We can consider the sequence of probability measures $\lambda_{n, G_{2}-G_{1}}$ on $\Sigma_{A}(n \geq 1)$ by normalizing $e^{S_{n}\left(G_{2}-G_{1}\right)} \lambda_{G_{1}}$.
Then the averages of the pushforwards of the measures $\lambda_{n, \phi_{2}-\phi_{1}}$ is an equilibrium state for $\phi_{2}$. This is formulated more precisely in the following theorem.

Theorem 3.9. Let $\mu_{G_{1}}$ be the equilibrium state for a Hölder function $G_{1}$. We can define a family of measures $\lambda_{n, G_{2}-G_{1}}$ supported on $W_{\delta}^{u}(x) \subset \Sigma_{A}\left(x \in \Sigma_{A}\right)$ by

$$
\frac{d \lambda_{n, G_{2}-G_{1}}}{d \lambda_{G_{1}}}(\underline{y})=\frac{e^{S_{n}\left(G_{2}-G_{1}\right)(\underline{y})}}{\int_{W_{\delta}^{u}(\underline{x})} w^{S_{n}\left(G_{2}-G_{1}\right)(\underline{y})} d \lambda_{G_{1}}^{u}(\underline{y})}, \quad n \geq 1 .
$$

for $\underline{y} \in \Sigma_{A}$. Then the averages

$$
\mu_{n, G_{2}-G_{1}}=\frac{1}{n} \sum_{i=0}^{n-1}\left(\sigma^{i}\right)^{*} \lambda_{n, G_{2}-G_{1}}, \quad n \geq 1,
$$

of pushforwards (supported on $\sigma^{i} W_{\delta}^{u}(\underline{x})$ ) converges to the unique equilibrium state $\mu_{G_{2}}$.
As before, if we assume that $G_{1}: X_{A} \rightarrow \mathbb{R}$ is Hölder continuous but only assume $G_{2}: X_{A} \rightarrow \mathbb{R}$ to be continuous then we have the following.

Theorem 3.10. Assume $G_{1}$ is Hölder continuous and $G_{2}$ is a continuous function. Let $\mu_{G_{1}}$ be the equilibrium state for the Hölder continuous function $G_{1}$. We can define a family of measures $\lambda_{n, G_{2}-G_{1}}$ supported on $W_{\delta}^{u}(\underline{x}) \subset \Sigma_{A}$ by

$$
\frac{d \lambda_{n, G_{2}-G_{1}}}{d \lambda_{G_{1}}}(\underline{y})=\frac{e^{S_{n}\left(G_{2}-G_{1}\right)(\underline{y})}}{\int_{W_{\delta}^{u}(\underline{x})} w^{S_{n}\left(G_{2}-G_{1}\right)(\underline{y})} d \lambda_{G_{1}}^{u}(\underline{y})}, \quad n \geq 1 .
$$

Then the weak star limit points of the averages

$$
\mu_{n, G_{2}-G_{1}}=\frac{1}{n} \sum_{i=0}^{n-1}\left(\sigma^{i}\right)^{*} \lambda_{n, G_{2}-G_{1}}, \quad n \geq 1,
$$

of pushforwards (supported on $\sigma^{i} W_{\delta}^{u}(\underline{x})$ ) are equilibrium states for $G_{2}$.
On a more philosophical level we might take $G_{1}=0$ and therefore $\mu_{G_{1}}$ to be the measure of maximal entropy. Therefore Theorem 3.10 might be viewed as a new construction of Gibbs measure for (other) Hölder continuous potentials.

To illustrate Theorem 3.9 we can consider a very simple example in the case of a full shift on two symbols and simple locally constant functions $G_{1}, G_{2}: \Sigma_{A} \rightarrow \mathbb{R}$ (i.e., functions that only depend on finitely many terms from $\underline{x}=\left(x_{n}\right)_{n=-\infty}^{\infty}$ and thus are automatically Hölder continuous).

Example 3.11. Let $\Sigma=\{0,1\}^{\mathbb{Z}}$ and let $\sigma: \Sigma \rightarrow \Sigma$ be the full shift on two symbols given by $\sigma\left(x_{n}\right)_{n \in \mathbb{Z}}=$ $\left(x_{n+1}\right)_{n \in \mathbb{Z}}$. Let $G_{1}: X \rightarrow \mathbb{R}$ be the constant function $G_{1}=-\log 2$, say ${ }^{3}$, then the associated unique equilibrium measure is the Bernoulli measure $\mu_{G_{1}}=\left(\frac{1}{2}, \frac{1}{2}\right)^{\mathbb{Z}}$. For $p \in(0,1)$, we shall consider the locally constant potential, $G_{2}: \Sigma \rightarrow \mathbb{R}$ defined at $\underline{x}=\left(x_{n}\right)_{n=-\infty}^{+\infty}$ by

$$
\varphi_{2}(x)= \begin{cases}\log p & x_{0}=0 \\ \log (1-p) & x_{0}=1\end{cases}
$$

then the associated unique equilibrium measure associated to $G_{2}$ is the Bernoulli measure $\mu_{G_{2}}=$ $(p, 1-p)^{\mathbb{Z}}$. Given any point $\underline{x}=\left(x_{n}\right)_{n=-\infty}^{\infty} \in \Sigma$ we can let

$$
W_{\text {loc }}^{u}(\underline{x})=\left\{y=\left(y_{n}\right)_{n=-\infty}^{\infty}: y_{i}=x_{i} \text { for } i \leq-1\right\}
$$

which we can identify as $W_{\text {loc }}^{u}(\underline{x})=\left\{x_{-}\right\} \times \Sigma^{+}$where $\Sigma^{+}=\{0,1\}^{\mathbb{Z}_{+}}$and $x_{-}=\left(x_{n}\right)_{n=-\infty}^{-1}$. The induced measure $\mu_{G_{1}}$ on $\Sigma$ corresponds to the Bernoulli measure $\left(\frac{1}{2}, \frac{1}{2}\right)^{\mathbb{Z}_{+}}$on $\Sigma^{+}=\{0,1\}^{\mathbb{Z}^{+}}$. We can explicitly write

$$
\begin{aligned}
e^{S_{n} G_{2}(\underline{y})-S_{n} G_{1}(\underline{y})} & \left.=\frac{1}{2^{n}} p^{\#\left\{0 \leq i \leq n-1: y_{i}=0\right\}}(1-p)^{\#\{0 \leq i \leq n-1}: y_{i}=1\right\} \\
& =\frac{\mu_{G_{2}}\left[y_{0}, \ldots, y_{n-1}\right]}{\mu_{G_{1}}\left[y_{0}, \ldots, y_{n-1}\right]}
\end{aligned}
$$

where we denote $\left[y_{0}, \cdots, y_{n-1}\right]=\left\{\left(z_{k}\right)_{k=-\infty}^{\infty}: z_{i}=y_{i}\right.$ for $\left.0 \leq i \leq n-1\right\}$. We can also assume the simplifications

1. $P\left(G_{1}\right)=P\left(G_{2}\right)=0$;
2. We can write

$$
\begin{aligned}
& \int_{W_{\text {loc }}^{u}(\underline{x})} e^{S_{n} G_{2}(\underline{y})-S_{n} \varphi_{1}(\underline{y})} d \mu_{G_{1}}^{u}(\underline{y}) \\
& =\sum_{\left[y_{0}, \ldots, y_{n-1}\right]} \mu_{G_{1}}\left(\left[y_{0}, \ldots, y_{n}\right]\right) \frac{\left.\mu_{G_{2}}\left[y_{0}, \ldots, y_{n-1}\right]\right)}{\left.\mu_{G_{1}}\left[y_{0}, \ldots, y_{n-1}\right]\right)}=1 ;
\end{aligned}
$$

and

[^2]3. if we let $A=\left[z_{-M}, \cdots, z_{N}\right]$ then for $n \geq M$ we can write
\[

$$
\begin{aligned}
\left(\sigma^{n}\right)^{*} \lambda_{n}(A) & =\sum_{\left[y_{0}, \ldots, y_{n-1}\right]} \mu_{\phi_{1}}\left(\sigma^{-n}(A) \cap\left[y_{0}, \ldots, y_{n-1}\right]\right) \frac{\left.\mu_{G_{2}}\left[y_{0}, \ldots, y_{n-1}\right]\right)}{\left.\mu_{G_{1}}\left[y_{0}, \ldots, y_{n-1}\right]\right)} \\
& =\sum_{\left[y_{0}, \ldots, y_{n-1}\right]} \frac{1}{2^{N+M+1}} p^{\#\left\{-M \leq i \leq N: z_{i}=0\right\}}(1-p)^{\#\left\{-M \leq i \leq N: z_{i}=1\right\}} \\
& =\mu_{G_{2}}(A)
\end{aligned}
$$
\]

Thus we finally conclude that $\left(\sigma^{n}\right)^{*} \lambda_{n}(A)=\mu_{G_{2}}(A)$ for $n \geq M$, which is consistent with Theorem 1.1.
The proof of Theorem 3.10 follows the same pattern as the proof of Gibbs measures for attractors. The following result relates the difference of the two pressures $P\left(G_{2}\right), P\left(G_{1}\right)$ to a certain growth rate and is key to the proof of Theorem 3.10.

Proposition 3.12. Let $G_{1}, G_{2}: \Sigma_{A}^{+} \rightarrow \mathbb{R}$ be (Hölder) continuous potentials. Then

$$
P\left(G_{2}\right)=P\left(G_{1}\right)+\lim _{n \rightarrow \infty} \frac{1}{n} \log \int_{W_{l o c}^{u}(\underline{x})} e^{S_{n}\left(G_{2}-G_{1}\right)(\underline{y})} d \mu_{G_{1}}^{u}(\underline{y}) .
$$

### 3.3 Rigidity and flexability of pressure

The questions of rigidity (i.e., how much data specifies the system) and flexibility (i.e., which characteristic properties can be achieved) are very popular in analysis and geometry. For example, a classical example is the Kac problem: "Can you hear the shape of a drum?" (i.e, whether the spectrum of the laplacian on a Riemann surface, with negative Euler characteristic, determines the metric). The answer in this case is "no". More recent dynamical formulations and results are due to Erchenko and Katok (for Anosov systems) [9].

Inspired by these developments, we want to recall a wellknown problem.
Definition 3.13. Given a Hölder continuous function $G: \Sigma_{A} \rightarrow \mathbb{R}$ we can consider the function $p_{G}: \mathbb{R}^{+} \rightarrow \mathbb{R}$ defined by

$$
p_{G}(t):=P(t G)=\sup \left\{h(\mu)+t \int G d \mu: \mu \in \mathcal{M}_{\sigma}\left(\Sigma_{A}\right)\right\} \text { for } t>0 .
$$

This can be viewed as a one parameter sub-family of $P: C^{\alpha}\left(\Sigma_{A}\right) \rightarrow \mathbb{R}$ restricted to $\mathbb{R}^{+} G:=$ $\{t G: t \in \mathbb{R}\} \subset C^{\alpha}(\Sigma)$.
Question. Does a knowledge of the function $p_{G}(t)$ on $\mathbb{R}$ determine $G \in C^{\alpha}\left(\Sigma_{A}\right)$ ?
The answer at this level of generality is clearly " $n o$ ": Clearly we can replace $G$ by $G=G+u \circ \sigma-u$ for any $u \in C^{\alpha}\left(\Sigma_{A}\right)$ and get the same function (i.e., $p_{G}(r)=p_{G+u \sigma-u}(t)$ ). Moreover, one can imagine examples with "automorphisms" on symbols that also give the same function, etc.

Each function $p_{G}: \mathbb{R}^{+} \rightarrow \mathbb{R}$ has a number of necessary properties:

1. The function $t \mapsto p_{G}(t)$ is real analytic (i.e., there is a power series expansion with non-zero radius of convergence at each point $t \in \mathbb{R})$. In particular, it is $C^{\infty}$ and the derivatives $p_{G}^{(n)}(t):=\frac{d^{n} p_{G}(t)}{d t^{n}}$ exist for all $n \geq 1 .{ }^{4}$

[^3]2. A second property is that $p_{G}(t)$ is convex, The convexity comes by showing that $\sigma^{2}(t):=p_{G}^{(2)}(t) \geq$ 0 . In particular one can write
\[

$$
\begin{aligned}
\sigma^{2}(t) & =\sum_{n \in \mathbb{Z}}\left(\int G \circ \sigma^{n} \cdot G d \mu_{t G}-\left(\int G d \mu_{t G}\right)^{2}\right) \\
& =\lim _{n \rightarrow+\infty} \frac{1}{n} \int\left(\sum_{j=0}^{n-1} G\left(\sigma^{j} \underline{x}\right)-n \int G d \mu_{t G}\right)^{2} d \mu_{t G}(x) \geq 0 .
\end{aligned}
$$
\]

3. Finally, the supporting line (i.e., tangent lines to $\left(t, p_{G}(t)\right)$ ) intersect the vertical axis in a closed bounded subinterval in $[0,+\infty)$.

