

Non-eclipse billiards and rigidity

“Can one hear the shape of scatterers when slightly deaf?”

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1 Introduction

Let $D_1, \dots, D_k \subset \mathbb{R}^2$ ($k \geq 3$) be bounded convex shapes, called “scatterers”, with analytic boundaries which satisfy the non-eclipse condition, i.e., for distinct $1 \leq i < j < \ell \leq k$ we have that the convex hull of $D_i \cup D_j$ is disjoint from D_ℓ . We define the billiard dynamics to be that of elastic collisions off the boundaries of the scatterers and straight line trajectories between them. Some trajectories can evidently miss the scatterers and escape without further collisions. However, we will restrict attention to the recurrent part of the system, i.e., the set X of those trajectories which bounce infinitely many times of the scatterers, both into the future and past. This simple model can be traced back to [1].

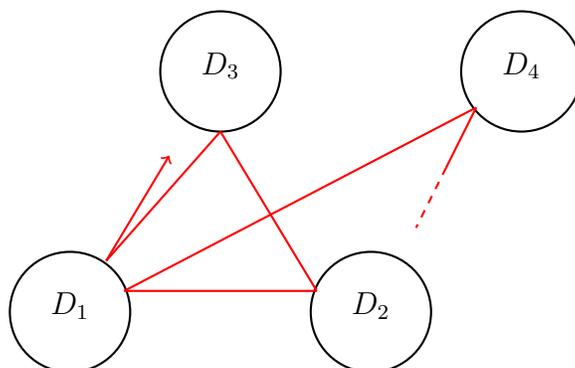


Figure 1: The trajectory is a straight line except for elastic collisions off the boundaries of the scatterers.

There are a countable infinity of periodic trajectories τ , which are dense in the space of all allowed trajectories X , and which can be naturally indexed by the sequence of labels of the scatterers encountered. We denote the length of the trajectory by $\ell(\tau)$ and let $(\ell(\tau))_\tau$ denote the *marked length spectrum*, i.e., the lengths associated to each periodic trajectory.

The following is a significant recent rigidity result.

Theorem 1.1 (J. de Simoi, V. Kaloshin and M. Leguil [3]). *For an open and dense set of reflecting obstacles D_1, \dots, D_k they are uniquely determined up to isometry by the marked length spectrum provided $D_1 \cup D_2$, say, has two axes of symmetry.*

The following is a natural question (suggested by the work of A. Gogolev and F. Rodriguez Hertz on length rigidity results for negatively curve surfaces [2]).

Question. Does the conclusion of Theorem 1.1, for example, hold with a knowledge of the lengths of fewer periodic trajectories? How could one select such a sub-family?

The following example illustrates a simple way to generate such a subfamily.

Example 1.2 (Winding numbers). *Every closed trajectory τ gives rise to an oriented closed curve in the plane. We can fix a point $p \in \mathbb{R}^2 \setminus \cup_{i=1}^k D_i$ which isn't contained in X and which lies inside some triangular trajectory, say. We can associate to a periodic trajectory an element*

$$[\tau] \in \pi_1(\mathbb{R}^2 \setminus \{p\}) = \mathbb{Z}$$

in the fundamental group of the plane with the point p removed. We can then restrict our attention to the subfamily of closed trajectories for which $[\tau] = 0$, say.

More generally, one could fix points $p_1, \dots, p_d \in \mathbb{R}^2 \setminus \cup_i D_i$ none of which is contained in X then we can associate to a closed trajectory an element $[\tau] \in \pi_1(\mathbb{R}^2 \setminus \cup_j \{p_j\}) = \mathbb{F}_d$. Let $\Gamma < \mathbb{F}_d$ be a normal subgroup then we can restrict our attention to closed trajectories for which $[\tau] = e \in \mathbb{F}_d/\Gamma$.

