The dimension of the Basilica Julia set

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1 Introduction

The Hausdorff dimension of Julia sets provides useful information about the complexity of the set. Therefore, it is useful to have a method to compute this value for examples which is both accurate and efficient. We will illustrate this with a well known and topical example.

Definition 1.1. *Given $c \in \mathbb{C}$ let $T_c : \mathbb{C} \to \mathbb{C}$ be the rational map $T_c(z) = z^2 + c$. The Julia set $\mathbb{J}_c$ can be defined to be the closure of the repelling periodic points, i.e.,*

$$\mathbb{J}_c = \overline{\{z \,:\, \exists n > 0, T^n z = z\}}.$$

This is a compact set. Historically, there has been interest in estimating the Hausdorff dimension $\dim_H(\mathbb{J}_c)$ [3].

Example 1.2. *Recently there has been more specific interest in the case of $c \in [-2, 2]$. One famous example is where $c = -1$, which is called the Basilica because of the apparent shape of $\mathbb{J}_{c=-1}$ (see Figure 1).*

We will describe a simple method to get the following rigorous bounds.

Theorem 1.3. *The dimension of the Basilica Julia set satisfies*

$$1.268352 \leq \dim_H(\mathcal{J}_{c=-1}) \leq 1.268353.$$

In principle, the same method would apply for any hyperbolic Julia set. However, the difficulty is in choosing appropriate domains for iterate function schemes for different examples.
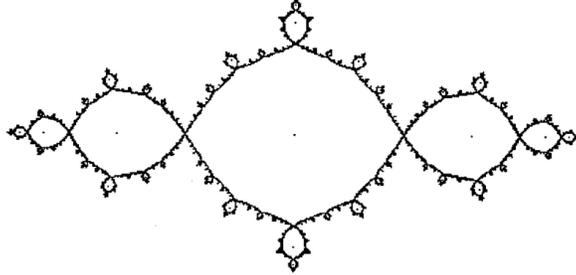
Figure 1: The Basilica Julia set $\mathbb{J}_{-1}$

2   An iterated function scheme

We first want to reduce the study of $\mathbb{J}_{-1}$ to that of the limit set X for an iterated function scheme (which corresponds to a quarter of the Julia set).

The Julia set $\mathbb{J}_{c=-1}$ has natural symmetries given by reflection in either the x-axis or the y-axis. Thus the dimension of $\mathbb{J}_{c=-1}$ is the same as that as the intersection

$$X = \mathbb{J}_{c=-1} \cap \{z = x + iy : x, y \geq 0\}$$

with the positive quadrant $x, y \geq 0$. We can write X as the limit set of a Markov iterated function scheme defined on the disjoint union [1] $A_1 \amalg A_2$ of two regions in the complex plane

$$A_1 = \left\{ re^{i\theta} : r_1 \leq r \leq r_2, \quad 0 \leq \theta \leq \frac{\pi}{2} \right\} \text{ and}$$
$$A_2 = \left\{ re^{i\theta} : R_1 \leq r \leq R_2, \quad 0 \leq \theta \leq \pi \right\}.$$

Example 2.1. *Possible choices are* $r_1 = \frac{1}{4}$, $r_2 = \frac{17}{20}$, $R_1 = \frac{1}{10}$, $R_2 = \frac{13}{20}$.

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[1]Points z in the overlap have two copies, one associated to each of the regions A_1 and A_2, which might formally be denoted $(z, 1)$ and $(z, 2)$. However, for simplicity of notation we just denote both by z
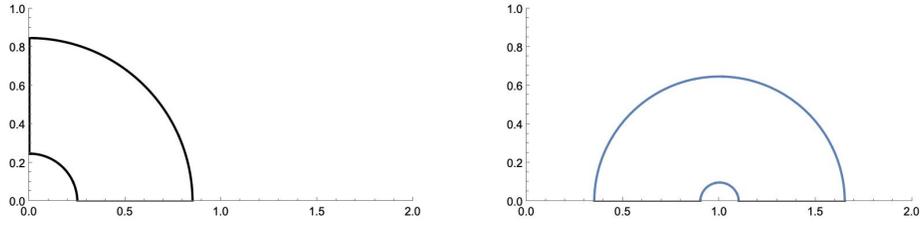
Figure 2: (i) The region A_1; and (ii) the region A_2.