Remark 3.14. The value $\sigma^{2}(t)$ has an alternative interpretation in terms of a central limit theorem. Let $\mu_{t G}$ be the Equilibrium measure for $t G \in C^{\alpha}\left(\Sigma_{A}\right) \rightarrow \mathbb{R}$ then for real numbers $\alpha<\beta$ we have that

$$
\begin{aligned}
& \mu_{t G}\left(\left\{x \in \Sigma_{A}: \alpha<\sum_{j=0}^{n-1} G\left(\sigma^{i} n x\right)-\int G d \mu_{t G}<\beta\right\}\right) \\
& \rightarrow \frac{1}{\sqrt{2 \pi \sigma^{2}}} \int_{\alpha}^{\beta} e^{-u^{2} / 2 \sigma^{2}} d u \text { as } n \rightarrow+\infty
\end{aligned}
$$

The following is one of the more recent formulations of a well known question:
Question [Kucherenko-Quas]. Let $\alpha>0$. Can a convex analytic function $F:(\alpha,+\infty) \rightarrow \mathbb{R}$ with its supporting lines intersecting the vertical axis in a closed bounded interval in $[0,+\infty)$ always be realised by the pressure function $p_{G}(t)=F(t)$ of some Hölder function $G$ on a subshift of finite type?

Apparently, excluding 0 from the domain of $F(t)$ helps eliminate extra conditions that $F(0)$ correspond to the topological entropy (and thus its exponential would be an algebraic integer).

Kucherenko and Quas established a somewhat surprising related result where they showed the existence of a continuous function $G$ corresponding to $F(t)$ [21]:

Theorem 3.15 (Kucherenko-Quas). Given a convex analytic function $F(t)$ with its supporting lines intersecting the vertical axis in a closed bounded interval in $[0,+\infty)$ then it can be realised by the pressure function $p_{G}(t)=F(t)$ of some continuous function $G$ on a subshift of finite type.

However, this leaves open the original question of whether a Hölder function can always be found. In this direction we have the following result relating the second, third and fourth derivatives of any such function $p(t)$ [26]:
Theorem 3.16. For any Hölder continuous function $G$ on a full shift space there exists a constant $M=M(G)>0$ such that

$$
\sqrt{2 \pi^{3}}\left(p_{G}^{(2)}(t)\right)^{3 / 2}\left|p_{G}^{(3)}(t)\right| \leq 9\left|p_{G}^{(3)}(t)\right|+2\left|p_{G}^{(4)}(t)\right|+3 \sqrt{2 \pi^{3}} M\left(p_{G}^{(2)}(t)\right)^{5 / 2}
$$

for any $t>0$.
The proof is based on calculating formulae for $P^{(n)}(t)(n=3,4)$ which is based on expressions of the form

$$
\begin{aligned}
P^{(3)}(t) & =\lim _{n \rightarrow+\infty} \frac{3}{n} \int\left(\sum_{j=0}^{n-1} G\left(f^{j} x\right)-n \int G d \mu_{t}\right)^{3} H(x) d \mu(x) \\
& +\lim _{n \rightarrow+\infty} \frac{1}{n} \int\left(\sum_{j=0}^{n-1} G\left(f^{j} x\right)-n \int G d \mu_{t}\right)^{3} d \mu(x)
\end{aligned}
$$

for some function $H$, and similar expressions for $P^{(4)}(t)$.
In particular, to answer the question of Kucherenko and Quas in the negative, it suffices to present an example of a function $F(t)$ whose derivatives do not satisfy point 1 above. An explicit example is

$$
F(t)=\frac{2 t^{2}+3 t+t e^{-t^{2}}+e^{-t^{2}}}{t}
$$

Kucherenko and Quas had an alternative approach to reach the same conclusion. [22].

## 4 Resonances

We now turn to the problem on how quickly mixing occurs in the hyperbolic diffeomorphism setting.

### 4.1 Speed of mixing

Given a diffeomorphism $f: M \rightarrow M$ and a $f$-invariant probability measure $\mu$ we can associate to continuous "test functions" $F_{1}, F_{2}: M \rightarrow \mathbb{R}$ a correlation function

$$
\rho(k)=\int F_{1} \circ f^{k} \cdot F_{2} d \mu-\int F_{1} d \mu \int F_{2} d \mu, \quad k \geq 0 .
$$

If $\mu$ is an equilibrium state for a Hölder continuous function and $f$ is Anosov then the transformation is mixing, i.e., $\rho(k) \rightarrow 0$ as $k \rightarrow+\infty$. Moreover, it is well known that when $F_{1}, F_{2}: M \rightarrow \mathbb{R}$ are Hölder continuous, then this convergence is exponentially fast, i.e., there exist $C>0$ and $0<\theta<1$ such that $|\rho(k)| \leq C \theta^{k}$ for $k \geq 1$.
Question. When can we say more about the convergence of $\rho(k) \rightarrow 0$ as $k \rightarrow+\infty$ ?
Ideally, we want to find cases such that for any $\epsilon>0$, there exist:

1. sequences of complex numbers $\left\{\rho_{n}\right\}_{n=1}^{\infty}$ converging to 0 ; and
2. polynomials $\left\{p_{n}\right\}_{n=1}^{N}$
such that

$$
\begin{equation*}
\int F_{1} \circ f^{m} \cdot F_{2} d \mu-\int F_{1} d \mu \int F_{2} d \mu=\sum_{n=1}^{N} p_{n}(m) \rho_{n}^{m}+\mathcal{O}\left(\epsilon^{m}\right), \text { for } m \geq 0 \tag{4.1}
\end{equation*}
$$

(where the degree of $p_{n}$ is determined by the multiplicity of $\rho_{n}$ ).
Definition 4.1. The values $\left\{\rho_{n}\right\}_{n=1}^{\infty}$ can be called resonances
The following powerful result gives the existence of such expansions [11].
Theorem 4.2 (after Gouëzel-Liverani). For any $C^{\infty}$ mixing Anosov diffeomorphism $f: M \rightarrow M$; a Gibbs measure $\mu$ for a $C^{\infty}$ potential $G: M \rightarrow \mathbb{R}$ and $C^{\infty}$ test functions $F_{1}, F_{2}: M \rightarrow \mathbb{R}$ an expansion of the above form (4.1) holds. Moreover, the values $\left\{\rho_{n}\right\}$ are independent of the functions $F_{1}, F_{2}: M \rightarrow \mathbb{R}$.

### 4.2 Examples

We can consider two examples of toral automorphisms described earlier. The first is particularly well known.

Example 4.3 (Arnol'd CAT maps). In the case of the (linear) Arnol'd CAT map $f: \mathbb{T}^{1} \times \mathbb{T}^{1} \rightarrow \mathbb{T}^{1} \times \mathbb{T}^{1}$ given by $f(z, w)=\left(z^{2} w, z w\right)$ and the area $\mu$ we can consider $F_{1}, F_{2}: \mathbb{T}^{1} \times \mathbb{T}^{1} \rightarrow \mathbb{R}$. In this case, for any $\epsilon>0$ :

$$
\int F_{1} \circ f^{k} \cdot F_{2} d \mu-\int F_{1} d \mu \int F_{2} d \mu=\mathcal{O}\left(\epsilon^{k}\right), \text { for } k \geq 0
$$

(i.e., there are no non-zero resonances). This example is easily analyzed using Fourier series for

$$
F_{1}(z, w)=\sum_{(n, m) \in \mathbb{Z}^{2}} a_{(n, m)} z^{n} w^{m} \text { and } F_{2}(x, y)=\sum_{(n, m) \in \mathbb{Z}^{2}} b_{(n, m)} z^{n} w^{m}
$$

and then

$$
\begin{aligned}
\int F_{1} \circ f^{m}(z, w) \cdot F_{2}(x, y) d z d w-\int & F_{1}(x, y) d x d y \int F_{2}(z, w) d z d w \\
& =\sum_{(n, m) \in \mathbb{Z}^{2}} a_{n, m} b_{(n, m)\left(\begin{array}{ll}
2 & 1 \\
1 & 1
\end{array}\right)^{n} .}
\end{aligned}
$$

The assumption $C^{\infty}\left(\mathbb{T}^{1} \times \mathbb{T}^{1}\right)$ guarantees that $\left|a_{(n, m)}\right|,\left|b_{(n, m)}\right|=O\left(\left(n^{2}+m^{2}\right)^{-N}\right)$ for all $N>0$ which leads to an expression (4.1) where all the $\rho_{n}$ vanish.

We would like to also examples where the values $\rho_{n}$ are not identically zero. Interesting concrete examples were introduced by Slipantschuk, Bandtlow and Just based on toral automorphisms and Blaschke products [44]. We want to review some of these (and related) examples although we will take a slightly different viewpoint.

Example 4.4 (Slipantschuk, Bandtlow and Just). Given $\lambda$ with $|\lambda|<1$, the Anosov diffeomorphism $B_{\lambda}: \mathbb{T}^{1} \times \mathbb{T}^{1} \rightarrow \mathbb{T}^{1} \times \mathbb{T}^{1}$ defined by

$$
B_{\lambda}(z, w)=\left(\left(\frac{z+\lambda}{1-\bar{\lambda} z}\right) z w,\left(\frac{z+\lambda}{1-\bar{\lambda} z}\right) w\right)
$$

is area preserving. Moreover, the resonances $\left\{\rho_{n}\right\}$ with respect to smooth functions $f, g: \mathbb{T}^{2} \rightarrow \mathbb{R}$ take the form

$$
\{0,1\} \cup\left\{\lambda^{l}, \bar{\lambda}^{l}: l \in \mathbb{N}\right\} .
$$

We can also consider one parameter families based on of the simple(r) linear Anosov map $(z, w) \mapsto$ $(z w, z)$ [37].

Example 4.5. For $\lambda \in \mathbb{C}$ with $|\lambda|<1$ the Anosov map $T_{\lambda}: \mathbb{T}^{2} \rightarrow \mathbb{T}^{2}$ defined by

$$
T_{\lambda}:(z, w) \mapsto\left(\left(\frac{z+\lambda}{1+\bar{\lambda} z}\right) w, z\right) .
$$

is area preserving. (When $\lambda=0$ this reduces to the linear map $T_{0}(z, w)=(z w, z)$.) For $\lambda_{1}=\sqrt{\lambda} a$ square root of $\lambda$ we have

$$
\begin{equation*}
\{0,1\} \cup\left\{\omega^{m} \overline{\lambda_{1}} \lambda^{n} \mid m, n \in \mathbb{N}_{0}, m+n \geq 1, \omega= \pm 1\right\} \tag{4.2}
\end{equation*}
$$

Remark 4.6. Checking that the (perturbed) maps are still Anosov merely involves checking a condition of Pujals-Shub [37].


Figure 2: (i) A plot of the resonances of $T_{\lambda}$, for $\lambda=0.8 e^{31 i \pi / 50}$; (ii) The resonances of $T_{\lambda} \circ T_{\mu}$, for $\lambda=0.9 e^{i \pi / 4}, \mu=0.65 e^{6 i \pi / 5}$; (iii) The resonances of $B_{\lambda}$, for $\lambda=0.99 e^{37 i \pi / 50}$.

We can also build new examples from these
Example 4.7. For $\lambda, \mu$ with $|\lambda|,|\mu|<1$ then under mild hypotheses $T_{\lambda} \circ T_{\mu}$ has resonances

$$
\{0,1\} \cup\left\{\lambda^{m} \mu^{n}, \lambda^{m} \bar{\mu} \mu^{n}, \bar{\lambda}^{m} \bar{\mu} \mu^{n}, \bar{\lambda}^{m} \mu^{n} \mid(m, n) \in \mathbb{N}_{0}^{2} \backslash\{(0,0)\}\right\} .
$$

The proof of these results is based on writing a suitable infinite matrix for the Koopman operator(s) $F \circ F \circ T_{\lambda}$, etc., acting on suitable Banach spaces [44], [37].

## 5 Geodesic flows

An historically important example in ergodic theory is the geodesic flow on a surface of negative curvature. This is a flow (hence the name) which takes place not on the two dimensional space $V$ but on the three dimensional space of tangent vectors of length 1 (with respect to the Riemannian metric $\rho)$.


Figure 3: A geodesic arc on $V$

### 5.1 Definition of the geodesic flow

We can now introduce some dynamics. We actually want to define a flow on the compact three dimensional manifold

$$
M=S V:=\left\{v \in T V:\|v\|_{\rho}=1\right\}
$$

which is the sphere bundle. To define a geodesic flow $\phi_{t}: M \rightarrow M(t \in \mathbb{R})$ we can take $v \in M$ and choose the unique (unit speed) geodesic $\gamma_{v}: \mathbb{R} \rightarrow V$ such that $\dot{\gamma}_{v}(0)=v$. We can then define $\phi_{t}(v):=\dot{\gamma}_{v}(t)$.