2 Coding the trajectories

We can associate a subshift of finite type $\sigma : \Sigma \rightarrow \Sigma$ on

$$\Sigma = \{x = (x_n)_{n=-\infty}^{\infty} \in \{1, \dots, k\} : x_n \neq x_{n+1} \text{ for } n \in \mathbb{Z}\},$$

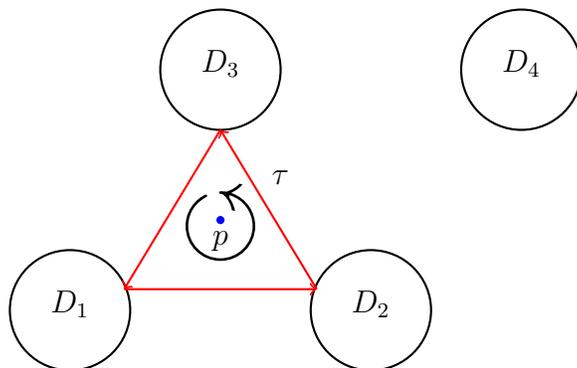


Figure 2: The illustrated (in red) periodic trajectory τ circles the point p and $[\tau]$ represents a generator $1 \in \pi_1(\mathbb{R}^2 \setminus \{p\}) = \mathbb{Z}$, say.

with the usual Tychanoff product topology, defined by $(\sigma x)_n = x_{n+1}$ (for $n \in \mathbb{Z}$). This shift is clearly transitive (i.e., there exists a dense orbit). Moreover, the periodic trajectories τ are easily seen to be in a bijection with (up to cyclic permutation) strings $\underline{i} = (i_0, \dots, i_{p-1}) \in \{1, \dots, k\}^p$ with $i_j \neq i_{j+1}$ for $0 \leq j \leq p-1$ and $i_0 \neq i_{p-1}$. We can denote the length of the trajectory by $\ell(\tau)$.¹

Let Γ be a discrete group. The following technical assumptions will be useful later.

Hypothesis 2.1. *We can associate to each ordered pair (i, j) with $i \neq j$ an element $\phi(i, j) \in \Gamma$ such that:*

1. $\Gamma = \langle \phi(i, j) : i \neq j \rangle$, i.e., Γ is finitely generated by the edge labelling; and
2. $\phi(i, j) = \phi(j, i)^{-1}$.

We can consider the skew product map $\hat{\sigma} : \Sigma \times \Gamma \rightarrow \Sigma \times \Gamma$ given by

$$\hat{\sigma}(x, g) = (\sigma x, g\phi(x_0, x_1))$$

where $x = (x_n)_{n=-\infty}^{\infty} \in \Sigma$, i.e., it associates an element in Γ to parts of trajectories between consecutive collisions according to the labels of the scatterers.

¹One could formulate the flow as a suspension flow under a Hölder function and view this as an Axiom A* flow.

Hypothesis 2.2. *Let us assume that the map $\widehat{\sigma} : \Sigma \times \Gamma \rightarrow \Sigma \times \Gamma$ is transitive.*

The main result is the following.

Theorem 2.3. *A knowledge of the lengths $\ell(\tau)$ of periodic trajectories τ associated to σ -periodic orbits $\{x, \sigma x, \dots, \sigma^{p-1}x\}$ with*

$$\phi^n(x) := \phi(x_0, x_1)\phi(x_1, x_2) \cdots \phi(x_{p-1}, x_0) = e$$

is equivalent to a knowledge of all of the lengths of periodic trajectories.

An important property of the billiard system is that it has an involution i which reverses the direction of a trajectory at each point and preserves the parameterization. In particular, if τ is a closed trajectory then its image $i(\tau)$ under the involution is again a closed trajectory with $\ell(\tau) = \ell(i\tau)$.

Let us proceed with the proof of Theorem 2.3 as follows. Choose an arbitrary closed trajectory β (of least period $\ell(\beta)$) and a point $x \in \beta$. For any $n \in \mathbb{N}$ we can flow for time $n\ell(\beta)$ to get an iterated closed orbit denoted β^n . If $i : X \rightarrow X$ is the involution then $i(\beta^n)$ is a closed trajectory where the direction is reversed.

