Zeros of the Selberg zeta function for non-compact surfaces

Abstract

In the present paper we provide a rigorous mathematical foundation for describing the zeros of the Selberg zeta functions Z_X for certain compact surfaces X with boundary, corresponding to convex cocompact Fuchsian groups Γ without parabolic points. For definiteness, we consider the case of symmetric "pairs of pants" where we show how Z_X can be approximated by complex trigonometric polynomials on suitably large domain (in Theorem 3.5). As our main application, we explain the striking empirical results of Borthwick [5] (Theorem 1.4).

Keywords Selberg zeta function · non-compact surface · configuration of zeros

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1 Introduction

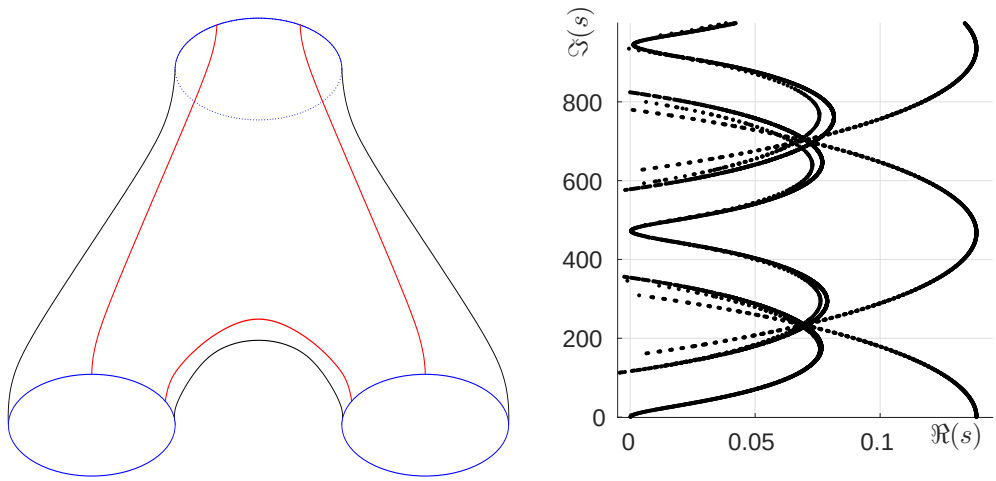
The Selberg zeta function Z_X associated to a compact Riemann surface X with negative Euler characteristic and without boundary is a well known and much studied complex function. It is a function of a single complex variable defined in terms of the lengths $\ell(\gamma)$ of the closed geodesics γ on the surface by analogy with the Riemann zeta function in number theory.

Definition 1.1. *We can formally define the Selberg zeta function by*

$$Z_X(s) = \prod_n \prod_\gamma (1 - e^{-(s+n)\ell(\gamma)}), \tag{1.1}$$

where the product is taken over all closed geodesics γ on X.

It was shown by Selberg in 1956 that for such surfaces the zeta function Z_X has an analytic extension to the entire complex plane and that the non-trivial zeros can be described in terms of the spectrum of the Laplace–Beltrami operator [16] and the Selberg trace formula.



(a) (b)

Figure 1: (a) A pair of pants X; (b) The zeros of the associated zeta function Z_X in a critical strip; observe the difference in scales on the axes. The individual zeros are so close in the plot that they almost appear to be six continuous curves.

Theorem 1.2 (Selberg)**.** *Let X be a compact Riemann surface with negative Euler characteristic and without boundary. Then the function Z_X has a simple zero at $s = 1$ and for any zero s in the critical strip $0 < \Re(s) < 1$ we have that either $s \in [0,1]$ is real, or $\Re(s) = \frac{1}{2}$.*

In the case of infinite area surfaces the situation is somewhat different, since the original trace formulae approach of Selberg no longer applies. However, it follows from the dynamical method of Ruelle that providing the surface X appears as a quotient space of $\mathbb{H}^2$ by a convex cocompact Fuchsian group then the analogous zeta function Z_X still has an analytic extension to the entire complex plane. Unfortunately, this construction provides little effective information on the location of the zeros, other than that the first zero at $s = \delta \in (0,1)$, the Hausdorff dimension of the limit set (of the Fuchsian group).

In interesting experimental work, Borthwick has studied the location of the zeros in classical examples of infinite area surfaces, such as pairs of pants [5]. In particular, for the case of symmetric pairs of pants he has observed a number of very interesting properties, some of which we will explain mathematically in this note. The starting point for our analysis has its roots in quite general results of Grothendieck and Ruelle. The study of the Selberg zeta function via the action of the Fuchsian group on the boundary $\partial\mathbb{H}^2$ dates back to work of the first author approximately twenty five years ago [14], which provided the formulae used by Borthwick. However, it is only with the advent of superior computational resources, and the ingenuity of those that employ them, that the striking features seen in Figure 1 have been revealed.

The plot in Figure 1 is fairly typical for the distribution of zeros in the critical strip for a sur-

face X with three boundary components each of which is sufficiently long. More precisely, a contemporary personal computer allows one to study surfaces with the length of boundary geodesics $\ell(\gamma) \geq 8$.

We denote by $\mathscr{S}_X$ the zero set of the function Z_X, i.e. $\mathscr{S}_X := \{s \in \mathbb{C} \mid Z_X(s) = 0\}$. Based on empirical evidence, one can make the following observations.

Qualitative Observations. Let X be a pair of pants with boundary geodesics of the length[1] $2b$. Then the elements of $\mathscr{S}_X$ follow a specific pattern, moreover,

O1 : The vertical spacing of zeros is approximately $\frac{\pi}{b}$.

O2 : The pattern of zeros appears to lie on four distinct curves, which seem to have a common intersection point at $\frac{\delta}{2} + i\frac{\pi}{2}e^b$.

O3 : The vertical apparent periodicity of the pattern of zeros is approximately πe^b.

However, we should stress that these qualitative observations are not completely rigorous for any fixed $b > 0$. For example the apparent periodicity in **O3** is not a true periodicity as we see from the behaviour of the zeros near the vertical line $\Re(s) = \delta$ in Figure 2. In fact, since the geodesic flow restricted to the non-wandering set is mixing, we know that there is only one zero with $\Re(s) = \delta$. Naud [11] showed an even stronger result: for any $\varepsilon > 0$ there is only finite number of zeros satisfying $\Re(s) > \frac{\delta}{2} + \varepsilon$. Nevertheless, partial numerical evidence illustrating the second observation **O2**, yet showing how it can be reconciled with the rigorous results, is given in Table 1. Namely, we analyze values of zeros closest to the right boundary $\Re(z) = \delta$ of the critical strip:

$$\mathscr{E} := \left\{ s_0 \in \mathscr{S}_X \mid \text{ for all } s \in \mathscr{S}_X \text{ such that } |s - s_0| < \frac{\pi}{2}e^b \text{ we have } \Re(s) < \Re(s_0) \right\}.$$

We see that for all $s \in \mathscr{E}$ satisfying $\Im(s) < 10^3$ there exist an $s' \in \mathscr{E}$ such that[2]

$$\left| s - s' + \pi e^b \right| \leq 3.$$

We will show that **O1–O3** hold asymptotically, in a suitable sense, for $b \to \infty$.

The key to understanding the apparent asymptotic behaviour as $b \to \infty$ is the approximation of Z_X by simpler functions with these precise properties, on suitable domains. Using estimates on the remainder of an absolutely convergent series formulation for Z_X, obtained in Theorem 3.5 we get an approximation to Z_X by determinants of certain explicit finite matrices, whose zero sets are easy to calculate analytically.

We will be studying the zeta function restricted to a compact subset of the critical strip, $0 \leq \Re(s) \leq \delta$. To formulate our results precisely we need the following notation.

[1] This normalization makes formulae in subsequent calculations shorter.

[2] We have, in fact, verified this for larger values of $\Im(s)$, but we omit the numerics here.

Endpoints for strings of zeros with $\Re(z) \approx \delta$		
$2b = 3\pi$, $z_0 = \delta = 0.146949$, $\pi \exp(b) = 349.715115$		
k	z_k	$z_k - z_{k-1}$
1	$0.146928 + i351.330281$	$2.093 \cdot 10^{-5} + i351.33028$
2	$0.146866 + i702.660561$	$6.278 \cdot 10^{-5} + i351.33028$
3	$0.146761 + i1053.990842$	$1.047 \cdot 10^{-4} + i351.33028$
$2b = 8$, $z_0 = \delta = 0.172887$, $\pi \exp(b) = 171.525147$		
k	z_k	$z_k - z_{k-1}$
1	$0.172785 + i172.781$	$-1.0196 \cdot 10^{-4} + i172.781053$
2	$0.172481 + i346.345$	$-3.0451 \cdot 10^{-4} + i173.564643$
3	$0.171974 + i519.126$	$-5.0674 \cdot 10^{-4} + i172.781053$
4	$0.171262 + i691.907$	$-7.1224 \cdot 10^{-4} + i172.781053$
5	$0.170343 + i865.472$	$-9.1839 \cdot 10^{-4} + i173.564643$
6	$0.169219 + i1038.253$	$-11.2437 \cdot 10^{-4} + i172.781054$

Table 1: Empirical estimates on zeros near the right boundary $\Re(z) = \delta$.