### 5.2 Topological entropy

For any flow $\phi_{t}: M \rightarrow M$ we can associate the topological entropy $h(\phi) \geq 0$ (of the time one flow $\phi_{t=1}$ ) defined as follows [?].
Definition 5.1. Given $T>0$ and $\epsilon>0$ we let $N(T, \epsilon)$ be the cardinality of the smallest finite set $X=X(\epsilon) \subset M$ such that for any $v \in M$ there exists $v^{\prime} \in X$ such that $\sup _{0 \leq t \leq T} d\left(\phi_{t} v, \phi_{t} v^{\prime}\right)<\epsilon$. The topological entropy is then given by

$$
h(\phi):=\lim _{\epsilon \rightarrow 0} \limsup _{T \rightarrow+\infty} \frac{1}{T} \log N(T, \epsilon) .
$$

This value is always non-zero and finite.
Let us henceforth assume that $V$ has a Riemannian metric of (strictly) negative curvature. In the case of geodesic flows the topological entropy has a simple geometric interpretation.
Theorem 5.2 (Manning's volume entropy, [27]). Let $\widetilde{V}$ be the universal cover for $V$ (with the lifted metric $\widetilde{\rho})$. Fix any point $x_{0} \in \widetilde{M}$ we let $B\left(x_{0}, R\right):=\left\{x \in \widetilde{M}: d\left(x, x_{0}\right)<R\right\}$ and then

$$
h(\phi)=\lim _{R \rightarrow+\infty} \frac{1}{R} \log \operatorname{Area}_{\widetilde{\rho}}\left(B\left(x_{0}, R\right)\right) .
$$

Proof. In negative curvature when we consider the lift of the Riemannian metric to the universal cover $\widetilde{V}$. Then $N(R, \epsilon)$ can be used to give bounds on the area of an annulus in $\widetilde{V}$ with radius $R$ and width approximately $\epsilon$. However, in negative curvature this has the same rate of growth as the area $\operatorname{Area}_{\tilde{\rho}}\left(B\left(x_{0}, R\right)\right)$ of a ball of radius $R$

The following classical result shows the importance of topological entropy as an invariant.
Lemma 5.3. It two flows $\phi_{1, t}: M_{1} \rightarrow M_{1}$ and $\phi_{2, t}: M_{2} \rightarrow M_{2}$ are topologically conjugate then the have the same topological entropy, i.e., $h\left(\phi_{1}\right)=h\left(\phi_{2}\right)$.

### 5.3 Entropies of measures

Let $\mu$ be a $\phi$-invariant probability measure (i.e., $\mu\left(\phi_{t} B\right)=\mu(B)$ for any Borel sets $B \subset M$ and $\mu(M)=1$ ). We can then associate the entropy $0 \leq h(\phi, \mu) \leq h(\phi)$ of the measure $\mu$ (of the time one flow $\phi_{t=1}$ ).
Definition 5.4 (After A. Katok [15]). Given $T>0, \delta>0$ and $\epsilon>0$ we let $N(T, \epsilon, \delta)$ denote the cardinality of the smallest finite set $S=S(T, \epsilon, \delta) \subset M$ such that

$$
\mu\left(\left\{v \in M: \exists v^{\prime} \in S \text { with } \sup _{0 \leq t \leq T} d\left(\phi_{t} v, \phi_{t} v^{\prime}\right)<\epsilon\right)>1-\delta\right.
$$

The entropy of the measure $\mu$ is then given by

$$
h(\phi, \mu):=\lim _{\delta \rightarrow 0} \lim _{\epsilon \rightarrow 0} \limsup _{T \rightarrow+\infty} \frac{1}{T} \log N(T, \epsilon, \delta) .
$$

We return to concentrating on geodesic flows. Our main example so far of an invariant measure so far is the Liouville measure:

Example 5.5 (Liouville measure). We recall that the Liouville measure $\nu$ is the $\phi$-invariant probability measure equivalent to the volume. In the particular case that $\rho_{0}$ is a metric of constant negative curvature $\kappa=-1$ then $h(\phi, \mu)=h(\phi)=1$.

There is another natural $\phi$-invariant probability measure:
Example 5.6 (Measure of maximal entropy). There exists unique $\phi$-invariant probability measure $\mu_{\max }$ such that $h\left(\phi, \mu_{\max }\right)=h(\phi)$. In the particular case, that $\rho$ is a metric of constant negative curvature then $\mu_{\max }$ is equal to the Liouville measure. (Moreover, they agree only when the metric $\rho$ has constant negative curvature).

There is now (another) classical result due to Katok relating entropies for different metrics.
Let us consider two metrics $\rho_{1}, \rho_{2}$ on a compact surface $V$. Let $h\left(\phi_{2}\right)$ be the topological entropy for the geodesic flow for $\left(V, \rho_{1}\right)$. Let $h\left(\phi_{1}, \mu_{1}\right)$ be the entropy of the geodesic flow for $\left(V, \rho_{1}\right)$ with respect to the measure $\mu_{1}$. We can then consider

$$
\int_{\|v\|_{\rho_{1}}=1}\|v\|_{\rho_{2}} d \mu_{1}(v)
$$

which measures the average change in the lengths of tangent vectors between different measures.
Lemma 5.7. There is an inequality

$$
h\left(\phi_{2}\right) \geq\left(\int_{\|v\|_{\rho_{1}}=1}\|v\|_{\rho_{2}} d \mu_{1}(v)\right)^{-1} h\left(\phi_{1}, \mu_{1}\right)
$$

Proof. The idea of the proof is to get a lower bound on the topological entropy $h\left(\phi_{2}\right)$ by constructing orbit segments for $\phi_{2, t}$. This is done using ergodic theory for the geodesic flow $\phi_{1, t}: M_{1} \rightarrow M_{1}$ and $\nu_{1}$ and the function $F: M_{1} \rightarrow \mathbb{R}$

$$
M_{1} \ni v \mapsto F(v)=\|v\|_{\rho_{2}} \in \mathbb{R}
$$

If we assume (for simplicity) that $\mu_{1}$ is ergodic then by the Birkhoff ergodic theorem then for almost every $\left(\mu_{1}\right) v \in M_{1}$ and sufficiently large $T$ :

$$
\frac{1}{T} \int_{0}^{T} F\left(\phi_{1, t}(v) d t=\frac{1}{T} \int_{0}^{T}\left\|\phi_{1, t}(v)\right\|_{\rho_{2}} d t \rightarrow\left(\int_{\|v\|_{\rho_{1}}=1}\|v\|_{\rho_{2}} d \mu_{1}(v)\right) \text { as } T \rightarrow+\infty\right.
$$

Thus for large $T$ we have "most" orbit segments of $\phi_{1}$ - length approximately $T$ correspond to orbit segments of $\phi_{2}$ - length

$$
T\left(\int_{\|v\|_{\rho_{1}}=1}\|v\|_{\rho_{2}} d \mu_{1}(v)\right)
$$

We can use these to get a lower bound on $h\left(\phi_{2}\right)$.
This leads to the main rigidity result on entropy.
Theorem 5.8 (Katok Entropy Rigidity Theorem [16]). The topological entropy is minimised on metrics of constant area at metrics of constant negative curvature (i.e., If $\rho_{2}$ is a metric of negative curvature and $\rho_{1}$ is a metric of constant negative curvature with $\operatorname{Area}_{\rho_{1}}(V)=\operatorname{Area}_{\rho_{2}}(V)$ then $\left.h\left(\phi_{2}\right) \geq h\left(\phi_{1}\right)\right)$.

Proof. By Koëbe's Theorem we can assume that $\rho_{2}$ is conformally equivalent to a metric $\rho_{1}$ of constant negative curvature, i.e., $\rho_{2}=f(x) \rho_{1}$, where $f: V \rightarrow \mathbb{R}^{+}$is a strictly positive smooth function.

Let $\nu_{1}$ be the Liouville measure for $M_{1}$ (i.e., $V$ with $\rho_{1}$ ). By conformality we can write

$$
\int_{\|v\|_{\rho_{1}=1}}\|v\|_{\rho_{2}} d \nu_{1}(v)=\int_{V} f(x) d \sigma_{1}(x) \text { and } \int_{V} f(x)^{2} d \sigma_{1}(x)=\sigma_{2}(V)=1
$$

where $\sigma_{1}$ and $\sigma_{2}$ are the normalised areas on $V$ (associated to $\rho_{1}$ and $\rho_{2}$, respectively). Thus

$$
\int_{V} f(x) d \sigma_{1}(x) \leq\left(\int_{V} f(x)^{2} d \sigma_{1}(x)\right)^{\frac{1}{2}}=1
$$

with equality if and only if $\rho=1$.
If $\rho_{1}$ is a metric of constant negative curvature then we know by Example 5.6 that $h\left(\phi_{1}, \mu_{1}\right)=h\left(\phi_{1}\right)$. We can then apply Lemma 5.7.

Remark 5.9. There are higher dimensional analogues of the Katok's theorem due to Besson-ContrerasGallot [4].

### 5.4 Smoothness of entropy

Assume that we change the metric smoothly then we might expect the entropy to vary smoothly.
To this end we need to make sense of smooth changes of metrics. We can interpret the metric as maps $\rho \in \Gamma\left(V, \mathcal{S}_{2}\right)$ where $\mathcal{S}_{2}$ are positive symmetric $2 \times 2$-matrices and denote by $\phi^{\rho}$ the associated geodesic flow.

Theorem 5.10 (Katok, Knieper, Pollicott and Weiss [19]). Given a $C^{\infty}$ family $(-\epsilon, \epsilon) \ni \lambda \mapsto \rho_{\lambda} \in$ $C^{\infty}\left(V, \mathcal{S}_{2}\right)$ the map

$$
(-\epsilon, \epsilon) \ni \lambda \mapsto h\left(\phi^{\rho_{\lambda}}\right) \in \mathbb{R}^{+}
$$

is $C^{\infty}$.
There is also an interpretation for the derivative:
Theorem 5.11 (Katok, Knieper and Weiss [18]). We can write the first derivative

$$
\left.\frac{d}{d \lambda} h\left(\phi^{\rho_{\lambda}}\right)\right|_{\lambda=0}=-\frac{1}{2} \int_{M_{0}} \frac{d}{d \lambda}\|v\|_{\rho_{\lambda}}^{2} d \mu_{\max }(v)
$$

where $\mu_{\max }$ is the unique probability measure (such that $h\left(\phi, \mu_{\max }\right)=h(\phi)$ ).

### 5.5 The Anosov property and Lyapunov exponents

The negative curvature gives rise to the the Anosov property through the negative curvature. One way to see this is via the Jacobi and Riccati equations.

Let $v \in V$ and let $\gamma_{v}: \mathbb{R} \rightarrow V$ be the associated geodesic on $V$. Let us then denote by $\kappa(t):=$ $\kappa\left(\gamma_{v}(t)\right)<0$ the curvature at $\gamma_{v}(t) \in V$ (i.e., after time $t$ along the (geodesic) orbit). The expansion and contraction in $E^{u}$ and $E^{s}$ along the geodesic (or orbit) can be seen through these solutions to the Jacobi equations.

Definition 5.12 (Jacobi equation). Consider solutions $J_{v}: \mathbb{R} \rightarrow \mathbb{R}$ on the real line to

$$
J_{v}^{\prime \prime}(t)+\kappa(t) J_{v}(t)=0 .
$$

The size of solutions $|J(t)|$ either grow or contract exponentially (for $E^{u}$ and $E^{s}$ ) depending on the initial conditions. If we define $a_{v}(t)=J_{v}^{\prime}(t) / J_{v}(t)$ then the Jacobi equation reduces to the Riccati equation.

Definition 5.13 (Riccati Equation). Consider solutions $a_{v}: \mathbb{R} \rightarrow \mathbb{R}$ on the real line to

$$
\begin{equation*}
a_{v}^{\prime}(t)+a_{v}(t)^{2}+\kappa(t)=0 . \tag{5.1}
\end{equation*}
$$

These determine the rate of growth (or contraction) for $E^{u}$ and $E^{s}$ along the (geodesic) orbit for $v$.
Example 5.14 $(\kappa=-1)$. In the case of constant negative curvature $\kappa=-1$ then one sees that there are two solutions to (5.1):

1. $a_{v}=1$ corresponding to an expansion $e^{t}$ in $E^{u}$; and
2. $a_{v}=-1$ corresponding to a contraction $e^{-t}$ in $E^{u}$.

We can consider the average expansion along a typical (geodesic) orbit of the positive solution. Let $\mu$ be any $\phi$-invariant (ergodic) probability measure then by the Birkhoff ergodic theorem for a.e., $(\mu)$ $v \in V$

$$
\lim _{T \rightarrow+\infty} \frac{1}{T} \int_{0}^{T} a_{v}(t) d \mu(v)=\int_{M} a_{v} d \mu(v) .
$$

This is the (positive) Lyapunov exponent.