Let $T(z) = \sqrt{1+z}$ and $R(x+iy) = -x+iy$. We can then define four maps

$$
\begin{aligned}
T_1 &= T : A_1 \to A_2 \\
T_2 &= T \circ R : A_1 \to A_2 \\
T_3 &= T : A_2 \to A_2 \\
T_4 &= T \circ R : A_2 \to A_1
\end{aligned}
\qquad \text{where} \qquad
\begin{aligned}
T_1(x+iy) &= \sqrt{1+x+iy} \\
T_2(x+iy) &= \sqrt{1-x+iy} \\
T_3(x+iy) &= \sqrt{1+x+iy} \\
T_4(x+iy) &= \sqrt{1-x+iy}
\end{aligned}
$$

In particular, we see that

$$T_4(A_2) \subset \text{int } (A_1) \text{ and } T_1(A_1) \cup T_2(A_1) \cup T_3(A_2) \subset \text{int } (A_2)$$
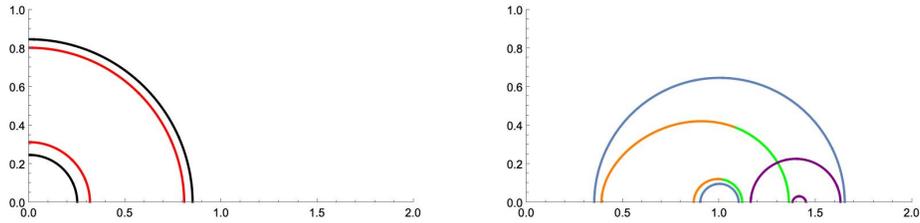

Figure 3: (i) The region A_1 contains the image $T_4(A_2)$ [red]; and the region A_2 contains the images $T_1(A_1)$ [green], $T_2(A_1)$ [orange] and $T_3(A_2)$ [purple]

We can then interpret X as the limit set for the associated iterated function system for the domain(s) A_1, A_2 and contractions T_1, T_2, T_3, T_4 as follows.

Lemma 2.2. *The set X consists of all possible accumulation points of sequences*

$$\{T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_n}(x_0), \quad n \geq 1\},$$

where $(i_n)_{n=1}^{\infty} \in \{1, 2, 3, 4\}^{\mathbb{N}}$ satisfy

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$$i_n \in \begin{cases} \{1,2\} & \text{if } x_0 \in A_1 \\ \{3,4\} & \text{if } x_0 \in A_2 \end{cases} \text{ and } i_k \in \begin{cases} \{1,2\} & \text{if } T_{i_{k+1}} \circ T_{i_{k+2}} \cdots \circ T_{i_n}(x_0) \in A_1 \\ \{3,4\} & \text{if } T_{i_{k+1}} \circ T_{i_{k+2}} \circ \cdots T_{i_n}(x_0) \in A_2 \end{cases} \text{ for } k \geq 2$$

3   Method for estimating the dimension

There are various methods available to estimate $\dim(X)$ cf. [3], [2], [4]. We will use the last of these. In particular, we want to use a characterization of the dimension $d = \dim_H(X)$ in terms of a family of transfer operators on the space of continuous functions on the disjoint union $A_1 \amalg A_2$ with the supremum norm.

It is notationally convenient to represent a function $f \in C(A_1 \amalg A_2, \mathbb{R})$ as a pair of function (f_1, f_2) where $f_1 \in C(A_1, \mathbb{R})$ and $f_2 \in C(A_2, \mathbb{R})$.

Definition 3.1. *For each $t \in \mathbb{R}$ we can associate an operator*

$$\mathcal{L}_t : C(A_1 \amalg A_2, \mathbb{R}) \to C(A_1 \amalg A_2, \mathbb{R})$$

defined for $f = (f_1, f_2)$ by $\mathcal{L}_t f = ((\mathcal{L}_t f)_1, (\mathcal{L}_t f)_2)$ where

$$(\mathcal{L}_t f)_1(z) = (2|T_1(z)|)^{-d} f_2(T_1 z) + (2|T_2(z)|)^{-d} f_2(T_2 z) \text{ where } z \in A_1$$
$$(\mathcal{L}_t f)_2(z) = (2|T_3(z)|)^{-d} f_2(T_3 z) + (2|T_4(z)|)^{-d} f_1(T_4 z) \text{ where } z \in A_2.$$

Remark 3.2. *For any quadratic map $T(z) = z^2 + c$ we see trivially observe that $|T'(z)|^{-d} = (2|z|)^{-d}$ which is how the weights in the above expressions arise.*

The simple criterion we want to bound $d = \dim_H(\mathbb{J}_{-1})$ is the following.

