Maximizing dimension for Bernoulli measures and the Gauss map

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1 Introduction

Let $T : (0,1] \to (0,1]$ be the usual Gauss map defined by

$$T(x) = \begin{cases} \frac{1}{x} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

For each infinite probability vector in $\mathcal{P} = \{\underline{p} = (p_k)_{k=1}^{\infty} \in [0,1]^{\mathbb{N}} : \sum_k p_k = 1\}$ we can associate a natural T-invariant measure $\mu_{\underline{p}} := \nu_{\underline{p}} \pi^{-1}$, where $\nu_{\underline{p}}$ is the usual countable state Bernoulli measure on $\mathbb{N}^{\mathbb{Z}}$ and $\pi : \mathbb{N}^{\mathbb{N}} \to [0,1)$ is the usual continued fraction expansion $\pi(x_n) = [x_1, x_2, x_3, \cdots]$. For such measures we can define the *entropy* and *Lyapunov exponents* by

$$h(\mu_{\underline{p}}) = -\sum_{n=1}^{\infty} p_n \log p_n \text{ and } \lambda(\mu_{\underline{p}}) = \int \log |T'| d\mu_{\underline{p}}(x),$$

whenever they are finite, and the *dimension* of $\mu_{\underline{p}}$ by $d(\mu_{\underline{p}}) = \frac{h(\mu_{\underline{p}})}{\lambda(\mu_{\underline{p}})} > 0$. Kifer, Peres and Weiss [2] observed that $d(\mu_{\underline{p}})$ is uniformly bounded away from 1 (making use of a thermodynamic approach of Walters) [1] i.e.,

$$D := \sup \left\{ d(\mu_{\underline{p}}) : h(\mu_{\underline{p}}), \lambda(\mu_{\underline{p}}) < +\infty \right\} < 1. \tag{1.1}$$

We will give a simple proof of the following result.

Theorem 1.1. *There exists $\underline{p}^{\dagger} \in \mathcal{P}$ with $h(\mu_{\underline{p}^{\dagger}}), \lambda(\mu_{\underline{p}^{\dagger}}) < +\infty$ such that:*

1. *$\mu_{\underline{p}}^{\dagger}$ realises the supremum in (1.1), i.e., $d(\mu_{\underline{p}}^{\dagger}) = D$; and*
2. *$p_k^{\dagger} \asymp k^{-2D}$.*

This answers a question I was asked by K. Burns. I posed the question to my graduate student N. Jurga who, in collaboration PDRA S. Baker, gave an elementary proof. The proof presented below uses thermodynamical ideas and has the merit of being very short and easy to generalize.

[1] In [2] they showed $D < 1 - 10^{-7}$, but Jenkinson and the author have improved this to $D < 1 - 5 \times 10^{-5}$

2 Proof of Theorem 1.1

We can begin with the following standard lemma from [3] (see also [1], Lemma 3.2).

Lemma 2.1. *If* $\frac{h(\mu_{\underline{p}})}{\lambda(\mu_{\underline{p}})} > \frac{1}{2}$ *then* $h(\mu_{\underline{p}}), \lambda(\mu_{\underline{p}}) < +\infty$.

Since it is easy to exhibit $\underline{p} \in \mathcal{P}$ with $h(\mu_{\underline{p}}), \lambda(\mu_{\underline{p}}) < +\infty$ and $d(\mu_{\underline{p}}) > \frac{1}{2}$ we can also write:

$$D = \sup_n \sup \left\{ d(\mu_{\underline{p}^*}) \,:\, \underline{p}^* \in \mathcal{P}_n \right\}, \tag{2.1}$$

where $\mathcal{P}_n$ consists of the probability vectors $\underline{p}^* = (p_k^*)_{k=1}^{\infty}$ satisfying $p_k^* = 0$, for $k > n$. For each n, the function $\mathcal{P}_n \ni \underline{p}^* \mapsto d(\mu_{\underline{p}^*})$ is easily seen to be smooth and since $\sum_{k=1}^n p_k^* = 1$ we can use the method of Lagrange multipliers to deduce that a critical point satisfies

$$\frac{\partial d(\mu_{\underline{p}^*})}{\partial p_i} = \frac{\partial d(\mu_{\underline{p}^*})}{\partial p_j} \text{ for } i \neq j. \tag{2.2}$$

The logarithmic derivatives of $d(\mu_{\underline{p}^*})$ obviously take the form

$$\frac{1}{d(\mu_{\underline{p}^*})}\frac{\partial d(\mu_{\underline{p}^*})}{\partial p_i} = \frac{1}{h(\mu_{\underline{p}^*})}\frac{\partial h(\mu_{\underline{p}^*})}{\partial p_i} - \frac{1}{\lambda(\mu_{\underline{p}^*})}\frac{\partial \lambda(\mu_{\underline{p}^*})}{\partial p_i} \text{ for } 1 \leq i \leq n. \tag{2.3}$$

We can rewrite the right hand side of (2.3) using the following two lemmas.