Notation 1.3. *We will be using the following.*

- *The compact part of the critical strip of the height T we denote by*

$$\mathscr{R}(T) = \{s \in \mathbb{C} \mid 0 \leq |\Re(s)| \leq \delta \text{ and } |\Im(s)| \leq T\}.$$

- *The compact part of the normalized critical strip of the height T we denote by*

$$\widehat{\mathscr{R}}(T) = \{s \in \mathbb{C} \mid 0 \leq |\Re(s)| \leq \ln 2 \text{ and } |\Im(s)| \leq T\}.$$

- *Let $\widehat{\mathscr{S}_X} := \{s = \sigma + it \mid Z_X\left(\frac{\sigma}{b} + ite^b\right) = 0\}$ be the zeros of Z_X mapped by $\sigma + it \mapsto b\sigma + ie^{-b}t$. (The surface X has an explicit dependence on b.)*

- *We will be using a set of regularly spaced points on the lines $\Re(s) = 0$ and $\Re(s) = \ln 2$ given by $\mathscr{L} = \ln 2 + i\pi\mathbb{Z} \cup i\pi\mathbb{Z}$.*

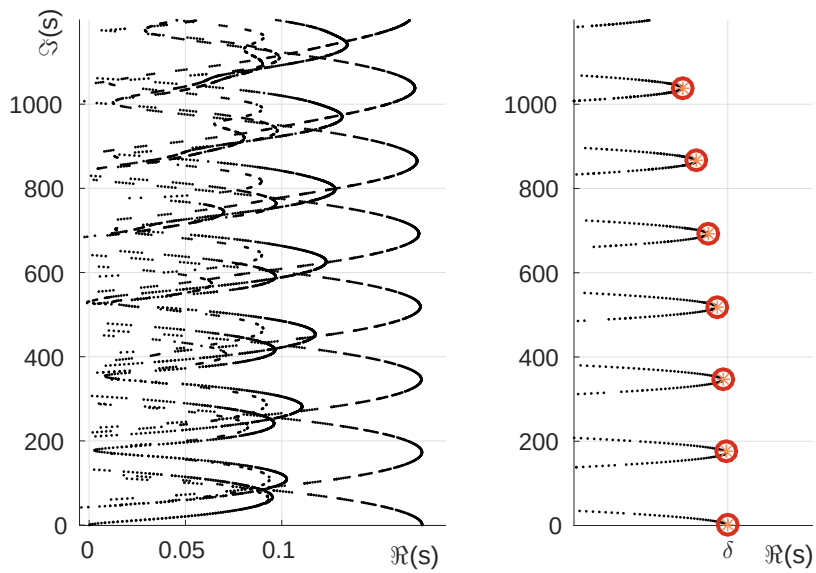


Figure 2: A more careful look at the apparent periodicity near $\Re(s) = \delta$, the case $\ell(\gamma_0) = 8$.

- *Let* $\mathscr{C} = \cup_{j=1}^{4} \mathscr{C}_j$, *where*

$$\mathscr{C}_1 = \left\{ \frac{1}{2} \ln |2 - 2\cos(t)| \mid t \in \mathbb{R} \right\};$$
$$\mathscr{C}_2 = \left\{ \frac{1}{2} \ln |2 + 2\cos(t)| \mid t \in \mathbb{R} \right\};$$
$$\mathscr{C}_3 = \left\{ \frac{1}{2} \ln \left| 1 - \frac{1}{2} e^{2it} - \frac{1}{2} e^{it} \sqrt{4 - 3e^{2it}} \right| \mid t \in \mathbb{R} \right\};$$
$$\mathscr{C}_4 = \left\{ \frac{1}{2} \ln \left| 1 - \frac{1}{2} e^{2it} + \frac{1}{2} e^{it} \sqrt{4 - 3e^{2it}} \right| \mid t \in \mathbb{R} \right\}.$$

The following result provides a rigorous explanation of the Observations **O1** and **O2** for large values of b.

Theorem 1.4. *The zero sets* $\mathscr{S}_X$ *and* $\widehat{\mathscr{S}_X}$ *satisfy the following.*

1. *There exist a subsequence* $s_n \in \mathscr{S}_X$ *and* $\varkappa > 1$ *such that*

$$\sup_{s_n \in \mathscr{S}_X \cap \mathscr{R}(e^{\varkappa b})} d(bs_n, \mathscr{L}) = O\Big(\frac{1}{\sqrt{b}}\Big),$$

2. *The affinely transformed zeros* $\widehat{\mathscr{S}_X}$ *are close to* $\mathscr{C}$, *more precisely, for any* $\varkappa \geq 1$ *we have*

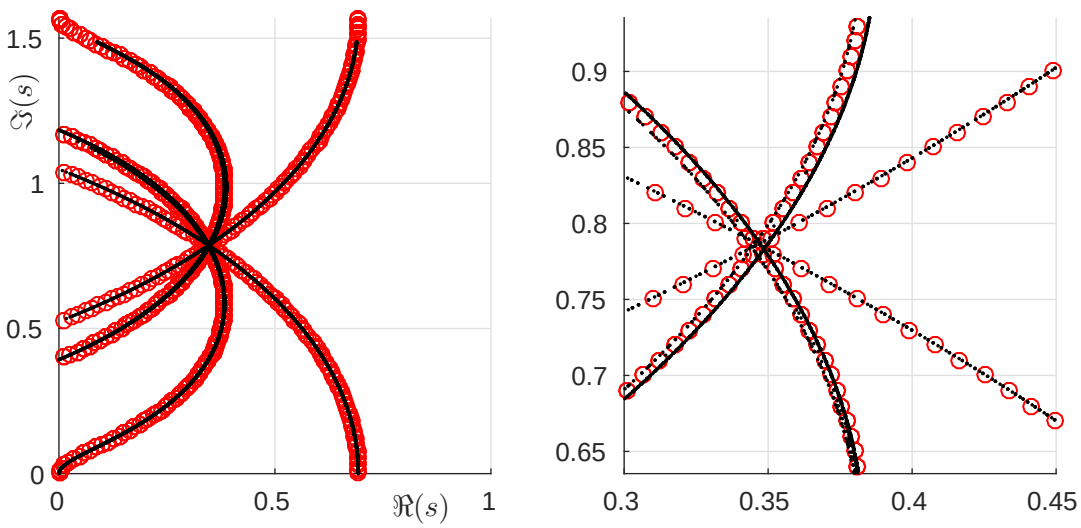
$$\sup_{s \in \widehat{\mathscr{S}_X} \cap \widehat{\mathscr{R}}(e^{\varkappa b})} d(s, \mathscr{C}) = O\Big(\frac{1}{\sqrt{b}}\Big),$$

Observation **O3** corresponds to the following remark.

Remark 1.5. The four curves $\mathscr{C}_1, \cdots, \mathscr{C}_4$ have common points, more specifically

$$\bigcap_{j=1}^{4} \mathscr{C}_j = \left\{ \frac{\ln 2}{2} + i\pi\left(\frac{1}{2}+k\right) \right\}, \quad k \in \mathbb{Z}.$$

The set $\mathscr{C}$ is invariant with respect to vertical translation by $\sigma + it \to \sigma + i(t+\pi)$.



(a) (b)

Figure 3: (a) The dashed lines correspond to the curves $\mathscr{C}_i$ and the circles correspond to the points of $\widehat{S}_X$ for $b = 5$; (b) A zoomed version in a neighbourhood of $\left(\frac{\ln 2}{2}, \frac{\pi}{4}\right)$.

A version of Theorem 1.4 part (1) for a fixed domain $\mathscr{R}(k_0)$ was also observed by Weich [17].

An element in the approach to this program touches on a more obvious question of how accurately we can approximate Z_X using finitely many closed geodesics. In particular, we need to estimate the approximation error in terms of the lengths of the boundary geodesic.

A similar analysis can be carried out for a punctured torus and for less symmetric surfaces. However, for simplicity of exposition we will concentrate on this example and leave the analysis in these other cases to another occasion.

This project has had a long gestation period, having begun after the first author heard the original empirical results of David Borthwick presented at a conference on Quantum Chaos in Roscoff in June of 2013. We are grateful to him for sharing his original Matlab code with us. We are grateful to F. Bykov for writing a new program. A preliminary announcement of these results was made by the first author at the conference "Spectral problems for hyperbolic dynamical systems" held in Bordeaux in May of 2014.

2 The zeta function and closed geodesics

In this section we make some preliminary estimates on the lengths of geodesics used in its definition. Let

$$\mathbb{D}^2 = \{z = x + iy \colon |z| < 1\}$$

denote the Poincaré disc equipped with the usual Poincaré metric $ds^2 = \frac{dx^2+dy^2}{(1-x^2-y^2)^2}$. This is a manifold with constant negative curvature -1. We can fix a value $0 < \alpha \leq \frac{\pi}{3}$ and consider a Fuchsian group $\Gamma_\alpha = \langle R_1, R_2, R_3 \rangle$ generated by reflections R_1, R_2, R_3 with respect to three disjoint equidistant geodesics $\beta_1, \beta_2, \beta_3$, with end points $e^{(\frac{2\pi}{3} j \pm \alpha) i} \in \partial\mathbb{D}^2$, $j = 1, 2, 3$, respectively. We can

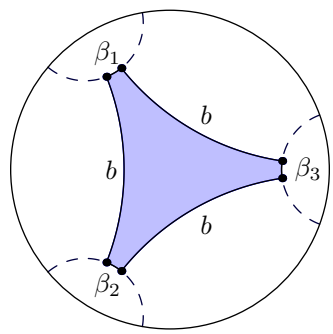


Figure 4: Three circles of reflection in Poincaré disk with pairwise distance b.

write $X = \mathbb{D}^2/\Gamma$, and assume that the surface X possess a metric of constant negative curvature -1, carried over from the Poincaré disk. In particular, X is a pair of pants i.e. topologically a sphere minus three disjoint disks. The length of the boundary geodesics of X we denote[3] by $2b$. In the notation introduced above, let us associate a closed geodesic γ to each conjugacy class in Γ. We shall denote by $\ell(\gamma)$ its length in the hyperbolic metric of the surface, and by $\omega(\gamma) \in 2\mathbb{N}$ its word length, i.e. the period of the cutting sequence, or the number of times it crosses the boundary of the fundamental domain of Γ.