### 5.6 Ricci flow and entropy

Given a metric $\rho$ and the associated curvature function $\kappa: V \rightarrow \mathbb{R}$ then by a slight abuse of notation we can also write $\kappa: M \rightarrow \mathbb{R}$ where $v \in T_{x} V$ and $\kappa(v):=\kappa(\pi(v))$ where $\pi: M \rightarrow V$ is the canonical projection. The average curvature satisfies:

$$
\bar{\kappa}:=\int \kappa(v) d \nu(v)=-\pi(g-1),
$$

where $\nu$ is the (normalised) Liouville measure on $M$, using the Gauss-Bonnet theorem.
Example 5.15 (Constant curvature metrics revisited). In the case of metrics $\rho_{0}$ of constant curvature $\kappa(x)=\bar{\kappa}$ we have that the entropy is

$$
h\left(\mu^{\rho_{0}}\right)=\sqrt{|\bar{\kappa}|} .
$$

By Katok's theorem we have that for other metrics $\rho$ of variable negative curvature and the same total area we have that

$$
h\left(\phi^{\rho}\right)>h\left(\mu^{\rho_{0}}\right)=\sqrt{|\bar{\kappa}|} .
$$

It is fashionable to study how families of metrics $\rho^{t}$ evolve under the Ricci flow. Recall that a Riemannian metric can be thought of as $\rho=\left\{\|\cdot\|_{\rho, x}\right\}_{x \in V}$, where $\|\cdot\|_{\rho, x}$ is a norm on $T_{x} V=\{x\} \times \mathbb{R}^{2}$. With a suitable choice of coordinates we can write each norm in terms of (positive definite) $2 \times 2$ matrices $\left(g_{i j}(x)\right)$ through the associated definite quadratic form

$$
\|v\|_{\rho, x}^{2}=g(x)(v, v):=v^{T}\left(g_{i j}(x)\right) v
$$

We can now define the flow on the space of metrics (of fixed area).
Definition 5.16. We can define the Ricci flow on the space of metrics (of constant area) by

$$
\begin{equation*}
\frac{d}{d t} g_{i j}^{t}(x)=-2\left(\kappa^{t}(x)-\bar{\kappa}\right) g_{i j}^{t} \text { for } x \in V \tag{5.2}
\end{equation*}
$$

where $\kappa^{t}(x)$ is the curvature of $\rho(t):=\left(g_{i j}^{t}\right)$.
There is a connection between solutions $\rho_{t}=\left(g_{i j}^{t}(x)\right)$ to the Ricci equation and the topological entropy.

Theorem 5.17 (Manning [28]). Starting from a metric $\rho=\left(g_{i j}\right)$ with non-constant negative curvature then the topological entropy is strictly decreasing along the solution $\rho_{t}$ to the Ricci equation (5.2).

To prove the entropy is decreasing along the orbit $\rho_{t}$ one can use the formula for the derivative of the topological entropy (along the solution to the Ricci equation):

$$
\left.\frac{d}{d t} h\left(\phi^{\rho^{t}}\right)\right|_{t=0}=-\frac{1}{2} \int_{M}\left(\left.\frac{d}{d t} g_{i j}\right|_{t=0}\right) d \mu_{\max }(v)=\int_{V}(\kappa-\bar{\kappa}) d \mu_{\max }(v)
$$

where $\mu_{\text {max }}$ is the measure of maximal entropy. We want to show the derivative is negative, i.e., that

$$
-\int_{V} \kappa(v) d \mu_{\max }(v)>\bar{\kappa}
$$

Step 1. By Katok's theorem $\sqrt{\bar{\kappa}}<h\left(\phi^{\rho}\right)$.
Step 2. The solution $a_{v}:=a_{v}(0)>0$ to the Riccati equation (1) gives the Lyapunov exponent and we have an inequality:

Lemma 5.18 (Ruelle [40]). We can write

$$
h(\phi)=h\left(\phi, \mu_{\max }\right) \leq \int_{M} a_{v} d \mu_{\max }(v) .
$$

Step 3. By the Cauchy-Schwarz inequality we can write

$$
\int_{M} a_{v} d \mu_{\max }(v) \leq\left(\int_{M} a_{v}^{2} d \mu_{\max }(v)\right)^{\frac{1}{2}}
$$

Step 4. We can the substitute for $a_{v}^{2}$ from the Riccati equation and observe that

$$
\begin{aligned}
& \int_{M} a_{v}^{2} d \mu \max (v)=-\underbrace{\int_{M} \frac{d a}{d s} d \mu \max (v)}_{=0}-\int_{M} \kappa(x) d \mu \max (v) \\
&=-\int_{M} \kappa(x) d \mu \max (v) .
\end{aligned}
$$

Comparing the above inequalities the result follows.

## Part II <br> Estimating Dimension and Lyapunov exponents

We now turn to another class of problems associated to hyperbolic systems.

## 6 Introduction

In ergodic theory and dynamical systems there are natural characteristic values. We are particularly interested in the Hausdorff dimension of dynamically defined sets and Lyapunov exponents for measures. We will begin by considering Hausdorff dimension of dynamically defined sets. Later we will turn to Lyapunov exponents.

We take as our guiding philosophy that it is useful to know the "size" of some dynamically defined sets and this will correspond to the numerical value of their Hausdorff dimension. However, typically, the dimension doesn't have an explicit expression. Therefore, we may want to compute the value rigorously, efficiently and effectively.


Figure 4: (i) The "Douady rabbit" Julia set, and (ii) the "basilica" Julia set

### 6.1 Classical Examples of sets and their dimensions

We can consider some classical examples (with connections to various areas of mathematics):

1. Julia sets (Dynamics). The Julia set $J(T) \subset \widehat{\mathbb{C}}$ associate to (hyperbolic) rational maps $T: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$, i.e., the closure of the periodic repelling points

$$
J(T)=\overline{\left\{z \in \widehat{\mathbb{C}}: T^{n} z=z,\left|\left(T^{n}\right)^{\prime}(z)\right|>1\right\}}
$$

for which we additionally require that $\inf _{z \in J(T)}\left|T^{\prime}(z)\right| \geq \gamma>1$. For example, when $c \in \mathbb{C}$ with $|c|$ sufficiently small we can consider $T(z)=z^{2}+c$. When $c=0$ then $J(T)$ is a circle, but otherwise $J(T)$ is a quasi-circle. On the other hand, for $c=1$ we have that $J(T)$ is the so-called Basilica Julia set.
2. Fuchsian group limit sets (Geometry). Let $\mathbb{D}=\{z \in \mathbb{C}:|z|<1\}$ and $T_{i}: \mathbb{D} \rightarrow \mathbb{D}$ ( $i=1,2,3$ ) be Mobius maps which reflect in disjoint circular arcs (meeting the unit circle $\partial \mathbb{D}$ perpendicularly). Following an example of McMullen, assume they are each symmetrically placed and each subtends an angle $\theta$ at the origin. The accumulation points of the orbit of 0 are a Cantor
 5
3. Restricted digit continued fractions (Number Theory). Let $A \subset \mathbb{N}$ be a finite set and consider the set

$$
E_{A}=\left\{\left[a_{1}, a_{2}, a_{3}, \ldots\right]:=\frac{1}{a_{1}+\frac{1}{a_{2}+\cdots}}: a_{i} \in A\right\}
$$

of numbers in the unit interval which have infinite continued fraction expansions whose digits all lie in $A$. This is easily seen to be a Cantor set. Such sets have applications to the Density one Zaremba Conjecture [5], Diophantine approximation and Lagrange spectra [29], etc.

### 6.2 General setting

Each of these sets described above are all variants on the limit set for an iterated function scheme (or system). For convenience, we will consider the simplest one dimensional representative case:

[^4]

Figure 5: Equi-placed circles used for the hyperbolic reflections defining the limit set in the unit circle

- Let $T_{1}, \cdots, T_{k}:[0,1] \rightarrow[0,1](k \geq 2)$ be $C^{2}$ maps (of more generally $C^{1+\theta}$, i.e., the derivatives $T_{i}^{\prime}$ are $\theta$-Hölder continuous);
- The maps are contracting, i.e., $\left\|T_{i}^{\prime}\right\|_{\infty}=\sup _{0 \leq x \leq}\left|T_{i}^{\prime}(x)\right|<1(1 \leq i \leq k)$;
- The images are disjoint, i.e., $T_{i}([0,1]) \cap T_{j}([0,1])=\emptyset$, for $i \neq j$ (which is case of the strong separation condition)

Remark 6.1. In higher dimensions we would also assume that the maps are conformal which, of course, comes for free in one dimension.

The basic object of study is the following.
Definition 6.2. The limit set $X \subset[0,1]$ is the smallest non-empty closed set $X$ satisfying

$$
X=\cup_{i=1}^{k} T_{i}(X)
$$

The existance and uniqueness of the set $X$ exists by Hutchinson's theorem [10].
Example 6.3 (Restricted digit continued fractions, revisited). In the case of the example of continued fractions one considers maps $T_{a}:[0,1] \rightarrow[0,1]$ for $a \in A$ defined by $T_{a}(x)=\frac{1}{x+a}$. In particular, $\left|T^{\prime}(x)\right|=\frac{1}{|x+a|^{2}} .{ }^{6} \quad$ The other two examples are similar, except we need to work on regions in the complex plane.

In this setting the Hausdorff dimension of the set $X$ is equal to the easier to define Box (or Minkowski) dimension, whose defintion we now recall.

Definition 6.4. Given $\epsilon>0$, let $N(X, \epsilon) \in \mathbb{N}$ be the smallest number of $\epsilon$-balls needed to cover $X$ then we let

$$
\operatorname{dim}(X)=\lim _{\epsilon \rightarrow 0} \frac{\log N(X, \epsilon)}{\log \epsilon}
$$

The above lemma exists in all of the examples we will consider.
We want to address the following problem:

[^5]Figure 6: The two contractions $T_{1}, T_{2}:[0,1] \rightarrow[0,1]$.

Problem 6.5. How can we estimate $\operatorname{dim}(X)$ ?
We will discuss 3 different approachs:
(I) Approximation of $\left\{T_{i}\right\}_{i=1}^{k}$ by simpler affine maps. This is a classical approach which we include for illustration.
(II) Using "determinants". This is a method based on complex functions defined using compositions of contractions. It has the most interesting mathematics and works quite well for good approximations, but is most effective when the maps $\left\{T_{i}\right\}$ are real analytic; and
(III) Using a min-max method. This method has very simple mathematics and is suprisingly effective.

We can compare each of these different methods in the context of a simple example.
Example 6.6 (Comparative example $E_{1,2}$ ). Consider the example $X=E_{1,2}$ corresponding to the maps 7

$$
T_{1}(x)=\frac{1}{1+x} \text { and } T_{2}(x)=\frac{1}{2+x} .
$$

In particular, $X$ is the Cantor set consisting of points $x$ with infinite continued fractions all of whose digits are either 1 and 2 .

Using Approach (I) (and variants):
Good (1941) computed $\operatorname{dim}\left(E_{12}\right)$ to 2 decimal places.
Bumby (1985) computed $\operatorname{dim}\left(E_{12}\right)$ to 6 decimal places.
Falk and Nussbaum (2018) computed $\operatorname{dim}\left(E_{12}\right)$ to 12 decimal places.
Using Approach (II):
Jenkinson and Pollicott (2017) computed $\operatorname{dim}\left(E_{12}\right)$ to 100 decimal places.
Using Approach (III):
Pollicott and Vytnova (2017) computed $\operatorname{dim}\left(E_{12}\right)$ to 200 decimal places.
More details can be found in [36].
Perhaps one lesson here is that "getting a bigger computer" isn't always enough: One needs to develop appropriate approachs for different problems.

## 7 Different approachs to estimation dimension

In the next subsections we will describe in more detail the ideas behind these three approachs.

### 7.1 Approach I : Approximation by similarities

To set the scene, we begin again with the simplest approach. Consider first the special case $S_{i}$ : $[0,1] \rightarrow[0,1]$ of affine maps given by $S_{i} x=r_{i} x+d_{i}(i=1, \cdots, k)$ where $0<r_{i}<1$ and $r_{i}+d_{i}<d_{i+1}$ ( $i=1, \cdots, k-1$ ) and $r_{k}+d_{k}<1$. In this case we have the following classic result:

Theorem 7.1 (Moran, $1946[10])$. The dimension $\operatorname{dim}(X)$ of the limit set $X=X\left(\left\{S_{i}\right\}\right)$ is the unique solution to $f(t)=1$ where $f: \mathbb{R} \rightarrow \mathbb{R}$ is defined by $f(t)=\sum_{i=1}^{k} r_{i}^{t}$.