Lemma 2.2. $\frac{\partial h(\mu_{\underline{p}^*})}{\partial p_i^*} = \log p_i^* + 1$.

We denote the intervals $[i] := [\frac{1}{i+1}, \frac{1}{i}]$, for $i \geq 1$.

Lemma 2.3. $\frac{1}{\lambda(\mu_{\underline{p}^*})}\frac{\partial \lambda(\mu_{\underline{p}^*})}{\partial p_i} = \frac{1}{p_i^*}\frac{\int_{[i]} \log|T'| d\mu_{\underline{p}^*}}{\int \log|T'| d\mu_{\underline{p}^*}} - 1$

Proof. Following ([5], Question 5 (a) p.96) and ([4], Proposition 4.10), we can first rewrite

$$\lambda(\mu_{\underline{p}^*}) = \frac{\partial P(f_{\underline{p}^*} - t\log|T'|)}{\partial t}|_{t=0} \text{ and } \frac{\partial \lambda(\mu_{\underline{p}^*})}{\partial p_i} = \frac{\partial^2 P(f_{\underline{p}^*} - s/p_i^* - t\log|T'|)}{\partial s \partial t}|_{t=0,s=0} \tag{2.4}$$

where $f_{\underline{p}} = -\sum_{j=1}^n \chi_{[j]} \log p_j$ and $P(\cdot)$ denotes the pressure function. Following ([5], Question 5 (b) p.96) and ([4], Proposition 4.11) we have

$$\begin{aligned}\frac{\partial^2 P(f_{\underline{p}^*} - s/p_i^* - t\log|T'|)}{\partial s \partial t}|_{t=0,s=0} &= \frac{1}{p_i^*}\int \left(\chi_{[i]} - p_i^*\right)\left(-\log|T'| + \int \log|T'| d\mu_{\underline{p}^*}\right) d\mu_{\underline{p}} \\ &+ \frac{2}{p_i^*}\sum_{n=1}^{\infty}\int \mathcal{L}^n_{f_{\underline{p}^*}}\left(\chi_{[i]} - p_i^*\right)\left(-\log|T'| + \int \log|T'| d\mu_{\underline{p}^*}\right) d\mu_{\underline{p}}\end{aligned} \tag{2.5}$$

where $\mathcal{L}_{f_{\underline{p}^*}} : C^1([0,1]) \to C^1(0,1]$ is defined by $w(x) = \sum_{k=1}^{\infty} p_k w\left(\frac{1}{k+x}\right)$ [5]. Since $\mathcal{L}_{f_{\underline{p}^*}} 1 = 1$ we can deduce that the series in (2.5) vanishes and using (2.4) we can write

$$\frac{1}{\lambda(\mu_{\underline{p}^*})}\frac{\partial \lambda(\mu_{\underline{p}^*})}{\partial p_i} = \frac{1}{p_i^*}\int (\chi_i - p_i^*)\left(-\frac{\log|T'|}{\int \log|T'| d\mu_{\underline{p}^*}} + 1\right) d\mu_{\underline{p}} = \frac{1}{p_i^*}\frac{\int_{[i]} \log|T'| d\mu_{\underline{p}}}{\int \log|T'| d\mu_{\underline{p}}} - 1. \tag{2.6}$$

□

Applying Lemmas 2.2 and 2.3 to (2.3) we see that the critical point for $d(\mu_{\underline{p}^*})$ satisfies.

$$2D\log\left(\frac{i+1}{j+1}\right) \leq \log\left(\frac{p_j^*}{p_i^*}\right) \leq 2D\log\left(\frac{i}{j}\right) \text{ for any } n \geq 2 \text{ and } i > j. \tag{2.7}$$

Letting n tend to infinity, and using the tightness coming from the bounds on p_i^*, we can deduce that there exists a limit point $\underline{p}^\dagger \in \mathcal{P}$ satisfying both $D = d(\mu_{\underline{p}^\dagger})$ (using (2.1)) and (2.7). The proof of Theorem 1.1 follows immediately.

Remark 2.4. One easily can generalize this simple analysis to suitable f-expansions.

References

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[5] D. Ruelle, *Thermodynamic formalism*, Encyclopedia of Mathematics and its Applications, 5. Addison-Wesley Publishing Co., Reading, Mass., 1978.