Our starting point is the following important result of D. Ruelle from 1976.

Theorem 2.1 (after Ruelle)**.** *Let $\delta > 0$ be the largest real zero for Z_X. In the notation introduced above, the infinite product* (1.1) *converges to a non-zero analytic function for $\Re(s) > \delta$ and extends as an analytic function to $\mathbb{C}$.*

Proof. The first part follows from more general results on Axiom A flows which we can apply to the geodesic flow restricted to the recurrent part [13]. The latter part follows from applying ideas from the work of Ruelle [15], see also [13] for more details. □

[3] Later we will establish a connection between the angle α and the distance b between geodesics of reflection.

Our strategy for understanding the behaviour of the zeta function is based on the idea of quantitative approximation by functions of the following type.

Definition 2.2. *We call a complex function f a* trigonometric polynomial *if it may be written in the general form*

$$f(s): = c_1 e^{-s\lambda_1} + \cdots + c_n e^{-s\lambda_n}, \tag{2.1}$$

where $c_1, \cdots, c_n \in \mathbb{R}$ and $\lambda_1, \cdots, \lambda_n > 0$.

Trigonometric polynomials were famously studied by Lagrange in his work on planetary motions (cf. [9]) and in the work of Bohr on almost periodic functions [4].

We are going to approximate the zeta function Z_X by trigonometric polynomials Z_n, which appear as determinants of simple matrices C_X. The curves $\mathscr{C}_j$, $j = 1, \ldots, 4$ appear afterwards as submanifolds containing the zero set of the determinants.

To get an insight into how the matrices are constructed, we consider a related complex function $1/\zeta$ defined by

$$\frac{1}{\zeta(s)} = \frac{Z_X(s)}{Z_X(s+1)} = \prod_{\gamma}(1 - e^{-s\ell(\gamma)}), \tag{2.2}$$

which shares the same zeros as $Z_X(s)$ in the strip $0 < Re(s) < \delta$, since Z_X is real analytic for $\Re(s) > 1$. The function ζ is the exact form of the zeta function studied by Ruelle. Intuitively, an approximation to the function $1/\zeta$ could be obtained by replacing the lengths $\ell(\gamma)$ of closed geodesics with some approximate values. In the following paragraph we present two possible approximations to $\ell(\gamma)$.

2.1 Approximating the lengths of closed geodesics

In the present context the coefficients λ_j, $j = 1, \ldots, n$ in (2.1) can be expressed in terms of closed geodesics, which in turn can be written in terms of the boundary length $2b$. In particular, we have the following result.

Proposition 2.3. *Let the length of a boundary geodesic on X be $2b$. Then*

1. *A coefficient λ_n is given by finite sums of lengths $\ell(\gamma)$ of closed geodesics γ of word length $2 \leq \omega(\gamma) \leq 2k$ for $1 \leq k \leq n$;*

2. *The hyperbolic length $\ell(\gamma)$ of a closed geodesic γ and the word length $\omega(\gamma)$ are related by*

$$2\cosh(\ell(\gamma)) = \sinh^{\omega(\gamma)}(b) P_\gamma\left(\frac{\cosh^2(b)}{\sinh^2(b)}\right) \tag{2.3}$$

for some polynomial $P_\gamma \in \mathbb{Z}[x]$ of degree $\frac{\omega(\gamma)}{2}$.

Proof. The values for λ_n in part (1) will be a consequence of the expansion of the presentation for Z_X as a series in (3.1). Part (2) follows from a straightforward expansion (see Lemma 4.4 in §4.2). □

The large symmetry group of the symmetric pair of pants means that many of the polynomials P_γ are the same. This explains why in numerical computation there appears to be a lot of multiplicity in the coefficients λ_j.

2.1.1 First order approximation

We may first consider the simplest approximation $\ell_0(\gamma) = 2nb$ to the length $\ell(\gamma)$ of closed geodesics of the word length $\omega(\gamma) = 2n$. This has a simple geometric interpretation. To every closed geodesic we associate a periodic cutting sequence, which in turn uniquely corresponds to a periodic orbit of the subshift of finite type $\sigma : \Sigma_A \to \Sigma_A$ on the words of alphabet of three symbols corresponding to three reflections. The transtition matrix for the subshift is given by the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

i.e. $\Sigma_A = \{\underline{x} = (x_n)_{n=0}^\infty \in \prod_{n=0}^\infty \{1,2,3\} \colon A(x_n, x_{n+1}) = 1, \text{ for } n \geq 1\}$ and $(\sigma \underline{x})_n = x_{n+1}$ for $n \geq 1$. Closed geodesics correspond to periodic points for σ, and $\frac{1}{2n}\operatorname{tr}(A^{2n})$ counts the number of (non-prime) closed geodesics corresponding to $2n$-reflections. Using the standard formal matrix identity[4]

$$\det(I - zA^2) = \exp\left(-\sum_{n=1}^\infty \frac{z^n}{n}\operatorname{tr}(A^{2n})\right)$$

we can define ζ_0 by[5]

$$\frac{1}{\zeta_0(s)} \colon = \prod_\gamma \left(1 - e^{-s\ell_0(\gamma)}\right) = \det(I - e^{-sb}A^2) = 1 - 6e^{-2bs} + 9e^{-4bs} - 4e^{-6bs},$$

where the product is taken over prime closed geodesics [13].

Later we will see that the function $1/\zeta_0$ is a good approximation to the function Z_X in a small neighbourhood of the real line. We may exploit this idea, for instance, to deduce that the function

[4] The equality holds true when the power series on the right hand side converges. In this case, the resulting analytic function has an analytic extension to $\mathbb{C}$. For the values of z corresponding to the divergent formal series, the equality between the analytic extension and the determinant is understood.

[5] As above, the infinite product converges in the right half plane $\Re(s) > 0$ and has an analytic extension to the left half-plane.

Z_X has a double zero at $s = 0$. Namely, we may rewrite the complex function

$$\frac{1}{\zeta_0(s)} = \det(I - e^{-2bs}A^2) = (1 - 4e^{-bs})(1 - e^{-bs})^2$$

and observe that $s = 0$ is a double zero of $\frac{1}{\zeta_0(s)}$.

2.1.2 Second order approximation

Now we would like to consider the next level of approximation to the lengths of closed geodesics, which turns out to be of the form $\ell_1(\gamma) = 2nb + c(\gamma)e^{-b}$, where the coefficient $c(\gamma) \in \mathbb{N}$ is defined using the cutting sequence.

We observe the following simple fact:

Lemma 2.4. *Consider a regular hyperbolic hexagon whose even sides are of length $b > 1$ and whose odd sides are of length ε_b then*

$$\varepsilon_b = 2e^{-b/2} + e^{-b} + O(e^{-3b/2}) \quad \text{as } b \to +\infty.$$

Proof. We recall that (cf. [3], Theorem 7.19.2)

$$\cosh \varepsilon_b (\sinh b)^2 = \cosh b + (\cosh b)^2.$$

The result follows by expanding the both sides in e^{-b} as $b \to +\infty$ and comparing the expansions. □

Now we are ready to describe the coefficient $c(\gamma)$.

Lemma 2.5. *Assume that γ corresponds to a cutting sequence of period $2n$ specified by*

$$\cdots j_1 j_2 j_3 j_4 \cdots j_{2n-1} j_{2n} \cdots,$$

where $j_k \neq j_{k+1}$ for $1 \leq k \leq 2n$ and $j_{2n} \neq j_1$. Then

$$c(\gamma) = \#\{1 \leq k \leq 2n \colon j_k \neq j_{k+2} \mod 2n\}$$

Proof. We will consider the geodesic segment $\gamma_{j_k j_{k+1} j_{k+2}} \subset \gamma$ corresponding to the subsequence of the cutting sequence $\cdots j_k j_{k+1} j_{k+2} \cdots$ and estimate its length. We will consider two cases separately: $j_k = j_{k+2}$ and $j_k \neq j_{k+2}$.

Let us first consider the case when $j_k = j_{k+2}$, see geodesic segment γ_{131} in Figure 5 for example. We can approximate this geodesic segment by the geodesic segment consisting of the sides of the two hexagons (and thus necessarily of length $2b$) up to an error $O(e^{-2b})$ (independently of j_{k-1} and j_{k+3}).