Proposition 3.3. *Let $t_- < t_+$.*

1. *If we can find $f^- = (f_1^-, f_2^-)$ such that*

$$\min_{i=1,2} \inf_{z \in A_i} \frac{(\mathcal{L}_{t_-} f^-)_i(z)}{f_i^-(z)} > 1 \text{ then } d > t_-.$$

2. *If we can find $f^+ = (f_1^+, f_2^+)$ such that*

$$\max_{i=1,2} \sup_{z \in A_i} \frac{(\mathcal{L}_{t_2} f^+)_i(z)}{f_i^+(z)} < 1 \text{ then } d < t_+.$$

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Sketch proof. It follows from the work of Bowen and Ruelle that $t = \dim_H(X)$ is the unique value such that $\mathcal{L}_t$ has spectral radius $\rho(t) = 1$. Moreover, for all $z \in A_1 \amalg A_2$ and $n \geq 1$ we have

$$\mathcal{L}_{t_-}^n f^-(z) > f^-(z) \text{ and } \mathcal{L}_{t_+}^n f^+(z) < f^+(z).$$

In particular, we can deduce that

$$\lim_{n \to +\infty} \left(\mathcal{L}_{t_-}^n f^- \right)^{1/n} = \rho(t_-) \geq 1 \geq \rho(t_+) = \lim_{n \to +\infty} \left(\mathcal{L}_{t_+}^n f^+ \right)^{1/n}$$

Since $t \mapsto \rho_t$ is differentiable and strictly monotone decreasing. The result then follows from the intermediate value theorem. $\qquad\square$

4   Practical implementation

There are probably various methods to construct functions $f = (f_1, f_2)$ for Proposition 3.3, but we will proceed as follows.

Step 1. Given $M > 1$ we can present functions on $f_1 : A_1 \to \mathbb{R}$ and $f_2 : A_2 \to \mathbb{R}$ as follows.

1. For $z = re^{i\theta} \in A_1$ with $r_1 \leq r \leq r_2$ and $0 \leq \theta \leq \pi/2$:

   (a) we introduce *Chebychev pairs for A_1* (ρ_i, θ_j) and points $z_{ij} = \rho_i e^{i\theta_j} \in A_1$ given by

   $$\rho_i = r_1 + \frac{(r_2 - r_1)}{2} \left(1 + \cos\left(\frac{\pi}{M} \left(i + \frac{1}{2} \right) \right) \right) \text{ for } i = 0, \cdots, M - 1 \text{ and}$$
   $$\theta_j = \frac{\pi}{4} \left(1 + \cos\left(\frac{\pi}{M} \left(j + \frac{1}{2} \right) \right) \right) \text{ for } j = 0, \cdots, M - 1;$$

   (b) we introduce the associated *Lagrange functions for A_1* given by $f_{1,n,m}(r, t) = \phi_n(r)\psi_m(t)$ (for $0 \leq n, m \leq M - 1$) where

   $$\phi_n(r) = \frac{\prod_{i=0}^{M-1}(r - \rho_i)}{\prod_{i=0, i \neq n}^{M-1}(\rho_n - \rho_i)} \text{ and } \psi_n(t) = \frac{\prod_0^{M-1}(t - \theta_i)}{\prod_{j=0, j \neq n}^{M-1}(\theta_n - \theta_j)}.$$

2. For $z = Re^{i\theta} \in A_2$ with $R_1 \leq R \leq R_2$ and $0 \leq \theta \leq \pi$

(a) we introduce *Chebychev pairs for* A_2 (Ω_i, Θ_j) and points $w_{ij} = 1 + \Omega_i e^{i\Theta_j} \in A_2$ given by

$$\Omega_i = R_1 + \frac{(R_2 - R_1)}{2}\left(1 + \cos\left(\frac{\pi}{M}\left(i + \frac{1}{2}\right)\right)\right) \text{ for } i = 0, \ldots, M - 1 \text{ and}$$

$$\Theta_i = \frac{\pi}{2}\left(1 + \cos\left(\frac{\pi}{M}\left(i + \frac{1}{2}\right)\right)\right) \text{ for } j = 0, \cdots, M - 1;$$