Figure 7: An illustration of the plot of the function $f(t)$ in Moran's theorem

For a more general $C^{2}$ iterated function scheme $\left\{T_{i}\right\}_{i=1}^{k}$ we can set

$$
\underline{r}_{i}:=\inf _{0 \leq x \leq 1}\left|T_{i}^{\prime}(x)\right| \leq \sup _{0 \leq x \leq 1}\left|T_{i}^{\prime}(x)\right|=: \bar{r}_{i}, \quad(i=1, \cdots, k) .
$$

and associate two affine iterated function schemes $\left\{\underline{T}_{i}\right\}_{i=1}^{k}$ and $\left\{\bar{T}_{i}\right\}_{i=1}^{k}$ whose contractions are given by $\left\{\underline{r}_{i}\right\}_{i=1}^{k}$ and $\left\{\bar{r}_{i}\right\}_{i=1}^{k}$, respectively. (The translational components are unimportant, except in as much as they are chosen to give disjoint images.) The values $\underline{d} \leq \bar{d}$ can easily be seen to give upper and lower bounds to the true dimension of the set $X$ [33].



Figure 8: For nonlinear contractions $\left\{T_{i}\right\}$ : (a) the linear maps $\left\{\bar{T}_{i}\right\}$ with stronger contractions; and (b) the linear maps $\left\{\underline{T}_{i}\right\}$ with weaker contractions

Proposition 7.2. Solving for $0<\underline{d} \leq \bar{d}<1$ in the equations

$$
\sum_{i=1}^{k}\left(\underline{r}_{i}\right)^{\underline{d}}=1=\sum_{i=1}^{k}\left(\bar{r}_{i}\right)^{\bar{d}}
$$

gives bounds $\underline{d} \leq \operatorname{dim}(X) \leq \bar{d}$.

[^6]These bounds are easy to establish, but may not be as good as required. There are a couple of basic ways to improve these bounds.
(a) Fix $m \geq 2$. Replace $\left\{T_{i}\right\}_{i=1}^{k}$ by the $2^{m}$ contractions

$$
T_{i_{1}} \circ \cdots \circ T_{i_{m}}:[0,1] \rightarrow[0,1] \quad\left(i_{1}, \cdots, i_{m} \in\{1, \cdots, k\}\right) .
$$

We then have two simple observations:
(i) The two families of contractions have the same limit set, i.e., $X\left(\left\{T_{i}\right\}\right)=X\left(\left\{T_{i_{1}} \circ \cdots \circ T_{i_{m}}\right\}\right)$; and
(ii) New bounds $\underline{d}_{m} \leq \operatorname{dim}(X) \leq \bar{d}_{m}$ associated to the iterated function scheme with contractions $\left\{T_{i_{1}} \circ \cdots \circ T_{i_{m}}\right\}$ satisfy $\bar{d}_{m}-\underline{d}_{m} \rightarrow 0$ [33].

These give us ways to improve the bounds but typically we only expect slow convergence with an upper bound $\bar{d}_{m}-\underline{d}_{m}=O(1 / m)$
(b) A second improvement comes by replacing $[0,1]$ by a union of image intergals $T_{i_{1}} \circ \cdots \circ T_{i_{m}}[0,1]$ and approximating

$$
T_{i}: T_{i_{1}} \circ \cdots \circ T_{i_{m}}[0,1] \rightarrow T_{i} \circ T_{i_{1}} \circ \cdots \circ T_{i_{m}}[0,1]
$$

$\left(i=1, \cdots, k\right.$ and $\left.1 \leq i_{1}, \cdots, i_{m} \leq k\right)$ by similarities (i.e., affine maps) cf.[30]. This might improve matters by obtaining exponential convergence of $\bar{d}_{m}-\underline{d}_{m} \rightarrow 0$ as $m \rightarrow+\infty$.

However, we will turn instead to the other two methods.

### 7.2 The Bowen-Ruelle pressure formula

The remaining two approachs make use of the following standard definition and result.
Definition 7.3. We can define a pressure function $P: \mathbb{R} \rightarrow \mathbb{R}$ by ${ }^{8}$

$$
P(t):=\lim _{n \rightarrow+\infty} \frac{1}{n} \log \left(\sum_{i_{1}, \cdots, i_{n} \in\{1, \cdots, k\}}\left|\left(T_{i_{1}} \circ \cdots \circ T_{i_{n}}\right)^{\prime}(0)\right|^{t}\right) \text { for } t \in \mathbb{R}
$$

The pressure function has a number of standard properties, which we now briefly summarized below.

Lemma 7.4. The function $P: \rightarrow \mathbb{R}$ satisfies the following:

1. $P(t)$ is $C^{\infty}$ (in fact, real analytic);
2. $P(t)$ is strictly decreasing; and
3. $P(t)$ is convex (i.e., $P^{\prime \prime}(t) \geq 0$ for all $t \in \mathbb{R}$ ).

The role of $P(t)$ in estimating $\operatorname{dim}(X)$ is explained by the following very useful result.
Proposition 7.5 (Bowen-Ruelle). The value $0<\operatorname{dim}(X)<1$ is the unique zero of $P(t)$, i.e., $P(\operatorname{dim}(X))=0$.

The basic idea for Proposition 7.5 was originally established by Bowen for limit set of certain Kleinian groups (i.e., Quasi-Fuchsian groups) [7]. However, after Bowen's early demise Ruelle put forward the generalization to other settings [38], [48].

[^7]

Figure 9: A plot of the function $P(t)$

Remark 7.6. This is a natural generalization of Moran's theorem. For similarities with contractions $\left\{r_{i}\right\}_{i=1}^{k}$ we have

$$
P(t)=\log \left(\sum_{i=1}^{k} r_{i}^{t}\right) .
$$

In particular $P(t)=0$ is equivalent to $\sum_{i=1}^{k} r_{i}^{t}=1$.

### 7.3 Approach II : Determinants

This approach is described in a recent survey [14], but we will describe the basic ideas. We need an additional hypothesis:

- We additionally assume that the maps $T_{i}:[0,1] \rightarrow[0,1]$ are real analytic.

In particular, there is a neighbourhood in the complex plane $[0,1] \subset U \subset \mathbb{C}$ and analytic extensions $T_{i}: U \rightarrow U$ (with the closure of the image satisfying $\overline{T_{i}(U)} \subset U$ ).


Figure 10: The image of the extension of $T_{i}$ to $U$ has the closure of its image $T_{i}(U)$ wholly contained inside $U$


Figure 11: For each $t$ we can plot $d(z, t)$. When $t=\operatorname{dim}(X)$ the zero in $z$ appears at 1 .

Example 7.7 (Restricted digits: Continued fractions). Given $T_{i}(x)=1 /(x+i)(i=1, \cdots, k)$ and let

$$
U=\left\{z \in \mathbb{C}:|z-1|<\frac{3}{2}\right\} .
$$

For $\underline{i}=\left(i_{1}, \cdots, i_{n}\right) \in\{1,2, \ldots, k\}^{n}$ we can denote:
(a) $|\underline{i}|=n$ (length of string);
(b) $T_{\underline{i}}=T_{i_{1}} \circ \cdots \circ T_{i_{n}}:[0,1] \rightarrow[0,1]$; and
(c) $x_{\underline{i}}=T_{\underline{i}}\left(x_{\underline{i}}\right)$ is the unique fixed point.

We can combine this data to define a family of complex functions.
Definition 7.8. We can formally define the determinant function for any $t \in \mathbb{R}$ by

$$
\mathbb{C} \ni z \mapsto d(z, t):=\exp \left(-\sum_{n=1}^{\infty} \sum_{|\underline{i}|=n} \frac{\left|T_{\underline{i}}^{\prime}\left(x_{\underline{i}}\right)\right|}{1-T_{\underline{i}}^{\prime}\left(x_{\underline{i}}\right)}\right)
$$

Remark 7.9. These are reminiscent (and closely connected) to dynamical zeta functions, with a few minor changes [35].

The following are basis properties of the function $d(z, t)[?]$.
Lemma 7.10. Given $t \in \mathbb{R}$ :

1. the function $d(z, t)$ converges for $|z|$ sufficiently small;
2. The function $z \mapsto d(z, t)$ extends analytically to $\mathbb{C}$; and
3. The function $z \mapsto d(z, t)$ has a zero at $\exp (-P(t))$.

This lemma essentially reduces the problem of finding the value $\operatorname{dim}(X)$ to that of finding the value $t_{0}(=\operatorname{dim}(X))$ for which the function $\mathbb{C} \ni z \mapsto d\left(z, t_{0}\right)$ has a zero at $z=1$. (This uses the Bowen-Ruelle pressure formula in Proposition 7.5.)

Of course it would be impossible to compute $d(z, t)$ precisely using a computer since it contains data on infinitely many periodic points and an infinite amount of data. Therefore, we need to appoximate this function. We can consider the power series expansion in $z$ of the form

$$
d(z, t)=1+\sum_{n=1}^{\infty} a_{n}(t) z^{n}
$$

where $a_{n}(t) \in \mathbb{R}$ can be seen to be expressed in terms of the periodic points of period at most $n$. We can break up the power series expansion as:

$$
\begin{aligned}
d(z, t):=1 & +\underbrace{a_{1}(t) z+a_{2}(t) z^{2}+\cdots+a_{25}(t) z^{25}}_{(I)} \\
& +\underbrace{a_{26}(t) z^{26}+\cdots+a_{500} z^{500}}_{(I I)} \\
& +\underbrace{a_{501}(t) z^{26}+a_{502} z^{502}+\cdots}_{(I I I)}
\end{aligned}
$$

where:

1. (I) can be expressed in terms of periodic points of period at most 25 . This will be the approxiamtion to $d(z, t)$ we use;
2. (II) can be bounded using numerical estimates on the terms $a_{n}(t)$; and
3. (III) can be more crudely estimates using the original estimates of Ruelle (after Grothendeick).

More details can be found in [14].
Remark 7.11. The complex function $d(z, t)$ is called a determinant because we can write it as $d(z, t)=$ $\operatorname{det}\left(T-z \mathcal{L}_{t}\right)$ where $\mathcal{L}_{t}$ is a transfer operator acting on Banach spaces (or Hilbert spaces) of analytic functions.

### 7.4 Approach III : "min-max"

We now describe a method employed by the author and P. Vytnova. In this approach the transfer operator is more to the fore. ${ }^{9}$

Let $C^{1}([0,1], \mathbb{R})$ be the Banach space of $C^{1}$ functions $f: C^{1}([0,1], \mathbb{R})$ with norm

$$
\|f\|=\underbrace{\sup _{0 \leq x \leq 1}|f(x)|}_{=:\|f\|_{\infty}}+\underbrace{\sup _{0 \leq x \leq 1}\left|f^{\prime}(x)\right|}_{=:\|f\|_{\infty}} .
$$

The following basic result connects the function $P(t)$ to the operators.
Lemma 7.12. The operator $\mathcal{L}_{t}$ has a simple maximal eigenvalue $e^{P(t)}$, i.e., the rest of the spectrum of the operator is contrained in a disc centred at 0 of radius strictly smaller than $e^{P(t)}$.

Thus the problem of finding the dimension $\operatorname{dim}(X)$ is reduced to that of finding $t_{0}$ such that $\mathcal{L}_{t_{0}}$ has 1 as its largest eigenvalue.

Approach: Given two (nearby) values $0<t_{1}<t_{2}<1$ we want to check if $t_{0}=\operatorname{dim}(X) \in\left[t_{1}, t_{2}\right]$. By the continuity and monotonicity of $t P(t)$ and the Bowen-Ruelle theorem it suffices to show $P\left(t_{1}\right)>$


Figure 12: If $P\left(t_{1}\right)>P\left(t_{2}\right)$ then $t_{1} \leq \operatorname{dim}(X) \leq t_{2}$
$0>P\left(t_{2}\right)$, by the intermediate value theorem. To confim this condition it is useful to use the following very simple result.

Lemma 7.13. We can estimate the pressure as follows.

1. Assume $f_{1}>0, f_{1} \in C^{0}([0,1], \mathbb{R})$ with

$$
\mathcal{L}_{t_{1}} f_{1}(x) \geq f_{1}(x) \text { for all } x \in[0,1]
$$

then $P\left(t_{1}\right) \geq 0$
2. Assume $f_{2}>0, f_{2} \in C^{0}([0,1], \mathbb{R})$ with

$$
\mathcal{L}_{t_{2}} f_{2}(x) \leq f_{2}(x) \text { for all } x \in[0,1]
$$

then $P\left(t_{2}\right) \leq 0$


Figure 13: (i) The existance of $f_{1}$ such that $\mathcal{L}_{t_{1}} f_{1} \geq f_{1}$; (ii) The existance of $f_{2}$ such that $\mathcal{L}_{t_{2}} f_{2} \leq f_{2}$

Proof. We give the proof of part 1. Since $\mathcal{L}_{t_{1}}$ is a positive operator we have that

$$
f_{1} \leq \mathcal{L}_{t_{1}} f_{1} \leq \mathcal{L}_{t_{1}}^{2} f_{1} \leq \cdots \leq \mathcal{L}_{t_{1}}^{n} f_{1} \leq \cdots
$$

[^8]Taking $n$th roots:

$$
\underbrace{\lim _{n \rightarrow+\infty}\left\|f_{1}\right\|_{\infty}^{\frac{1}{n}}}_{=1}=\underbrace{\lim _{n \rightarrow+\infty}\left\|\mathcal{L}_{t_{1}}^{n} f_{1}\right\|_{\infty}^{\frac{1}{n}}}_{=e^{P(t)}}
$$

The proof of part 2 is similar.
At first sight if may seem that we haven't gained very much, since we have replaced the problem of estimating $P(t)$ to one of finding functions $f_{1}$ and $f_{2}$. More precisely, we are left with the problems:
(a) Given $t_{1}<t_{2}$, how to we find $f_{1}, f_{2}>0$ in $C([0,1])$ ? and
(b) How do we choose (nearby) $t_{1}<t_{2}$ ?