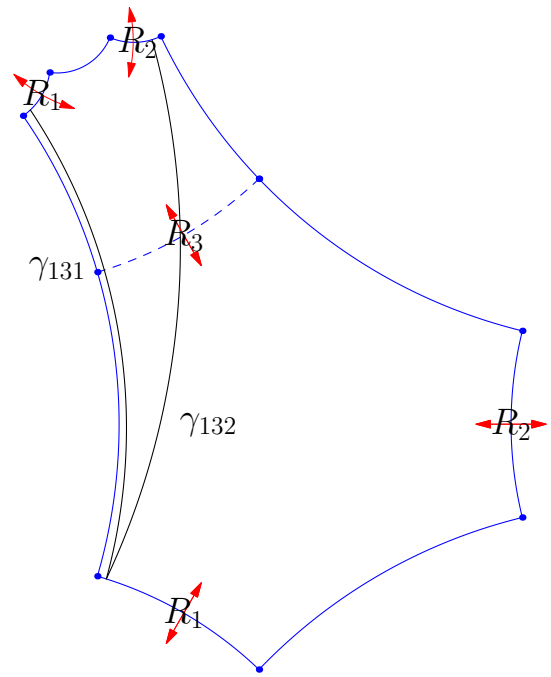


Figure 5: A geodesic segment γ_{131}, running along the side of two hexagons when $R_{j_{k+2}} = R_{j_k}$; and a geodesic segment γ_{132}, traversing two hexagons when $R_{j_{k+2}} \neq R_{j_k}$.

Let us now consider the case $j_k \neq j_{k+2}$, see the geodesic segment γ_{132} in Figure 5 for example. We may approximate the length of this segment by double the length of the hypothenuse of the hyperbolic right-angled triangle one leg of which is equal to the edge of the fundamental domain of the length b and another leg equal to a half of the edge of fundamental domain of the length ε_b. We can use the hyperbolic cosine law to write $\cosh\left(\frac{1}{2}\ell(\gamma_{j_k j_{k+1} j_{k+2}})\right) = \cosh\left(\frac{\varepsilon_b}{2}\right)\cosh(b)$. Therefore we can write

$$\begin{aligned}
\ell(\gamma_{j_k j_{k+1} j_{k+2}}) &= 2\ln\left(\cosh b \cosh\frac{\varepsilon_b}{2} + \sqrt{\cosh^2 b \cosh^2\frac{\varepsilon_b}{2} - 1}\right) = \\
&= 2\ln\left(2\cosh b\cosh\frac{\varepsilon_b}{2} + \frac{1}{2}e^{-b} + O(e^{-2b})\right) = \\
&= 2\ln(2\cosh b) + 2\ln\left(\cosh\frac{\varepsilon_b}{2}\right) + 2\ln\left(1 + \frac{1}{2}e^{-2b} + O(e^{-3b})\right)
\end{aligned} \tag{2.4}$$

We have asymptotic expansions

$$\ln(2\cosh b) = b + \ln(1 + e^{-2b}) = b + e^{-2b} + O(e^{-4b}) \tag{2.5}$$

and

$$\ln\left(\cosh\frac{\varepsilon_b}{2}\right) = \ln\left(\frac{e^{e^{-b/2}+O(e^{-b})} + e^{-e^{-b/2}+O(e^{-b})}}{2}\right) = \ln\left(1 + \frac{e^{-b}}{2} + O(e^{-2b})\right). \tag{2.6}$$

Substituting (2.5) and (2.6) into (2.4), we conclude $\ell(\gamma_{j_k j_{k+1} j_{k+2}}) = 2b + e^{-b} + O(e^{-2b})$, and the Lemma follows. Again, a routine calculation shows that the choice of j_{k-1} and j_{k+3} will change $\ell(\gamma_{j_k j_{k+1} j_{k+2}})$ by not more than $O(e^{-2b})$. □

We may associate to a closed geodesic a periodic orbit of a subshift of finite type on the space of sequences of six symbols $R_{j_1} R_{j_2}$, where $1 \leq j_1, j_2 \leq 3$ and $j_1 \neq j_2$. The transition matrix is given by

$$\widehat{B}^{j_3 j_4}_{j_1 j_2} = \begin{cases} 0 & \text{if } j_2 = j_3, \\ 1 & \text{otherwise.} \end{cases}$$

As one would expect, we have that $\frac{1}{2n} \operatorname{tr} \widehat{B}^n$ is equal to the number of closed geodesics of word length $\omega(\gamma) = 2n$.

In order to keep the additional information of the second order approximation to the length of closed geodesics, consider the matrix

$$B^{j_3 j_4}_{j_1 j_2}(z) = \begin{cases} 0 & \text{if } j_2 = j_3; \\ 1 & \text{if } j_1 = j_3 \text{ and } j_2 = j_4; \\ z^2 & \text{if } j_1 \neq j_3 \text{ and } j_2 \neq j_4; \\ z & \text{otherwise.} \end{cases}$$

With a suitable labeling, we obtain

$$B(z) = \begin{pmatrix} 1 & z & 0 & 0 & z^2 & z \\ z & 1 & z^2 & z & 0 & 0 \\ 0 & 0 & 1 & z & z & z^2 \\ z^2 & z & z & 1 & 0 & 0 \\ 0 & 0 & z & z^2 & 1 & z \\ z & z^2 & 0 & 0 & z & 1 \end{pmatrix} \tag{2.7}$$

Lemma 2.6. *The coefficients of the polynomial*

$$\frac{1}{2n} \operatorname{tr}(B^n(z)) = d_{2n} z^{2n} + d_{2n-1} z^{2n-1} + \cdots + d_1 z + d_0$$

are given by

$$d_k = \#\{\gamma \colon \omega(\gamma) = 2n \text{ and } c(\gamma) = k\}.$$

Proof. Follows immediately from the definition. □

Now we return to the complex function $\frac{1}{\zeta}$ given by (2.2) and consider the first approximation to the length of the closed geodesics:

$$\ell_1(\gamma) = 2nb + c(\gamma)e^{-b}.$$

Now we define

$$\frac{1}{\zeta_1(s)} := \prod_{\gamma}(1 - e^{-sl_1(\gamma)}) = \det(I - e^{-2sb}B(\exp(se^{-b})))$$

In the next section we will show that there exists a trigonometric polynomial Z_n which gives a good approximation to both the function Z_X and the determinant $\det(I - e^{-2sb}B(\exp(se^{-b})))$ on a large rectangle $0 < \Re(s) < \delta$ and $0 < \Im(s) < \exp(b)$. Combining this information with some technical estimates, we will deduce location of zeros of Z_X from the zeros of the determinant.

3 Approximation results

The Selberg zeta function, initially defined in terms of lengths of all closed geodesics, is not an object which can be easily computed numerically: one has to be satisfied with a finite (although large) set of geodesics.

Lemma 3.1. *There are exactly $p_n = 4^n + 2$ closed geodesics of the word length $\omega(\gamma) = 2n$.*

Proof. Follows by induction in n from the recurrent relation $p_n = 3 \cdot 4^{n-1} + p_{n-1}$, $p_1 = 6$. □

Naturally, a fundamental question arises: to what extend does an approximating function Z_n, which uses geodesics of word length at most $\omega(\gamma) \leq 2n$ actually reflect the features of Z_X?

In this section we construct trigonometric polynomials Z_n and quantify the error carefully.

3.1 The expansion

We can begin by considering the function in two complex variables

$$Z_X(s,z) = \prod_n \prod_{\gamma} \left(1 - z^{\omega(\gamma)} e^{-(s+n)\ell(\gamma)}\right), \tag{3.1}$$

which converges for $|z|$ sufficiently small and $\Re(s)$ sufficiently large. We can follow Ruelle [15] in re-writing the infinite product as a series

$$Z_X(s,z) = 1 + \sum_{n=1}^{\infty} a_n(s) z^n \tag{3.2}$$

by taking the Taylor expansion in z about 0. It is then easy to see that $a_n(s)$ is defined in terms of finitely many closed geodesics with word lengths at most n. In fact this series converges to a bianalytic function for both $z, s \in \mathbb{C}$:

Theorem 3.2 (Ruelle [15]). *Using the notation introduced above, there exists $C = C(s) > 0$ and $0 < \theta < 1$ such that $|a_n| \leq C\theta^{n^2}$ and thus the series* (3.2) *converges. In particular, we can deduce that $Z_X(z,s)$ is analytic in both variables.*

We next make an easy observation.

Lemma 3.3. *The odd coefficients vanish i.e., $a_1 = a_3 = \cdots = 0$.*

Proof. The fundamental domain for X consists of two hexagons glued across three of the alternating six edges. Since any closed geodesic γ crosses the fundamental domain an even number of times it follows that $\omega(\gamma) \in 2\mathbb{N}$ in the infinite product (3.1) and thus only the even terms $a_2, a_4, \cdots$ can be non-zero in (3.2). □

We can formally rewrite the zeta function (3.1) as

$$Z_X(s) = \exp\left(-\sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{\omega(\gamma)=m} \frac{e^{-s\ell(\gamma)}}{1-e^{-\ell(\gamma)}}\right) = \exp\left(-\sum_{m=1}^{\infty} b_m(s) z^m\right) \tag{3.3}$$

where

$$b_m(s) = \begin{cases} \frac{1}{m} \sum\limits_{\omega(\gamma)=m} \frac{e^{-s\ell(\gamma)}}{1-e^{-\ell(\gamma)}} & \text{if } m \text{ is even;} \\ 0 & \text{if } m \text{ is odd;} \end{cases} \tag{3.4}$$

and then expand the exponential as a power series and obtain coefficients a_n comparing (3.2) with (3.3), since these series converge provided $\Re(s) > 1$. In particular, we can easily check that the first three non-zero terms are:

$$\begin{aligned} a_2(s) &= -b_2(s); \\ a_4(s) &= -b_4(s) + \frac{b_2(s)^2}{2}; \\ a_6(s) &= -b_6(s) + b_2(s)b_4(s) - \frac{b_2^2(s)}{3}; \end{aligned}$$

and in general,

$$a_n(s) = -\frac{1}{n} \sum_{j=0}^{n-2} a_j(s) b_{n-j}(s).$$

Combining the latter with (3.4), we deduce that each of the coefficients $a_n(s)$ is a complex trigonometric polynomial.