The following simpler lemma will prove useful.

Lemma 2.4. *For each symbol $i \in \{1, 2, \dots, k\}$ we can choose a string $\xi(i) = x_1x_2 \dots x_n$ (where $n = n(i)$ depends on i) such that*

1. $x_1 \neq i$ and $x_n \neq i$;
2. $\phi(i, x_1)\phi(x_1, x_2)\phi(x_2, x_3) \cdots \phi(x_n, i) = e$

Proof. This follows from Hypothesis 2.2, part 2. □

Proceeding to the proof of the main result we can denote $\beta = (i_0, i_1, \dots, i_{m-1})$, say, with $i_j \neq i_{j+1}$ for $1 \leq j \leq m-1$ and $i_0 \neq i_{m-1}$, and then write

$$\beta^n = (i_0, i_1, \dots, i_{m-1}, i_0, i_1, \dots, i_{m-1}, \dots, i_0, i_1, \dots, i_{m-1})$$

(with the block representing β repeated n times) then

$$i(\beta^n) = (i_{m-1}, i_{m-2}, \dots, i_0, i_{m-1}, i_{m-2}, \dots, i_0, \dots, i_{m-1}, i_{m-2}, \dots, i_0)$$

where we use i this time to represent reversing the order of the sequence. In particular, we can consider the concatenation:

$$\gamma_n = \beta^n \xi(i_{m-1}) i(\beta^n) \xi(i_0)$$

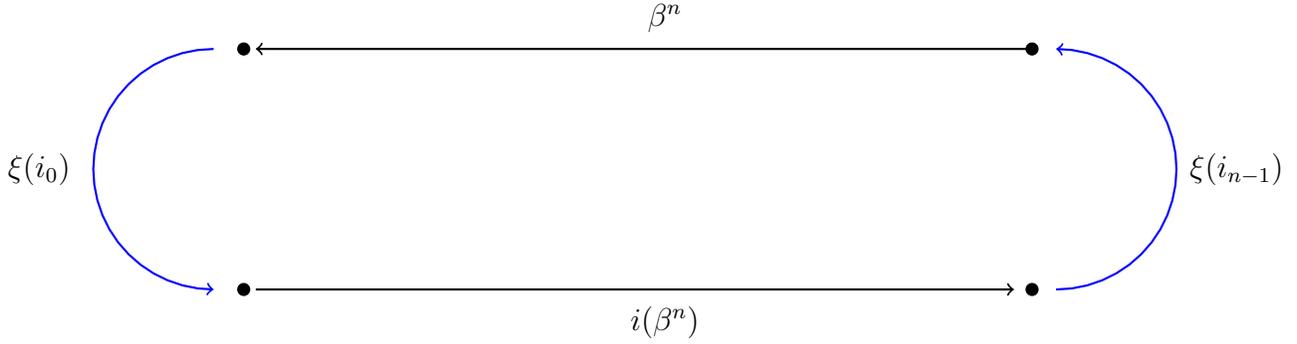


Figure 3: We construct a closed orbit γ_n out of the four orbit segments.

which by Hypothesis 2.1 represents a closed orbit for which the associated element in Γ is e .

We can now easily relate the lengths $\ell(\beta)$ and $\ell(\gamma_n)$ using the following lemma.

Lemma 2.5. *There exists $C > 0$ (independent of n) such that*

$$2n\ell(\beta) \leq \ell(\gamma_n) \leq 2n\ell(\beta) + C.$$

Proof. These bounds follow easily from the construction. In particular, C comes from a bound on the lengths associated to $\xi(\cdot)$ and a simple form of closing lemma. \square

Since C is independent of n we immediately have that

$$\ell(\beta) = \frac{1}{2} \lim_{n \rightarrow +\infty} \frac{\ell(\gamma_n)}{n}.$$

Recall that β is an arbitrary closed trajectory and γ_n is a closed trajectory whose length $\ell(\gamma_n)$ we are assuming is known. Thus a knowledge of the lengths of all of the closed orbits τ is equivalent to a knowledge of the closed orbits with the additional condition $\phi(\tau) = e$, as claimed.