The solutions to these two problems are actually surprisingly straightforward. We merely import some useful ideas from numerical analysis (for example colllocation) which allow us to construct the test functions $f_{1}$ and $f_{2}$ as polynomials.

We briefly recall the main idea(s). Fix $N \geq 2$ and consider Lagrange polynomials $p_{n}:[0,1] \rightarrow \mathbb{R}$ $(n=1, \cdots, N)$ and Chebychev points $\left(x_{n}\right)_{n=1}^{N} \subset[0,1]$, then $p_{n}\left(x_{m}\right)=\delta_{n, m}$ for $1 \leq n, m \leq N$. We can then associate $N \times N$ matrices $M_{i}(i=1,2)$ defined by

$$
M_{i}(r, s)=\left(\mathcal{L}_{t_{i}} p_{r}\right)\left(x_{s}\right) \text { for } 1 \leq r, s \leq N .
$$

We can then let $\left(v_{1}^{i}, \cdots, v_{N}^{i}\right)$ be the maximal (left) eigenvector.
We can then try to apply Lemma 7.13 with $f_{i}(x)=\sum_{r=1}^{N} v_{r}^{i} p_{r}(x)$ for $i=1,2$. In particular, we need to check that for $N$ sufficiently large:
(i) $f_{i}>0(i=0,1)$; and
(ii) $\mathcal{L}_{t_{1}} f_{1}>f_{1}$ and $\mathcal{L}_{t_{2}} f_{2}<f_{2}$.

We can use a bisection method to improve the choices of $t_{1}<t_{2}$. More precisely, starting from $0=t_{1}^{(1)}<t_{2}^{(1)}=1$ we can generate inductively sequences of pairs $t_{1}^{(n)}<t_{2}^{(n)}(n \geq 1)$ such that $0<t_{2}^{(n)}-t_{1}^{(n)} \leq 2^{-(n-1)}$ by successively replacing one of the points by their midpoint $\left(t_{2}^{(n)}+t_{1}^{(n)}\right) / 2$.

Example 7.14 (Another illustrative example: Feigenbaum attractor). Feigenbaum conjectured (and Lanford proved [23]) the existance of a real analytic real analytic unimodal map $g:[-1,1] \rightarrow[-1,1]$ such that
(i) $g(0)=1$;
(ii) $g^{\prime}(0)=0$ and $g^{\prime}(x)>0$ for $x<0$ and $g^{\prime}(x)<0$ for $x>0$;
(iii) $g(x)=g(-x)$ and $g(x)=\alpha(g \circ g)(x / \alpha)$ where $\alpha=-1 / g(1)$

The attractor $X=\overline{\cup_{n=0}^{\infty} g^{n}(0)}$ is the closure of the orbit of 0 , but it is also given by an iterated function scheme with contractions defined in terms of $g$ (and $\alpha$ ). Fortunately, the series expansion for $g(x)$ is known to high accuracy. (The other Feigenbaum constants are known rigorously to over a thousand decimal places.)

We can compare estimates on the dimension of $X$ using the different methods above.
Using Approach (I):
Grassberger (1985) computed $\operatorname{dim}(X)$ non-rigorously to 8 decimal places.
Bensimon et al (1986) computed $\operatorname{dim}(X)$ non-rigorously to 10 decimal places.


Figure 14: A plot of the Feigenbaum functon $g(x)$

## Using Approach (II):

Christiansen et al (1990) computed $\operatorname{dim}(X)$ non-rigorously to 27 decimal places.

## Using Approach (I):

Falconer (1990) computed $\operatorname{dim}(X)$ rigorously to 1 decimal place.
Thurlby (2023) computed $\operatorname{dim}(X)$ rigorously to 2 decimal places.

## Using Approach (III):

We can compute $\operatorname{dim}(X)$ (semi-) rigorously ${ }^{10}$ to 50 decimal place.

## 8 Estimating Lyapunov exponents

We now move onto a second class of numerical values. There are (at least) two natural settings for Lyapunov exponents. The first is associated to the dynamics of maps and the second associated to finite families of matrices. For definiteness we consider the following simple settings.

1. Associate to a map $f: M \rightarrow M$ and an invariant ergodic probability measure (e.g., expanding maps on $[0,1]$ and the absolutely continuous probability measure $\mu$ ).
2. Random matrix products. (e.g., Families of matrices $A_{1}, \cdots, A_{k} \in S L(2, \mathbb{R})(k \geq 2)$ chosen randomly with respect to a probability vector $\underline{p}=\left(p_{1}, \cdots, p_{k}\right)$ with $p_{i}>0$ and $\left.\sum_{i=1}^{k} p_{i}=1\right)$.

### 8.1 Lyapunov exponents for interval maps

We begin with the Lyapunov exponent for maps. Let $T:[0,1] \rightarrow[0,1]$ be a piecewise $C^{2}$ expanding map, i.e., there exist $0=x_{0}<x_{1}<\cdots<x_{n+1}=1$ such that:

- $T \mid\left(x_{j}, x_{j+1}\right)$ is $C^{2}$;
- There exists $\beta>1$, such that $\left|T^{\prime}(x)\right| \geq \beta$ for $x_{j}<x<x_{j+1}$ and $j=0, \cdots, n-1$ (the map $T$ is piecewise expanding); and
- $T\left(x_{k}, x_{k+1}\right)=(0,1)$, for $k=0,1, \cdots, n-1 .{ }^{11}$

[^9]The following standard result guarantees the existance of an absolutely continuous probability measure [24].

Lemma 8.1 (Li-Yorke, [24]). There exists an absolutely continuous probability $d \mu=\rho(x) d x$, where $\rho(x) \in C^{1}\left([0,1], \mathbb{R}^{+}\right)$and the measure is ergodic.

The following is a simple illustrative example.
Example 8.2 (Lanford map). Let $T:[0,1] \rightarrow[0,1]$ by

$$
T(x)=2 x+\frac{1}{2} x(1-x) \quad(\bmod 1) .
$$

Then $\left|T^{\prime}(x)\right| \geq \frac{3}{2}$ for $0 \leq x \leq 1$.
Given a map $T:[0,1] \rightarrow[0,1]$ as above and the absolutely continuous ergodic measure $\mu$ we can define the following numerical quantity.

Definition 8.3. We associate to $\mu$ the Lyapunov exponent

$$
\lambda(T, \mu)=\int \log \left|T^{\prime}(x)\right| d \mu(x)
$$

This is equal to the entropy $h(T, \mu)$ of the measure $\mu$ by the Pesin equality.
Remark 8.4. By the Birkhoff ergodic theorem, for a.e. ( $\mu$ ) $x$ we have that

$$
\lambda(T, \mu)=\lim _{n \rightarrow+\infty} \frac{1}{n} \log \left|\left(T^{n}\right)^{\prime}(x)\right| .
$$

In particular, we see that the derivative in the definition implies that $\lambda(T, \mu)$ measures the instability of neighbouring orbits.

To proceed, we define the corresponding pressure function (now for expanding maps, rather than contractions).

Definition 8.5. We can define a pressure function $p: \mathbb{R} \rightarrow \mathbb{R}$ for $T$ by

$$
p(t)=\lim _{n \rightarrow+\infty} \frac{1}{n} \log \sum_{T^{n} x=x}\left|\left(T^{n}\right)^{\prime}\right|^{-t}
$$

where the summation is over periodic orbits. ${ }^{12}$
This pressure function has the following extremely useful properties.
Lemma 8.6. The pressure function $p(t)$ has the following properties:
(i) $p(1)=0$;
(ii) $t \mapsto p(t)$ is real analytic and convex;
(iii) $\lambda(T, \mu)=-\left.\frac{d p}{d t}\right|_{t=1}$; and
(iv) for $\epsilon>0$ we have

$$
-\frac{p(1+\epsilon)}{\epsilon} \leq-\left.\frac{d p}{d t}\right|_{t=1} \leq \frac{p(1-\epsilon)}{\epsilon}
$$



Figure 15: The Lyapunov exponent is the absolute value of the slope of the tangent to $p(t)$ at $t=0$. This lies in the interval $\left[-\frac{p(1+\epsilon)}{\epsilon}, \frac{p(1-\epsilon)}{\epsilon}\right]$

Briefly, Part (iii) does for Lyapunov exponents what the Bowen-Ruelle theorem did for dimension. Part (iv) follows from convexity of $p(t)$ in Part (ii). This is the key inequality for the estimates we need.

To proceed we need to make a judicious choice of $\epsilon>0$ and to estimate $p(1+\epsilon)$ and $p(1-\epsilon)$ as accurately as possible. To this end we use (another) transfer operator. Let $T_{i}:[0,1] \rightarrow[0,1]$ $(i=1, \cdots, k)$ be the inverse branches for $T$, i.e., $T \circ T_{i}(x)=x$.

Definition 8.7. Given $t \in \mathbb{R}$ we define a transfer operator $\mathcal{L}_{t}: C^{1}([0,1], \mathbb{R}) \rightarrow C^{1}([0,1], \mathbb{R})$ by

$$
\mathcal{L}_{t} f(x)=\sum_{i=1}^{k}\left|T_{i}^{\prime}(x)\right|^{t} f\left(T_{i} x\right) .
$$

The role of the family of operators is explained by the following.
Lemma 8.8. $\mathcal{L}_{t}$ has a maximal eigenvalue $e^{p(t)}$.
To estimate $\lambda(T, \mu)$ we proceed as follows. Fix $\epsilon>0$ sufficiently small. If we can choose $0<\alpha<\beta$ with

$$
P(1-\epsilon) \leq \beta \text { and } P(1+\epsilon) \leq-\alpha .
$$

then by parts (iii) and (iv) of the lemma we can deduce the inequality

$$
\begin{equation*}
\frac{\alpha}{\epsilon} \leq \lambda(T, \mu) \leq \frac{\beta}{\epsilon} . \tag{12.1}
\end{equation*}
$$

To apply these bounds we need the following simple lemma.
Lemma 8.9. We have the the following bounds.

1. Assume there exists $f_{1} \in C([0,1], \mathbb{R})$ and $f_{1}>0$ such that

$$
\mathcal{L}_{1+\epsilon} f_{1} \leq e^{-\alpha} f_{1}
$$

implies $p(1+\epsilon) \geq-\alpha$.

[^10]2. Assume there exists $f_{2} \in C([0,1], \mathbb{R})$ and $f_{2}>0$ such that
$$
\mathcal{L}_{1-\epsilon} f_{2} \leq e^{\beta} f_{2}
$$
implies $p(1-\epsilon) \geq \beta$.



0

Figure 16: (i) The existance of $f_{1}$ implies $p(1+\epsilon) \geq-\alpha$; (ii) The existance of $f_{2}$ implies $p(1-\epsilon) \geq \beta$

Proof. For part 1 , since $e^{\alpha} \mathcal{L}_{1+\epsilon} f_{1} \leq f_{1}$ and by positivity of the operator $L_{1+\epsilon}$ gives that

$$
\cdots \leq e^{\alpha n} \mathcal{L}_{1+\epsilon^{n}} f_{1} \leq \cdots \leq e^{\alpha} \mathcal{L}_{1+\epsilon} f_{1} \leq f_{1}
$$

Thus

$$
e^{\alpha} e^{p(1+\epsilon)}=\lim _{n \rightarrow+\infty}\left\|e^{\alpha n} \mathcal{L}_{1+\epsilon^{n}} f_{1}\right\|_{\infty}^{\frac{1}{n}} \leq \lim _{n \rightarrow+\infty}\left\|f_{1}\right\|_{\infty}^{\frac{1}{n}}=1
$$

For part 2 , since $e^{-\beta} \mathcal{L}_{1-\epsilon} f_{2} \leq f_{2}$ and by positivity of the transfer operator

$$
\cdots \leq e^{-\beta n} \mathcal{L}_{1-\epsilon^{n}} f_{2} \leq \cdots \leq e^{-\beta} \mathcal{L}_{1-\epsilon} f_{2} \leq f_{2}
$$

Thus

$$
e^{-\beta} e^{p(1-\epsilon)}=\lim _{n \rightarrow+\infty}\left\|e^{-\beta n} \mathcal{L}_{1-\epsilon}^{n} f_{2}\right\|_{\infty}^{\frac{1}{n}} \leq \lim _{n \rightarrow+\infty}\left\|f_{2}\right\|_{\infty}^{\frac{1}{n}}=1
$$

The usefulness of this method is illustrated by some simple examples.
Example 8.10 (Lanford map revisited). Recall that $T:[0,1] \rightarrow[0,1]$ defined by $T(x)=2 x+\frac{1}{2} x(1-$ $x)(\bmod 1)$. We can choose

$$
\begin{aligned}
\epsilon & =10^{-180} \\
\alpha & =6.5766 \ldots 890 \times 10^{-181} \\
\beta & =6.5766 \ldots 898 \times 10^{-181} .
\end{aligned}
$$

Substituting into (12.1) leads to a value

$$
\lambda(T, \mu)=0.65766 \cdots
$$

which is accurate to 128 places.
Remark 8.11. As in the applications to $\operatorname{dim}(X)$ we can use ideas from numerical analysis to choose $f_{1}$ and $f_{2}$.