3.2 Tail estimates

We begin by defining a sequence of functions Z_n, approximating Z_X.

Definition 3.4. *We define the complex trigonometric polynomial* $Z_n(s) = \sum_{k=0}^{n} a_k(s)$*, where the terms* $a_k(s)$ *correspond to those in* (3.2)

Our main approximation result is the following.

Theorem 3.5. *Let X be a symmetric pair of pants with boundary geodesics of the length $2b$. Then we may approximate Z_X on the domain $\mathscr{R}(T)$ by the complex trigonometric polynomial Z_n so that* $\sup_{\mathscr{R}(T)} |Z_X - Z_{n-2}| \leq \eta(b,n,T)$ *where $T = T(b) = e^{\varkappa b}$ for some constant $\varkappa > 1$ independent of b and n, such that*

1. *for any $n \geq 14$ we have $\eta(b,n,T(b)) = O\left(\frac{1}{\sqrt{b}}\right)$ as $b \to \infty$*
2. *for any $b \geq 20$ we have $\eta(b,n,T(b)) = O\left(e^{-bk_1 n^2}\right)$ as $n \to \infty$.i*

for some $k_1 > 0$ which is independent on b and n.

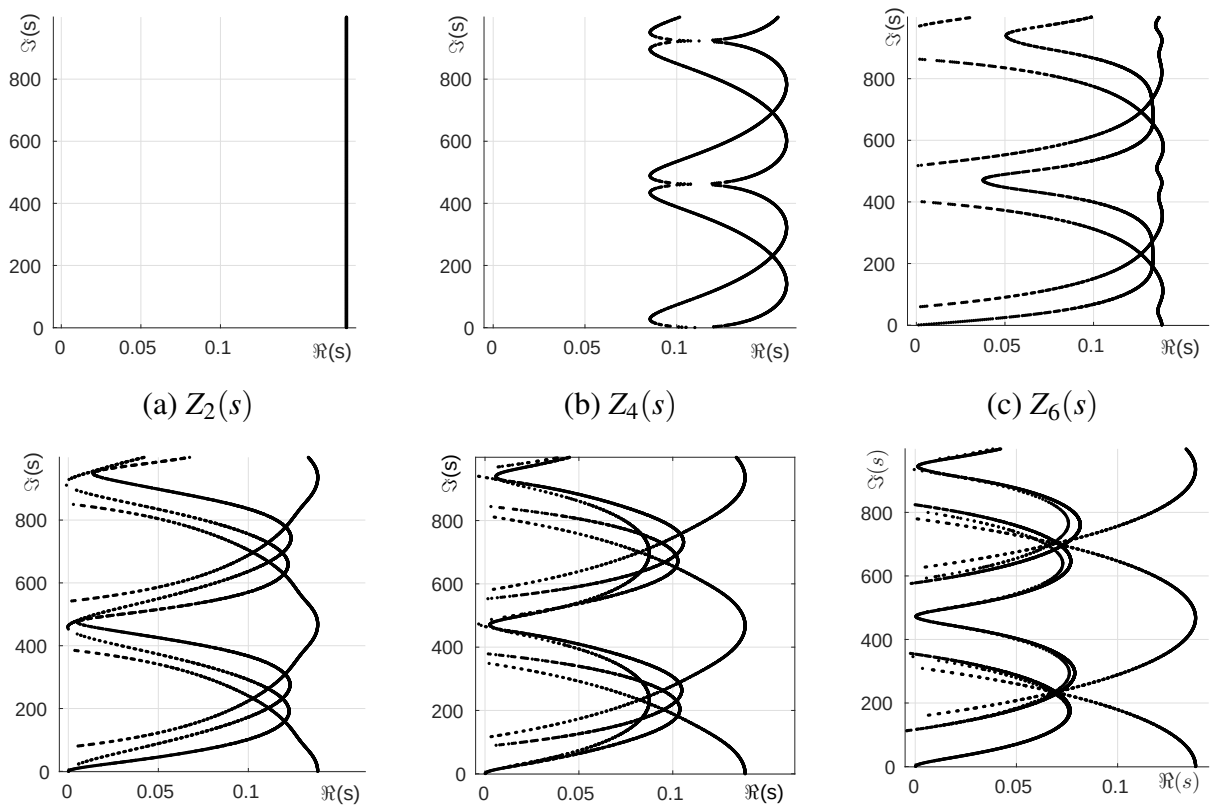
For a fixed b this theorem estimates the number of terms a_n needed to uniformly approximate Z_X to any given error, on an exponentially growing domain. On the other hand for a given n this Theorem estimates the difference between Z_X and Z_n as $b \to \infty$ on an exponentially growing domain.

Remark 3.6. The constant k_1 in Theorem 3.5 should satisfy inequality $0 < k_1 < \frac{2-\varkappa}{60}$, although this bound is not sharp. A sharp bound can be obtained using the same argument, but the formulae will be more complicated.

Example 3.7. *We can fix a surface by choosing the length of boundary geodesics $2b$ and plot the zeros for the approximating trigonometric polynomials $Z_{2n} = 1 + a_2 + \cdots + a_{2n}$ for $n = 1, 2, \cdots, 6$. For instance, in Figure 6 zeros of polynomials approximating Z_X with $b = 5$ are shown.*

In particular, this more refined result shows that numerical results for Z_n can be used to justify Observation **O2** in a domain $\mathscr{R}(e^{k_1 b})$. Since in practice n is bounded above by computational considerations, we may assume that it is fixed. Even with a modern computer, one will not be able to consider $n > 16$ in a reasonable time[6]. Moreover, in practical applications b cannot be chosen too large either due to computer restrictions because of accumulation of errors while dealing with

[6]The most time-consuming part is the Newton method used to locate a zero starting from a point of a lattice on $\mathscr{R}(T)$. The time taken by this calculation grows exponentially with n. The total number of the searches is proportional to the area of $\mathscr{R}(T)$, which is proportional to T.



(a) $Z_2(s)$ (b) $Z_4(s)$ (c) $Z_6(s)$

(d) $Z_8(s)$ (e) $Z_{10}(s)$ (f) $Z_{12}(s)$

Figure 6: Plots of the zeros $Z_{2n}(s)$ for $n = 1,2,\ldots,6$ when $b = 5$.

small numbers. The coefficients b_m defined by (3.4) involve a sum of $4^m + 2$ terms of the order $\exp(-2smb)$ with $0 < \Re(s) < 0.25$, say. The bound $4^m + 2$ is equal to number of closed geodesics of the word length $2m$, see Lemma 3.1.

4 Nuclear operators and analytic functions

The proof of Theorem 3.5 is based on the original approach in [15] (and the interpretation in [14]). We begin by recalling some abstract results, essentially due to Grothendieck, on nuclear operators. We then complete the section by relating the length of the boundary geodesics to the contraction on the boundary corresponding to reflections, generating the group Γ_α.

4.1 Nuclear Operators

The convergence of the series (3.2) in Theorem 3.2 will follow from estimates of Ruelle [15], after Grothendieck [6]. We summarize below the general theory.

Let $\mathscr{B}$ be a Banach space.

Definition 4.1. *We say that a linear operator $T : \mathscr{B} \to \mathscr{B}$ is nuclear (with exponentially decreasing coefficients) if there exist for each $n \geq 1$*

1. *$w_n \in \mathscr{B}$, with $\|w_n\|_{\mathscr{B}} = 1$;*
2. *$\nu_n \in \mathscr{B}^*$, with $\|\nu_n\|_{\mathscr{B}^*} = 1$;*
3. *$\lambda_n \in \mathbb{R}$, with $0 < \lambda < 1$, $C > 0$ satisfying $|\lambda_n| \leq C\lambda^n$*

such that

$$Tf = \sum_{n=1}^{\infty} \lambda_n w_n \nu_n(f). \tag{4.1}$$

Lemma 4.2 (after Grothendieck)**.** *A nuclear operator on a Banach space is trace class, and we can write*

$$\det(I - zT) = \exp\Big(-\sum_{n=1}^{\infty} \frac{z^n}{n} \operatorname{tr} T^n\Big).$$

We may also expand

$$\det(I - zT) = 1 + \sum_{n=1}^{\infty} a_n z^n,$$

where

$$a_n = \sum_{j_1 < \cdots < j_n} \det\Big([\nu_{j_k}(w_{j_l})]_{k,l=1}^n\Big) \lambda_{j_1} \ldots \lambda_{j_n}. \tag{4.2}$$

Applying estimates of Grothendieck and Ruelle, we obtain an explicit bound.

$$|a_n| \leq C^n n^{n/2} \lambda^{n(n+1)/2}, \tag{4.3}$$

where $n^{n/2}$ bounds the supremum norm of the matrix[7] $[\nu_{j_k}(w_{j_l})]_{k,l=1}^n$.

[7] We follow Ruelle in including the term $n^{n/2}$ although this can be improved upon by looking at Hilbert spaces of analytic functions. For instance, O. Bandtlow and O. Jenkinson [2] have shown that we can supress $n^{n/2}$ by working with Hardy spaces, but then λ would be different, too.