(b) we introduce the associated *Legrendre functions for* A_2 given by $f_{2,n,m}(R, T) = \Phi_n(R)\Psi_m(T)$ (for $0 \le n, m \le M - 1$) where

$$\Phi_n(R) = \frac{\prod_{i=0}^{M-1}(R - \Omega_i)}{\prod_{i=0,i\neq n}^{M-1}(\Omega_n - \Omega_i)} \text{ and } \Psi_n(T) = \frac{\prod_{i=0}^{M-1}(T - \Theta_i)}{\prod_{j=0,j\neq n}^{M-1}(\Theta_n - \Theta_j)}.$$

In particular we see that

$$f_{1,n,m}(z_{ij}) = \delta_{ni}\delta_{mj}(z) \text{ and } f_{2,n,m}(w_{ij}) = \delta_{ni}\delta_{mj}(z)$$

Step 2. Given $t > 0$ we can then associate $M^2 \times M^2$ matrices M_{11}^t, M_{12}^t, M_{21}^t and M_{22}^t whose rows and columns are indexed by pairs $(n, m), (k, l) \in \{0, 1 \cdots, M - 1\}^2$ and whose entries are defined as follows

$M_{11}^t((n, m), (k, l)) = 0$

$M_{12}^t((n, m), (k, l)) = (2|T_1(w_{kl})|)^{-d}f_{1,n,m}(T_1(w_{kl}))$

$M_{21}^t((n, m), (k, l)) = (2|T_2(z_{kl})|)^{-d}f_{2,n,m}(T_2 z_{kl}) + (2|T_1(z_{kl})|)^{-d}f_{2,n,m}(T_1 z_{kl})$

$M_{22}^t((n, m), (k, l)) = (2|T_3(w_{kl})|)^{-d}f_{2,n,m}(T_3 w_{kl})$

We can then associate the $(2M^2) \times (2M^2)$ matrix

$$M^t = \begin{pmatrix} M_{11}^t & M_{12}^t \\ M_{21}^t & M_{22}^t \end{pmatrix}$$

Step 3. Let $v^t = (v_{1,n,m}^t, v_{2,n,m}^t)_{(n,m)}$ be the left eigenvector associated to the largest (positive) eigenvector for M^t. Finally, we let

$$f_1(z) = \sum_{(n,m)} v_{1,n,m}^t f_{1,n,m} \text{ and } f_2(z) = \sum_{(n,m)} v_{2,n,m}^t f_{2,n,m}$$

By a simple bisection procedure we can narrow the choices of $0 < t_- < t_+ < 2$ to give the required estimate.

Example 4.1. *We can use Mathematica and work to 50 decimal places. We can take $M = 30$ then M^t is a 1800×1800 matrix. Let $t_1 = 1.268352$ and $t_2 = 1.268353$ then we associate functions $f^- = f_{t_1}$ and $f^+ = f_{t_2}$ using the construction described above.*
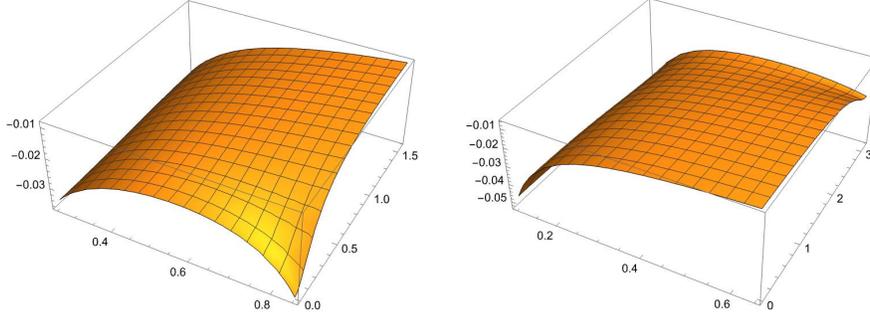

Figure 4: Plots of $f^+|A_1$ and $f^+|A_2$ (although the plots of $f^-|A_1$ and $f^-|A_2$ are very similar).

We can the estimate

$$\min_{i=1,2} \inf_{z \in A_i} \frac{(\mathcal{L}_{t_1} f^+)_i(z)}{f_i^+(z)} = 1.00000051367952\ldots > 1$$

and

$$\max_{i=1,2} \sup_{z \in A_i} \frac{(\mathcal{L}_{t_2} f^-)_i(z)}{f_i^-(z)} = 0.999999671522\ldots < 1$$

This confirms the bounds in Theorem 1.3.

Remark 4.2. *One can apply the same construction for $T_c(z) = z^2 + c$ with $c \in (-1.03, -0.97)$ using the same domains A_1 and A_2. However, for other values of c one will need to find new domains and new iterated function schemes.*

References

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