### 8.2 Lyapunov exponents for random matrix products

We now turn to the second notion of Lyapunov exponents. Fix $k \geq 2$. Assume we are given a finite collection of matrices

$$
A_{1}, \cdots, A_{k} \in S L(2, \mathbb{R}), \quad k \geq 2
$$

and a probability vector $\underline{p}=\left(p_{1}, p_{2}, \cdots, p_{k}\right)$ with $p_{i}>0$ and $\sum_{i=1}^{k} p_{i}=1$.
More generally we could consider matrices in $G L(2, \mathbb{R})$, but for simplicity we consider matrices in $S L(2, \mathbb{R})$. Given matrices in $G L(2, \mathbb{R})$ we can associate matrices in $S L(2, \mathbb{R})$ by scaling the entries for each matrix. The Lyapunov exponent for the new family of matrices differs from that of the original family by an explicit constant, which comes from a simple application of the Birkhoff ergodic theorem. Thus there is no loss in generality in considering $S L(2, \mathbb{R})$.

Definition 8.12. We can associate the Lyapunov exponent defined by

$$
\lambda=\lambda\left(\left\{A_{i}\right\}, \underline{p}\right)=\lim _{n \rightarrow+\infty} \sum_{i_{1}, \cdots, i_{n} \in\{1, \cdots, k\}}\left(p_{i_{1}} \cdots p_{i_{n}}\right) \frac{\log \left\|A_{i_{1}} \cdots A_{i_{n}}\right\|}{n} .
$$

The limit exists by subadditivity. We are restricting attention to the top Lyapunov exponents.
The value $\lambda$ features in the work of Bellman, Furstenberg, Kesten, Kingman, Guivarc'h, Conze, Le Page, etc. If the matrices don't correspond to rotations ${ }^{13}$ then it follows from a classical result of Furstenberg that $\lambda>0$.

More generally, we need to make assumptions on $\left\{A_{1}, \cdots, A_{k}\right\}$.
Definition 8.13. We say that $\left\{A_{1}, \cdots, A_{k}\right\}$ is strongly irreducible if there don't exist a finite set of directions whose union is preserved by all $A_{i}(1 \leq i \leq k)$.

We want to address the following problem.
Problem. How do we estimate $\lambda>0$ ?
By analogy with the previous case we can consider a transfer operator acting on a Banach space. In the present case we consider a Banach space of Holder functions on the unit circle $\mathbb{S} \subset \mathbb{R}^{2}$. For $\alpha>0$ sufficiently small $C^{\alpha}(\mathbb{S})$ denote the Banach space of $\alpha$-Holder continuous functions with the norm

$$
\|f\|_{\alpha}=\sup _{x \neq y} \frac{|f(x)-f(y)|}{|x-y|^{\alpha}}+\sup _{x \in \mathbb{S}}|f(x)| .
$$

We can associate to the matrices $A_{i}$ projective maps $T_{i}: \mathbb{S} \rightarrow \mathbb{S}$ defined by

$$
T_{i}(x)=\frac{A_{i}(x)}{\left\|A_{i}(x)\right\|_{2}} \text { for } i=1, \cdots, k
$$

where $\|\cdot\|_{2}$ is the usual pythagorian norm on $\mathbb{R}^{2}$.
Definition 8.14. We can then define a family of transfer operators $\mathcal{L}_{t}: C^{\alpha}(\mathbb{S}) \rightarrow C^{\alpha}(\mathbb{S})^{14}$ for $t \in \mathbb{R}$ by

$$
\mathcal{L}_{t} f(x)=\sum_{i=1}^{k} p_{i}\left|T_{i}^{\prime}(x)\right|^{t} f\left(T_{i} x\right)
$$

Of course, the operator also preserves more regular functions. However, by restricting to suitable $\alpha$-Hölder functions (for $\alpha>0$ sufficiently small) one can recover the valuable spectral gap in the following useful and important result of Le Page [25].

[^11]

Figure 17: The Lyapunov exponent is the absolute value of the slope of the tangent to $Q(t)$ at $t=0$. This is lies in the interval $\left[-\frac{Q(1+\epsilon)}{\epsilon}, \frac{Q(1-\epsilon)}{\epsilon}\right]$

Lemma 8.15 (Le Page). For $\alpha>0$ suitably small and $|t|$ sufficiently small there is a maximal eigenvalue $e^{Q(t)}$.

The value $Q(t)$ plays the role previously taken by the pressure. The relationship between $Q(t)$ and the value $\lambda$ of the Lyapunov exponent is described by the following simple lemma.

Lemma 8.16. For $|t|$ sufficiently small:
(i) $Q(0)=1$;
(ii) $t \mapsto Q(t)$ is real analytic and convex;
(iii) $\lambda=-\left.\frac{1}{2} \frac{d Q}{d t}\right|_{t=0}$; and
(iv) for $\epsilon>0$ sufficiently small we have

$$
\frac{Q(-\epsilon)}{\epsilon} \leq\left.\frac{d Q}{d t}\right|_{t=0} \leq \frac{Q(\epsilon)}{\epsilon} .
$$

We can now proceed to estimate $\lambda$ as we did in the case of maps. In particular, we have an analogous criterion in terms of positive continuous test functions $f_{1}, f_{2}: \mathbb{S} \rightarrow \mathbb{R}^{+}$. However, since we need to construct functions which are only Holder continuous function this approach proves to be less efffective than for for expanding maps.

As before, it is illustrative to consider some specific examples. The following example follows the elegant construction of Barany, Beardon, Carne, Diaconis, McMullen, etc. [3], [31].

Example 8.17 (Barymetric subdivision). Consider an (equilateral) triangle $T \subset \mathbb{R}^{2}$ in the plane. We can first subdivide $T$ into 6 subtriangles using medians. We then continue iteratively, then at the $n$th step there are $6^{n}$ triangles.

For a typical (Lebesgue) point $x \in T$ we can denote by $T_{n}(x) \in x$ the sub-triangle containing $x$ at the nth stage. This is well defined for almost all $x$ with respect to Lebesgue measure. We are interested in the following question.

Problem. What is the asymptotic shape of $T_{n}(x)$ as $n \rightarrow+\infty$ ?


Figure 18: Triangles are successively subdivided by medians.
Since we are not concerned with the size of the triangles, merely their shape, we can quantify the degeneration of such triangles by defining $\theta_{n}(x) \in\left[\frac{\pi}{3}, \pi\right)$ to be the largest internal angle of the triangle $T_{n}(x)$. The main result in this direction was the following:

Theorem 8.18 (Bárány-Beardon-Carne [3]). There exists $\lambda>0$, a.e., $x \in T$ such that

$$
\left|\theta_{n}(x)-\pi\right|=O\left(e^{-2 \lambda n}\right)
$$

In their proof, the value $\lambda$ is ingeniously identified as the Lyapunov exponent for the 6 matrices

$$
\begin{gathered}
A_{1}=\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right) A_{2}=\left(\begin{array}{cc}
\frac{2}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\
0 & \frac{3}{\sqrt{6}}
\end{array}\right) A_{3}=\left(\begin{array}{cc}
\frac{4}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\
\frac{3}{\sqrt{6}} & -\frac{3}{\sqrt{6}}
\end{array}\right) \\
A_{4}=\left(\begin{array}{cc}
\frac{2}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\
\frac{\sqrt{\sqrt{6}}}{\sqrt{6}} & 0
\end{array}\right) A_{5}=\left(\begin{array}{cc}
-\frac{2}{\sqrt{6}} & \frac{4}{\sqrt{6}} \\
-\frac{3}{\sqrt{6}} & \frac{3}{\sqrt{6}}
\end{array}\right) A_{6}=\left(\begin{array}{cc}
-\frac{4}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\
-\frac{3}{\sqrt{6}} & 0
\end{array}\right),
\end{gathered}
$$

which correspond to the affine maps from the original triangle to the first six subtriangles, chosen with equal probability. Using the method described above we can estimate

$$
0.007728<\lambda<0.07732 .
$$

A second application, this time to hyperbolic geometry rather than Euclidean geoemtry, is the following.

Example 8.19 (Drift in hyperbolic space). Let $\mathbb{D}^{2}=\{z=x+i y:|z|<1\}$ be the unit disk equipped with the usual Poincaré metric d defined by

$$
d s^{2}=\frac{d x^{2}+d y^{2}}{\left(1-x^{2}-y^{2}\right)^{2}} .
$$

In particular, the space $\mathbb{D}^{2}$ has constant curvature $\kappa=-1$. Let $\Gamma_{0}=\left\{g_{1}, \cdots, g_{k}\right\}$ be a finite set of isometries (which will take the form of Mobius maps which preserve the unit circle). Let $\underline{p}=$ $\left(p_{1}, \cdots, p_{k}\right)$ be a probability vector.

Starting from a reference point $x \in \mathbb{D}^{2}$ we can randomly apply elements $g \in \Gamma_{0}$ chosen with respect to $\underline{p}$. A typical sequence of steps $\underline{g}=\left(g_{i_{n}}\right)_{n=0}^{\infty} \in \Gamma_{0}^{\mathbb{Z}_{+}}$(with respect to the Bernoulli measure $\underline{p}^{\mathbb{Z}_{+}}$) has a limit (in the Euclidean sense) on the boundary (i.e., the unit circle $\mathbb{S}$ ) which we denote by:

$$
\xi=\xi(\underline{g})=\lim _{N \rightarrow+\infty} g_{i_{1}} g_{i_{2}} \cdots g_{i_{N}} x \in \partial \mathbb{D}^{2}=\mathbb{S}
$$



Figure 19: The eight generators $\left\{g_{i}\right\}_{i=1}^{8}$ are Mobius maps which preserve the unit circle and map the sides of the hyperbolic octogon to sides of the same colour.

The hitting measure on $\mathbb{R} \cup \infty$ is the push forward $\nu=\xi_{*}\left(\underline{p}^{\mathbb{Z}_{+}}\right)$. Moreover, the dimension of this measure is given by

$$
\operatorname{dim}_{H}(\nu)=\frac{\text { Random walk entropy }}{\text { Drift }}
$$

where the Random walk entropy is defined in [2] and

$$
\text { Drift }=\lim _{N \rightarrow+\infty} \frac{1}{N} \sum_{i_{1}, \cdots, i_{N}} p_{i_{1}} \cdots p_{i_{N}} d\left(g_{i_{1}} \cdots g_{i_{N}} x, x\right)
$$

measures Moreover, the isometries $\Gamma_{0}$ correspond to matrices in $S L(2, \mathbb{R})$ and the drift corresponds to $\lambda$ (up to a factor of 2 ).

For definiteness, we can consider a surface group of genus 2 and a regular octogonal fundamental domain we can choose $\Gamma_{0}$ to consist of the eight maps in the standard side pairing.

We can estimate

$$
\text { Drift }=1.69 \pm 10^{-2}
$$

Moreover, there exists an estimate by Gouezel-Matheus-Maucourant that the Random Walk entropy of $1.45 \pm 10^{-2}$ [12]. In particular, these combine with theabove estimate to give $\operatorname{dim}_{H}(\nu)=0.85 \pm 10^{-2}$.

### 8.3 Positive matrices

In the special case of matrices with positive entries it is possible to get much better estimates on the lyapunov exponent. The reason is that when we consider the operator $\mathcal{L}_{t}$ it actually preserves the space $C^{\omega}(\Delta)$ of analytic functions on a small neighbourhood in the complex plane of $\Delta=\mathbb{R}_{+}^{2} \cap \mathbb{S} \subset \mathbb{S}$, i.e., the intersection of the unit circle $\mathbb{S}$ with the positive quadrant $\mathbb{R}_{+}^{2}$. The greater regularity allows a more effective choice of polynomials $f_{1}, f_{2}: \Delta \rightarrow \mathbb{R}$.