4.2 Constructing the Banach space

Given four points $z_1 < w_1 < w_2 < z_2$ on the boundary $\partial\mathbb{H}^2$, we define the cross ratio by

$$[z_1, w_1, w_2, z_2] = \frac{(z_1 - w_2)}{(z_1 - w_1)} \frac{(w_1 - z_2)}{(w_2 - z_2)}.$$

We recall the following classical formula (cf. [3] §7.23).

Lemma 4.3. *Let L_1, L_2 be two disjoint geodesics in $\mathbb{H}^2$ with end points z_1, z_2 and w_1, w_2. The distance $d(L_1, L_2)$ between L_1 and L_2 satisfies $[z_1, w_1, w_2, z_2] = \tanh^2(d(L_1, L_2)/2)$.*

By assumption, the group $\Gamma = \langle R_1, R_2, R_3 \rangle$ is generated by reflections with respect to three disjoint geodesics, which we denote by β_1, β_2, and β_3, respectively. Without loss of generality, we may assume that the geodesic β_j has end points $e^{\left(\frac{2\pi j}{3} \pm \theta\right) i}$, for $j = 1, 2, 3$ and a small real number θ. More precisely, by straightforward calculation using Lemma 4.3 we get

Lemma 4.4. *Let β_1 and β_2 be two disjoint geodesics in $\mathbb{H}^2$ with end points $e^{\left(\pm\frac{2\pi}{3} \pm \theta\right) i}$ and assume that the distance $d(\beta_1, \beta_2) = b \gg 1$. Then $\sin\theta = \frac{1}{2\cosh b}$.*

Remark 4.5. In notations and under the hypothesis of the last lemma, we have an asymptotic relation

$$\theta = \frac{1}{2} e^{-b} \left(1 + e^{-2b} + o(e^{-3b})\right) \text{ as } b \to \infty.$$

To define the Banach space, we fix a small $\varphi < \theta$ and introduce three additional geodesics υ_j with end points $e^{\left(\frac{2\pi j}{3} \pm \varphi\right) i}$, $j = 1, 2, 3$. We may consider the disk $\mathbb{D}$ as a subset of $\mathbb{C}$ and formally extend the geodesics υ_j to circles $\overline{\upsilon}_j \subset \mathbb{C}$. Furthermore, let $\{U_j\}_{j=1}^3$ be three compact disks in $\mathbb{C}$ such that $\partial U_j = \overline{\upsilon}_j$ cf. Figure 7.

The Banach space of bounded analytic functions $f : \sqcup_{j=1}^3 U_j \to \mathbb{C}$ on the union $\sqcup_{i=1}^3 U_i$ we denote by $\mathscr{B}$. We supply it with the supremum norm $\|f\|_{\mathscr{B}} := \|f\|_\infty$.

4.3 Transfer operators

We can now define transfer operators $\mathscr{L}_s$, acting on the Banach space $\mathscr{B}$ of bounded analytic functions on $\sqcup_{j=1}^3 U_j$.

Definition 4.6. *For each $s \in \mathbb{C}$ we can define*

$$(\mathscr{L}_s f)(z) = \sum_{k=1}^{3} \chi_{U_k}(z) \sum_{j \neq k} (R_j'(z))^s f(R_j(z)), \tag{4.4}$$

where χ_{U_k} is the indicator function of U_k.

We can apply the general theory of nuclear operators to the transfer operators by virtue of the following

Lemma 4.7. *The operator $\mathscr{L}_s \colon \mathscr{B} \to \mathscr{B}$ is nuclear.*

Proof. We observe that the operators $f \mapsto f \circ R_j$ are nuclear and $R_j'(z)^s$ are analytic [15]. Thus $\mathscr{L}_s$ are nuclear, too. □

In particular, applying Lemma 4.2 to $T = \mathscr{L}_s$ allows us to recover Theorem 3.2, see [15] for details.

5 Proof of the approximation result

In this section we give a proof of Theorem 3.5. We will need the following simple technical estimate.

Lemma 5.1. *Let x_n be a sequence of real numbers satisfying $|x_n| \leq \exp\big(pn - qn^2\big)$ for some constants $p, q > 0$. Then for any $n > 1$ we have that*

$$\sum_{k=n}^{\infty} |x_k| \leq \frac{\sqrt{\pi}}{2\sqrt{q}} \exp\Big(\frac{p^2}{4q}\Big) \exp\Big(-q\Big(n - \frac{p}{2q}\Big)^2\Big)$$

Proof. The result follows by straightforward calculation using the classical bound fot the error function $\int_n^\infty \exp(-t^2)dt \leq \frac{\sqrt{\pi}}{2}\exp(-n^2)$. □

We now turn to the proof of Theorem 3.5. This follows the same lines as [8]. The central new idea is that the disks U_1, U_2 and U_3 used to define $\mathscr{B}$ are allowed to depend on b.

Proof. Without loss of generality we may assume that the geodesic β_j has end points $e^{(\frac{2\pi}{3}j \pm \theta)i} \in \partial\mathbb{D}$ for $j = 1, 2, 3$. We choose three additional geodesics υ_j with end points $e^{(\frac{2\pi}{3}j \pm \varphi)i} \in \partial\mathbb{D}$ for some $0 < \varphi < \theta$, that we will specify later, see Figure 7 for details. We may consider $\mathbb{D}$ as a subset of $\mathbb{C}$ with usual Euclidean metric, and then complete υ_j to full Euclidean circles $\overline{\upsilon}_j \in \mathbb{C}$. We define $U_j \in \mathbb{C}$ to be compact disks with $\partial U_j = \overline{\upsilon}_j$.

It turns out that the calculations are much easier in the upper half model of the hyperbolic space $\mathbb{H}^2$. We choose the map $S(z) = i\frac{1-z}{1+z}$ to change the coordinates. Then the geodesic β_j has end points $\frac{\sin(\frac{2\pi j}{3} \pm \theta)}{1+\cos(\frac{2\pi j}{3} \pm \theta)} \in \partial\mathbb{H}^2$, and its Euclidean radius and centre are given by, respectively

$$\varepsilon_j = \frac{1}{2}\left(\frac{\sin(\frac{2\pi j}{3} + \theta)}{1 + \cos(\frac{2\pi j}{3} + \theta)} - \frac{\sin(\frac{2\pi j}{3} - \theta)}{1 + \cos(\frac{2\pi j}{3} - \theta)}\right) \tag{5.1}$$

$$c_j = \frac{1}{2}\left(\frac{\sin(\frac{2\pi j}{3} + \theta)}{1 + \cos(\frac{2\pi j}{3} + \theta)} + \frac{\sin(\frac{2\pi j}{3} - \theta)}{1 + \cos(\frac{2\pi j}{3} - \theta)}\right). \tag{5.2}$$

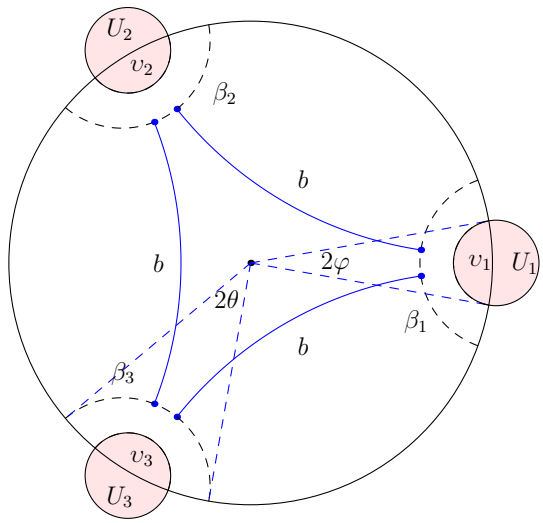


Figure 7: Three geodesics β_j in $\mathbb{D}$ giving rise to the reflections R_j; and three additional geodesics υ_j which are used to define domain of the analytic functions in $\mathscr{B}$.

The end points of υ_j are $\frac{\sin(\frac{2\pi j}{3}\pm\varphi)}{1+\cos(\frac{2\pi j}{3}\pm\varphi)} \in \partial\mathbb{H}^2$ and the euclidean radius is

$$r_j = \frac{1}{2}\left(\frac{\sin(\frac{2\pi j}{3}+\varphi)}{1+\cos(\frac{2\pi j}{3}+\varphi)} - \frac{\sin(\frac{2\pi j}{3}-\varphi)}{1+\cos(\frac{2\pi j}{3}-\varphi)}\right) \tag{5.3}$$

We can consider the Banach space $\mathscr{B}$ to be the space of bounded analytic functions on $\sqcup_{j=1}^3 U_j \subset \mathbb{C}$ with the supremum norm.

We see that the reflection R_j with respect to the geodesic β_j in $\mathbb{H}^2$ is given by $R_j(z) = \frac{\varepsilon_j^2}{z-c_j} + c_j$. We deduce that for any distinct j,k,l the image $R_j(U_k \cup U_l) \subset U_j$, provided $\frac{\varepsilon_j^2}{z-c_j} < r_j$ for any $z \in U_k \sqcup U_l$. We know that for all $z \in U_k \sqcup U_l$ we have $|z-c_j| > 1$, thus it is sufficient to chose θ and ε_j such that $\varepsilon_j^2 < r_j$.