Example 2.6 (Example 1.2 revisited). *We can now explain how Theorem 2.3 applies to Example 1.2. Fix distinct points $p, q \in \mathbb{R}^2 \setminus \cup_{i=1}^k D_i$. For each scatterer D_i (with $1 \leq i \leq k$) we can draw curves $c_i : [0, 1] \rightarrow \mathbb{R}^2 \setminus \cup_{i=1}^k D_i \cup \{p\}$*

with $c_i(0) = q$ and $c_i(1) \in \partial D_i$. For $i \neq j$ we can associate a close loop γ_{ij} (which does not pass through p) based at q by composing c_i with a segment ρ_{ij} of a trajectory from ∂D_i to ∂D_j and then with c_j^{-1} back to q (where there are also small connecting segments in ∂D_i and ∂D_j to close up the loop). We can then associate to $i \neq j$ the element $\phi(i, j) = [\gamma_{ij}] \in \pi_1(\mathbb{R}^2 \setminus \{p\}) = \mathbb{Z}$. Then for a periodic trajectory τ corresponding to a periodic sequence $\sigma^p x = x \in \Sigma$ we have that

$$\phi^p(x) = [\gamma_{x_0 x_1}] [\gamma_{x_1 x_2}] \cdots [\gamma_{x_{p-1} x_0}] = [\tau]$$

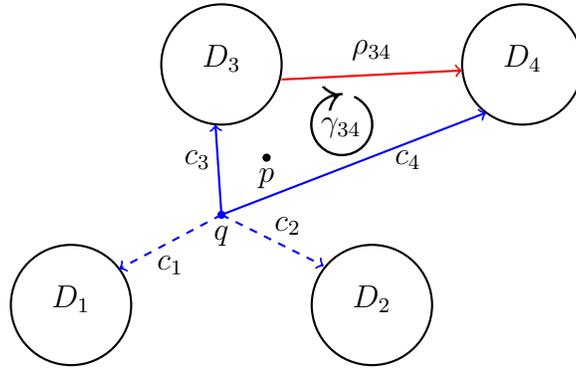


Figure 4: We can associate to the scatters D_3 and D_4 , for example, a closed curve γ_{34} .

It only remains to show that the skew product $\hat{\sigma} : \Sigma \times \Gamma \rightarrow \Sigma \times \Gamma$ is transitive and then apply Theorems 2.3 and 1.1. However, we can construct a transitive orbit $\xi = (x, 0) \in \Sigma \times \mathbb{Z}$ for $\hat{\sigma}$ using a standard method. We first choose an enumeration of the countable family of all pairs $((w_n, r_n))_{n=1}^{\infty}$ of allowed finite strings and integers. We can next associate to the triangle a word w_0 of three symbols and then there will be an associated weight $\phi^3(w_0) = 1$, say. Similarly if we reverse the order of the symbols to get a word \bar{w}_0 we see that $\phi^3(\bar{w}_0) = -1$.

We can then construct the sequence x by concatenating all of the words $(w_n)_{n=1}^{\infty}$, with suitable iterates of w_0 and \bar{w}_0 (and any extra symbols needed to form a bridge between to get an allowed sequence) as appropriate to have the alignment in the second terms $(r_n)_{n=1}^{\infty}$.

References

- [1] P. Gaspard and S. Rice, Scattering from a classically chaotic repeller, *The Journal of Chemical Physics* 90, 2225 (1989).
- [2] A. Gogolev and F. Rodriguez Hertz, Abelian Livshits theorems and geometric applications, in *A Vision for Dynamics in the 21st Century-the Legacy of Anatole Katok*, Cambridge University Press, Cambridge, 2024, pp. 139–167.
- [3] J. de Simoi, V. Kaloshin and M. Leguil, Marked length spectral determination of analytic chaotic billiards with axial symmetries, *Invent. math.* 233 (2023) 829–901