The following matrices $2 \times 2$ matrices were suggested by Vilma Orgoványi.
Example 8.20 (Positive matrices). We can consider the matrices

$$
M_{1}=\left(\begin{array}{ll}
1 & 0 \\
2 & 2
\end{array}\right), M_{2}=\left(\begin{array}{ll}
2 & 1 \\
1 & 2
\end{array}\right), M_{3}=\left(\begin{array}{ll}
2 & 2 \\
0 & 1
\end{array}\right)
$$

and $\underline{p}=\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. We can change the matrices for those that each have determinant 1 by considering

$$
A_{1}=\frac{1}{\sqrt{2}}\left(\begin{array}{ll}
1 & 0 \\
2 & 2
\end{array}\right), A_{2}=\frac{1}{\sqrt{3}}\left(\begin{array}{ll}
2 & 1 \\
1 & 2
\end{array}\right), A_{3}=\frac{1}{\sqrt{2}}\left(\begin{array}{ll}
2 & 2 \\
0 & 1
\end{array}\right) .
$$

For these matrices we can estimate the Lyapunov exponent $\lambda_{0}$ for $A_{1}, A_{2}$ and $A_{3}$ and $\underline{p}=\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

$$
0.539 \ldots<\lambda_{0}<0.566 \ldots
$$

Since $(2 \log 2+\log 3) /=0.828302 \cdots$ we have that the Lyapunov exponent $\lambda=\lambda_{0}+0.828302 \cdots$ for the original matrices $M_{1}, M_{2}$ and $M_{3}$ and $\underline{p}=\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ satisfies

$$
1.3678 \ldots \ldots<\lambda<1.3950 \ldots
$$

Example 8.21 (Positive matrices). We can consider the matrices

$$
A_{1}=\left(\begin{array}{lll}
1 & 2 & 3 \\
3 & 2 & 1 \\
1 & 9 & 1
\end{array}\right) \text { and } A_{2}=\left(\begin{array}{lll}
3 & 2 & 4 \\
1 & 7 & 1 \\
1 & 5 & 1
\end{array}\right)
$$

Let $\underline{p}=\left(\frac{1}{2}, \frac{1}{2}\right)$. We can estimate

$$
\lambda=3.76492 \pm 10^{-5} .
$$

As before, the estimates are based on transfer operators and test functions based on colocation. For $(x, y)$ in the triangle $\Delta$ with $x, y \geq 0$ and $x+y \leq 1$ we associate the vector $(x, y, 1-x-y)$ in the standard simplex. We then apply $F_{A_{1}}$ and $F_{A_{2}}$ which are the linear actions of $A_{1}$ and $A_{2}$, respectively to get vectors $F_{A_{1}}[x, y], F_{A_{2}}[x, y] \in \mathbb{R}^{3}$. We then project down to the triangle again to get maps $f_{A_{1}}[x, y]$ and $f_{A_{2}}[x, y]$ are representative maps on the triangle. We can find test functions by collocation on the triangle $\Delta .{ }^{15}$

### 8.4 Variations on themes

Many of the ideas described above could be applied in slightly different settings.

1. We could estimate the dimension of basic sets $\Lambda$ for hyperbolic surface diffeomorphisms or three dimensional hyperbolic flows. This would be approached by using Markov partitions and the approach in [?].
2. We could estimate the Lyapunov exponents for Anosov flows. For example, we could estimate the metric entropy for a given geodesic flow on a surface of negative curvature.
3. We could estimate the Lyapunov exponents of matrix valued cocycles rather than random matrix products.

## References

[1] J.-P. Aubin and I. Ekeland, Applied nonlinear analysis, Wiley, New York, 1984.
[2] A. Avez, Entropie des groupes de type fini, C. R. Acad. Sci. Paris 275A (1972) 1363-1366.
[3] I. Bárány, A. Beardon and T. Carne, Barycentric subdivision of triangles and semigroups of Möbius maps, Mathematika, 43 (1996) 165-171.

[^12][4] G. Besson, G. Courtois and S. Gallot, Entropies et rigidités des espaces localement symétriques de courbure strictement négative, GAFA 5 (1995) 731-799.
[5] J. Bourgain and A. Kontorivich, On Zaremba's conjecture, 180 (2014) 137-196.
[6] R. Bowen, Equilibrium states and the ergodic theory of Anosov diffeomorphisms, Lecture Notes in Mathematics 470, Springer, Berlin, 1975.
[7] R. Bowen, Hausdorff dimension of quasi-circles, Publ. Math. (IHES) 50 (1979) 11-25.
[8] V. Climenhaga, Y. Pesin and A. Zelerowicz, Equilibrium states in dynamical systems via geometric measure theory. Bull. Amer. Math. Soc. (N.S.) 56 (2019) 569-610.
[9] A. Erchenko and A. Katok, Flexibility of entropies for surfaces of negative curvature, Israel J. Math., 232 (2019) 631-676.
[10] K. Falconer, Fractal Geometry: Mathematical Foundations and Applications, Wiley, 2003.
[11] S. Gouëzel and C. Liverani, Banach spaces adapted to Anosov systems, Ergod. Th. and Dynam. Sys., 26 (2006) 189-217.
[12] S. Gouëzel, F. Mathéus and F. Maucourant, Sharp lower bounds for the asymptotic entropy, Groups Geom. Dyn. 9 (2015), 711-735. of symmetric random walks,
[13] M. Hirsch and C. Pugh, Stable manifolds and hyperbolic sets. 1970 Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968) pp. 133-163, Amer. Math. Soc., Providence, R.I.
[14] O. Jenkinson and M. Pollicott, Calculating Hausdorff Dimension of Julia Sets and Kleinian Limit Sets, Amer. J. Math., 124 (2002) 495-545.
[15] A. Katok, Lyapunov exponents, entropy and periodic orbits for diffeomorphisms, Publ. Math. (IHES) 51 (1980) 137-173.
[16] A. Katok, Entropy and closed geodesies, Ergod. Th. and . Dynam. Sys. 2 (1982) 339-367
[17] A. Katok and B. Hasselblatt, Introduction to the Modern Theory of Dynamical Systems, C.U.P., Cambridge, 1995.
[18] A. Katok, G. Knieper and H. Weiss, Formulas for the derivative and critical points of topological entropy for Anosov and geodesic flows, Comm. Math. Phys. 138 (1991) 19-31
[19] A. Katok, G. Knieper, M. Pollicott and H. Weiss, Differentiability and analyticity of topological entropy for Anosov and geodesic flows, Invent. Math, 98 (1989) 581-597.
[20] Y. Kifer, Large Deviations in Dynamical Systems and Stochastic Processes, Transactions of the American Mathematical Society 321, no. 2 (1990) 505-524.
[21] T. Kucherenko and A. Quas, Flexibility of the Pressure Function, preprint.
[22] T. Kucherenko and A. Quas, Asymptotic behavior of the pressure function for Hölder potentials, preprint.
[23] O. Lanford, A computer-assisted proof of the Feigenbaum conjectures, Bull. Amer. Math. Soc. (N.S.) 6 (1982) 427-434.
[24] A. Lasota and J. Yorke, On the existance of invariant mesures for piecewise monotonic transformation, Trans. Amer. Math. Soc., 186 (1973) 481-488.
[25] E. Le Page, Theoremes limites pour les produits de matrices aleatoires, In Probability Measures on Groups (Oberwolfach, 1981), LNM 928 (1982) 258-303.
[26] L. Ma and M. Pollicott, Rigidity of pressures of Hïder potentials and the fitting of analytic functions via them, preprint
[27] A. Manning, Topological entropy for geodesic flows, Annals of Math. 110 (1979), 567-573.
[28] A. Manning, The volume entropy of a surface decreases along the Ricci flow, Ergod. Th. and Dynam. Sys., 24 (2004) 171-176
[29] C. Matheus, C. Moreira, M. Pollicott and P. Vytnova, Hausdorff dimension of Gauss-Cantor sets and two applications to classical Lagrange and Markov spectra, Adv. Math., 409 (2022) 108693.
[30] C. McMullen, Hausdorff dimension and conformal dynamics, III: Computation of dimension, Amer. J. Math., 120 (1998) 691-721.
[31] C. McMullen, Barycentric subdivision, martingales and hyperbolic geometry, https://people.math.harvard.edu/ ctm/home/text/class/harvard/219/21/html/home/sources/mcmullen_bary.pdf
[32] M. Misiurewicz, A short proof of the variational principle for a $\mathbb{Z}_{+}^{N}$ action on a compact space, in International Conference on Dynamical Systems in Mathematical Physics (Rennes, 1975), pp. 147-157. Astérisque, No. 40, Soc. Math. France, Paris, 1976.
[33] J. Palis and F. Takens, Hyperbolicity and Sensitive Chaotic Dynamics at Homoclinic Bifurcations, CUP, Cambridge, 1995.
[34] D. Parmenter and M. Pollicott, Equilibrium measures for hyperbolic attractors defined by densities, Discrete and Continuous Dynamical Systems 42, (2022) 3953-3977.
[35] W. Parry and M. Pollicott, Zeta functions and the periodic orbit structure of hyperbolic dynamics, Asterisque, 187-188 (1990) 1-268.
[36] M. Pollicott and P. Vytnova, Hausdorff dimension estimates applied to Lagrange and Markov spectra, Zaremba theory, and limit sets of Fuchsian groups, Trans. Amer. Math. Soc. Ser. B 9 (2022), 1102-1159
[37] E. Pujals and M. Shub, Dynamics of two-dimensional Blaschke products, Ergodic Theory and Dynamical Systems 28 (2008) 575-585
[38] D. Ruelle, A measure associated with axiom-A attractors. Amer. J. Math. 98 (1976) 619-654.
[39] D. Ruelle, Thermodynamic Formalism: The Mathematical Structure of Equilibrium Statistical Mechanics, 2nd ed, C.U.P., Cambridge, 2004.
[40] D. Ruelle, An inequality for the entropy of differentiable maps, Bol. Soc. Bras. Math., 9 (1978) 83-87.
[41] D. Ruelle, Repellers for real analytic maps. Ergodic Theory Dynamical Systems 2, (1982) 99-107.
[42] Ya. Sinai, Markov partitions and Y-diffeomorphisms, Funct. Anal. and Appl., 2:1 (1968) 64-89.
[43] Y. Sinai, The asymptotic behaviour of the number of closed geodesies. A.M.S. Transl. 73 (1968), 227-250.
[44] J. Slipantschuk, O. Bandtlow and W. Just, Complete spectral data for analytic Anosov maps of the torus, Nonlinearity 30 (7), 2667
[45] S. Smale, Differentiable dynamical systems, Bull. Amer. Math. Soc. 73(6): 747-817.
[46] P. Walters, Ergodic Theory, Springer, Berlin, 1982.
[47] R. Williams, Expanding attractors, Publ. Math. (IHES) 43 (1974) 169-203.
[48] M. Zinsmeister, Thermodynamic Formalism and Holomorphic Dynamical Systems, AMS, Providence, 2000.


[^0]:    ${ }^{1}$ Sinai and Ruelle actually show the stronger result that the measures $f_{*}^{k} \lambda$ converge to $\mu_{S R B}$ in the weak star topology without the need to average. Moreover, the topological mixing hypothesis is not restrictive because of the Smale spectral decomposition theorem [45].

[^1]:    ${ }^{2}$ This follows the lines of the Misiurewicz proof of the variational principle [32].

[^2]:    ${ }^{3}$ We recall that adding a constant or coboundary doesn't change the equilibrium, so we could have taken $G_{1}=0$

[^3]:    ${ }^{4}$ The proof of this result by Ruelle used families of transfer operators $\mathcal{L}_{t}$ and characterized the pressure in terms of the isolated maximal eigenvalue of a such an operator operator.

[^4]:    ${ }^{5}$ In some problems it may be convenient to know when the limit set has dimension larger than $\frac{1}{2}$.

[^5]:    ${ }^{6}$ Formally, if $a=1$ then $T_{1}^{\prime}(0)=1$ and the map is not strictly contracting. However, this is easily overcome

[^6]:    ${ }^{7}$ Technically, when $a=1$ then $T_{1}$ isn't contracting because $T_{1}^{\prime}(0)=1$, but this is easily accommodated

[^7]:    ${ }^{8}$ We evaluate the derivatives at 0 for definiteness. We could replace it by any other value $0 \leq x_{0} \leq 1$

[^8]:    ${ }^{9}$ This slightly fanciful name is meant to invoke the spirit of the Courant-Fischer-Weyl min-max principle for eigenvalues of operators. However it is probably better to relate it to the Collatz-Wielandt formula and Birkhoff-Varga formula.

[^9]:    ${ }^{10}$ We did the estimates using Mathematica, which since it is not an open source code means we have to reply on its internal error estimates. Thus the "rigour" element might be questionable.
    ${ }^{11}$ More generally, we can consider Markov maps at the expense of more notation.

[^10]:    ${ }^{12}$ There are various other equivalent definitions

[^11]:    ${ }^{13}$ For $d \times d$ matrices we need proximality and strong irreducibility
    ${ }^{14}$ We could also consider the action on $\mathbb{R} P^{1}$.

[^12]:    ${ }^{15}$ Following Boyd, we can use collocation on $[0,1]^{2}$ by taking products of Lagrange polynomials and Chebyshev points and the projecting $\pi:[0,1]^{2} \rightarrow \Delta$ by $\pi(x, y)=(x /(1-y), y)$