Using (5.1) by straightforward calculation we may estimate $\frac{1}{2}\theta \le \varepsilon_j \le 2\theta + O(\theta^2)$ and $\frac{1}{2}\varphi \le r_j \le 2\varphi + O(\varphi^2)$ for small values of θ and φ. Hence it is sufficient to choose θ and φ such that $4\theta^2 < \frac{1}{2}\varphi$. Using Lemma 4.4, we see $\theta = e^{-b}(1+e^{-2b}+o(e^{-3b}))$. In particular, it is sufficient to choose $\varphi = e^{-b\varkappa}$ for some $1 < \varkappa < 2$ and b sufficiently large. Then

$$\frac{1}{2}e^{-b\varkappa} \le r_j \le 2e^{-b\varkappa} + O(e^{-2b\varkappa}), \tag{5.4}$$

$$\frac{1}{2}e^{-b} \le \varepsilon_j \le 2e^{-b} + O(e^{-2b}). \tag{5.5}$$

Using the Cauchy integral formula for $z \in U_k$ and $R_j(z) \in U_j$ we can write

$$f(R_j(z)) = \frac{1}{2\pi i}\int_{\partial U_j} \frac{f(\xi)}{\xi - R_j(z)} d\xi$$

and then

$$(R_j'(z))^s f(R_j(z)) = \frac{(R_j'(z))^s}{2\pi i}\int_{\partial U_j}\frac{f(\xi)}{\xi - R_j(z)}d\xi.$$

Since $\mathscr{L}_s$ is a nuclear operator, it satisfies (4.1). More precisely, we may write

$$\begin{aligned}(\mathscr{L}_s f)(z) &= \sum_{j=1}^{3}(R_j'(z))^s f(R_j(z))\sum_{k=1,k\neq j}^{3}\chi_{U_k}(z)\\ &= \sum_{j=1}^{3}(R_j'(z))^s\cdot\Big(\sum_{n=0}^{\infty}\frac{(R_j(z)-c_j)^n}{2\pi i}\int_{\partial U_j}\frac{f(\xi)}{(\xi-c_j)^{n+1}}d\xi\Big)\cdot\Big(\sum_{k=1,k\neq j}^{3}\chi_{U_k}(z)\Big)\\ &= \sum_{n=0}^{\infty}\lambda_n w_n(z)\nu_n(f),\end{aligned}$$

where, $w_n \in \mathscr{B}$, $\nu_n \in \mathscr{B}^*$, and $\lambda_n \in \mathbb{R}^+$ satisfy conditions of Definition 4.1. We may choose for any $j \in \{1,2,3\}$

$$w_{3n+j} \asymp (R_j'(z))^s\cdot\frac{(R_j(z)-c_j)^n}{2\pi i}\sum_{k=1,k\neq j}^{3}\chi_{U_k}(z) \tag{5.6}$$

$$\nu_{3n+j} \asymp \int_{\partial U_j}\frac{f(\xi)}{(\xi-c_j)^{n+1}}d\xi \tag{5.7}$$

with normalization $\|w_n\|_\infty = \|\nu_n\|_\infty = 1$. Then for any $j \in \{1,2,3\}$

$$|\lambda_{3n+j}| = \big\|(R_j')^s\,|_{U_k\cup U_l}\,\|\cdot\|(R_j-c_j)^n\,|_{U_k\cup U_l}\big\|_\infty\cdot\left\|\frac{1}{2\pi i}\int_{\partial U_j}\frac{f(\xi)}{(\xi-c_j)^{n+1}}d\xi\right\|_\infty, \tag{5.8}$$

where $k \neq j$ and $l \neq j$. We may observe that $\xi - c_j = r_j$ and conclude

$$|\lambda_{3n+j}| \leq \|R_j'(z)^s\,|_{U_k\cup U_l}\,\|\cdot\frac{\varepsilon_j^{2n}}{r_j^{n+1}}$$

or more precisely, using formulae (5.1) and (5.3), we obtain an upper bound

$$\begin{aligned}|\lambda_n| &\leq \max_j\|R_j'(z)^s\,|_{U_k\cup U_l}\,\|\cdot\max\Big\{\frac{1}{r_1}\frac{\varepsilon_1^{2n/3}}{r_1^{n/3}},\ \frac{1}{(\varepsilon_2 r_2)^{2/3}}\frac{\varepsilon_2^{2n/3}}{r_2^{n/3}},\ \frac{1}{\varepsilon_3^{4/3}r_3^{1/3}}\frac{\varepsilon_3^{2n/3}}{r_3^{n/3}}\Big\}\\ &\leq \max_j\|R_j'(z)^s\,|_{U_k\cup U_l}\,\|\cdot\frac{1}{\varepsilon_3^{4/3}r_3^{1/3}}\max_j\frac{\varepsilon_j^{2n/3}}{r_j^{n/3}}.\end{aligned}$$

Comparing this with the definition of the nuclear operator 4.2, we get explicit bounds for parameters λ and $C(s)$.

$$|\lambda_n| \leq \max_k\sup_{z\in U_k}\max_{j\neq k}\left|\frac{\varepsilon_j^{2s}}{(z-c_j)^{2s}}\right|\cdot\frac{1}{\varepsilon_3^{4/3}r_3^{1/3}}\cdot\max_j\frac{\varepsilon_j^{2n/3}}{r_j^{n/3}} \leq C(s)\lambda^n,$$

with the choices

$$\lambda = \Big(\max_j \frac{\varepsilon_j^2}{r_j}\Big)^{\frac{1}{3}} \tag{5.9}$$

$$C(s) = \max_k \sup_{z\in U_k} \max_{j\neq k} \left|\frac{\varepsilon_j^{2s}}{(z-c_j)^{2s}}\right| \cdot \frac{1}{\varepsilon_3^{4/3} r_3^{1/3}}. \tag{5.10}$$

Using the bounds (5.4) and (5.5) for ε_j and r_j, we conclude

$$\lambda = \Big(\max \frac{\varepsilon_j^2}{r_j}\Big)^{\frac{1}{3}} \leq 2e^{-\frac{2-\varkappa}{3}b}. \tag{5.11}$$

Furthermore, we see that for any $j \neq k$ for all $z \in U_k$ we have $|\arg(z-c_j)| \leq \arcsin(\frac{r_j}{c_k-c_j}) \leq 2\varphi$. Therefore for $s = \sigma + it$,

$$|(z-c_j)^{2s}| = \big|\exp\big(2(\ln|z-c_j| + i\arg(z-c_j))\cdot(\sigma+it)\big)\big| = |z-c_j|^{\sigma}\cdot\exp(-2\arg(z-c_j)t) \geq \exp(4\varphi t),$$

since by construction $\inf_{z\in U_k}|z-c_j| > 1$. Using (5.4) and (5.5), we deduce

$$\begin{aligned} C(s) = \max_k \sup_{z\in U_k} \max_{j\neq k} \left|\frac{\varepsilon_j^{2s}}{(z-c_j)^{2s}}\right| \cdot \frac{1}{\varepsilon_3^{4/3} r_3^{1/3}} &\leq \max_k \sup_{z\in U_k} \max_{j\neq k} \left|\frac{\varepsilon_j^{2\sigma}}{(z-c_j)^{2s}}\right| \cdot 4e^{b(\varkappa+4)/3} \\ &\leq 4\varepsilon_j^{2\sigma} e^{4\varphi t + b(\varkappa+4)/3} \leq e^{\ln 4 - 2b\sigma + 4\varphi t + b(\varkappa+4)/3}. \end{aligned} \tag{5.12}$$

Substituting bounds (5.11) and (5.12) into Ruelle's inequality (4.3) and taking into account $t < T$ for $s = \sigma + it \in \mathscr{R}(T)$, we obtain an upper bound

$$\begin{aligned} |a_n(s)| &\leq C^n(s)\lambda^{n(n+1)/2} n^{n/2} \\ &\leq \exp\Big((\ln 4 - 2b\sigma + \frac{b(\varkappa+4)}{3} + 4\varphi t)n - \frac{n(n+1)}{2}\Big(\frac{b(2-\varkappa)}{3} - \ln 2\Big) + \frac{n\ln n}{2}\Big) \\ &\leq \exp\Big((\ln 4 + \frac{b(\varkappa+4)}{3} + 4\varphi T)n - \frac{n(n+1)}{2}\Big(\frac{b(2-\varkappa)}{3} - \ln 2\Big) + \frac{n\ln n}{2}\Big), \end{aligned} \tag{5.13}$$

since $\exp(-2bn\sigma) \leq 1$.

In order to estimate the tail of the series $\sum_{n=14}^{\infty} a_n(s)$ using Lemma 5.1, it is sufficient to find a constant $k_2 < 1$ such that

$$n\Big(\ln 4 + \frac{b(\varkappa+4)}{3} + \frac{b(2-\varkappa)}{6}\Big) + \frac{(n+1)n}{2}\ln 2 + \frac{n\ln n}{2} < \frac{bn^2(2-\varkappa)}{6}k_2 \tag{5.14}$$

which is equivalent

$$\frac{\frac{15\ln 2}{b} + 10 + \varkappa}{k_2(2-\varkappa) - \frac{3\ln 2}{b}} < n - \frac{3\ln n}{bk_2(2-\varkappa) - 3\ln 2}. \tag{5.15}$$

It is clear that the inequality doesn't hold for any $n \leq 10$ but it does hold for all $b \geq 20$ and $n \geq 14$ with the choices $\varkappa = 1.05$ and $0.95 \leq k_2 < 1$. Therefore we obtain an upper bound

$$|a_n(s)| \leq \exp\Big(4\varphi T n - \frac{b(2-\varkappa)(1-k_2)}{6} n^2\Big), \quad \text{for all } n \geq 14. \tag{5.16}$$

We recall that $\varphi = e^{-\varkappa b}$ and applying Lemma 5.1 with the choices $p = 4e^{-\varkappa b}T$ and $q = \frac{b(2-\varkappa)(1-k_2)}{6}$, we get an estimate $|Z_X(s) - Z_{n-2}(s)| = \sum_{k=n}^{\infty} |a_n(\sigma + it)| \leq \eta(b,n,T(b))$, where $T(b) = k_0 e^{\varkappa b}$ for some $k_0 > 0$ and all $b \geq 20$, $n \geq 14$:

$$\begin{aligned} &\eta(b,n,T(b)) = \\ &= \frac{\sqrt{6\pi}}{2\sqrt{b(2-\varkappa)(1-k_2)}} \exp\Big(\frac{24k_0^2}{b(2-\varkappa)(1-k_2)}\Big) \exp\Big(-\frac{b(2-\varkappa)(1-k_2)}{6}\Big(n - \frac{12k_0}{b(2-\varkappa)(1-k_2)}\Big)^2\Big). \end{aligned} \tag{5.17}$$

Therefore we have the desired asymptotic estimates:

1. for any $n \geq 14$ we have $\eta(b,n,T(b)) = O\big(\frac{1}{\sqrt{b}}\big)$ as $b \to \infty$;
2. for any $b \geq 20$ we have $\eta(b,n,T(b)) = O\big(e^{-bk_1 n^2}\big)$; as $n \to \infty$.

hold with the choices, for example, $0 < k_1 \leq \frac{(2-\varkappa)(1-k_2)}{6}$, where $1 < \varkappa < 2$ and k_2 are chosen so that (5.14) holds. □

6 Asymptotic results for large b

We can now use the approximation result in Theorem 3.5 for fixed n and large b to prove the results in Theorem 1.4.

Theorem 6.1. *Given a $\varkappa > 1$ as in Theorem 3.5, the complex analytic function $Z_X\big(\frac{s}{b}\big)$ converges uniformly to $Z_6\big(\frac{s}{b}\big)$ on the domain $\widehat{\mathscr{R}}(e^{\varkappa b})$, more precisely,*

$$\sup_{s \in \widehat{\mathscr{R}}(e^{\varkappa b})} \left| Z_X\left(\frac{s}{b}\right) - Z_6\left(\frac{s}{b}\right) \right| = O\left(\frac{1}{\sqrt{b}}\right) \text{ as } b \to +\infty.$$

Proof. In Appendix we explain how to compute the lengths of closed geodesics. By a straightforward manipulation with some help from a computer we can show that $a_n(s) = \exp(-nbs - 2b)(1 + O(\exp(-2b))^s$ for $n = 8, 10, 12$ and therefore $a_n(s/b) = \exp(-ns - 2b)(1 + O(\exp(-2b))^{s/b} \to 0$ as $b \to \infty$. The result follows from Theorem 3.5 which gives the bound for $\|Z_X - Z_{14}\|_\infty$. □

Remark 6.2. Using the estimates for the hyperbolic length of closed geodesics of the word length $\omega(\gamma) \leq 6$, presented in the Appendix, we may compute the first few non-zero coefficients

$$\begin{aligned}
a_2(s) &= -\frac{6e^{-2bs}}{1-e^{-2b}} \\
a_4(s) &= 15e^{-4bs} - 6e^{-4bs}\left(1+e^{-b}+2e^{-2b}\right)^{2s} + O(e^{-b}) \\
a_6(s) &= -20e^{-6bs} - 2e^{-6bs}\left(1+3e^{-b}+3e^{-2b}\right)^{2s} - 6e^{-6bs}\left(1+2e^{-b}+3e^{-2b}\right)^{2s} + \\
&\quad + 24e^{-6bs}\left(1+e^{-b}+2e^{-2b}\right)^{2s} + O(e^{-b}).
\end{aligned}$$

We see that

$$Z_6\left(\frac{s}{b}\right) = 1 + a_2\left(\frac{s}{b}\right) + a_4\left(\frac{s}{b}\right) + a_6\left(\frac{s}{b}\right) \xrightarrow{b\to\infty} 1 - 6e^{-2s} + 9e^{-4s} - 4e^{-6s} = \det(I - e^{-2s}A^2).$$

Now we are ready to prove the first part of Theorem 1.4

Proof. We shall show that on the domain $\mathscr{R}(e^{\varkappa b})$ we have that the function $Z_X(s)$ vanishes at $s_n(b)$ such that $\lim\limits_{b\to\infty} s_n(b)\cdot b = (\ln 2 + i\pi n)$.

The function $\det(I - e^{-2s}A^2)$ vanishes at $\{i\pi n, \ln 2 + i\pi n\}$, for $n \in \mathbb{Z}$. For any sufficiently small $\eta > 0$ we have that the closed balls

$$\overline{U(\ln 2 + 2i\pi n, \eta)} = \{s \in \mathbb{C} \colon |s - (\ln 2 + 2i\pi n)| \leq \eta\}$$

contains no more zeros. Let us denote $\varepsilon = \inf_{s\in\partial U} |\det(I - e^{-2s}A^2)| > 0$. Using Theorem 6.1, we can now choose b sufficiently large so that we have

$$\inf_{s\in\partial U} |Z_X\left(\frac{s}{b}\right) - \det(I - e^{-2s}A^2)| < \frac{\varepsilon}{2}.$$

It then follows by Rouché's Theorem [1] that for any $n \in \mathbb{N}$ the function $Z_X(s)$ has exactly one zero $s_n(b)$, satisfying $|s_n(b) - \frac{1}{b}(\ln 2 + i\pi n)| < \eta$. □

Remark 6.3. Weich [17] has observed this result for a case of the domain of the fixed size, i.e. $\mathscr{R}(c)$, where c is independent of b.

This is consistent with an old result [10], which states that the largest real zero of Z_X satisfies $\delta \sim \frac{\ln 2}{b}$ as $b \to \infty$.

This implies asymptotic spacing of individual zeros suggested in **O1**. We now turn to the problem of describing the distribution of the zeros **O2** and **O3**. In Section 2.1.2 we introduced a matrix B, closely connected to the zeta function. In the following lemma we show its determinant appoximates the zeta function and quantify the error.

Lemma 6.4. *Let us recall* (2.7)

$$B(z) = \begin{pmatrix} 1 & z & 0 & 0 & z^2 & z \\ z & 1 & z^2 & z & 0 & 0 \\ 0 & 0 & 1 & z & z & z^2 \\ z^2 & z & z & 1 & 0 & 0 \\ 0 & 0 & z & z^2 & 1 & z \\ z & z^2 & 0 & 0 & z & 1 \end{pmatrix}$$

The real analytic function $Z_{12}\left(\frac{\sigma}{b} + ite^b\right)$ *converges uniformly to* $\det(I - \exp(-2\sigma - 2itbe^b)B(e^{it}))$, *and more precisely,*

$$\left|Z_{12}\left(\frac{\sigma}{b} + ite^b\right) - \det\left(I - \exp(-2\sigma - 2itbe^b)B(e^{it})\right)\right| = O\left(\exp(-b)\right) \text{ as } b \to +\infty.$$

Proof. This follows by straightforward calculation of the first 12 coefficients and the determinant. Let us introduce dummy variables $x\colon = \exp(-2\sigma - 2itbe^b)$ and $y\colon = e^{it}$ with $|x| < 1$ and $|y| = 1$. Then

$$\det\left(I - \exp(-2\sigma - 2itbe^b)B(e^{it})\right) = \det\left(I - xB(y)\right) = \sum_{k=0}^{6} x^k P_k(y^2), \tag{6.1}$$

where $P_k \in \mathbb{Z}[\cdot]$ are some polynomials with integer coefficients. More precisely, we can compute:

$$\begin{aligned} P_0(y) &\equiv 1, \\ P_1(y) &\equiv 6, \\ P_2(y) &= 15 - 6y^2, \\ P_3(y) &= 20 - 24y^2 + 6y^4 + 2y^6, \\ P_4(y) &= 15 - 36y^2 + 27y^4 - 6y^6, \\ P_5(y) &= -6(y^2 - 1)^4, \\ P_6(y) &= (y^2 - 1)^6. \end{aligned}$$

At the same time

$$Z_{12}\left(\frac{\sigma}{b} + ite^b\right) = \sum_{j=0}^{12} a_j\left(\frac{\sigma}{b} + ite^b\right)$$

With some help of a computer, one can deduce by straightforward manipulation that $a_n\left(\frac{\sigma}{b} + ite^b\right) = x^n P_n(y) + O(e^{-b})$. We shall illustrate this using formulae for the coefficients from Remark 6.2.

$$a_2\left(\frac{\sigma}{b} + ite^b\right) = -\frac{6\exp\left(-2\sigma - 2itbe^b\right)}{1 - e^{-2b}} = \frac{-6x}{1 - e^{-2b}} = -6x(1 + O(e^{-2b})) = xP_1 + O(e^{-b}).$$

Similarly for a_4:

$$\begin{aligned}
a_4\left(\frac{\sigma}{b}+ite^b\right) &= 15e^{-4\sigma-4itbe^b} - 6e^{-4\sigma-4itbe^b}\left(1+e^{-b}+2e^{-2b}\right)^{2\sigma/b+2ite^b} + O(e^{-b}) = \\
&= 15x^2 - 6x^2\left(1+e^{-b}+2e^{-2b}\right)^{2\sigma/b}\cdot\left(1+e^{-b}+2e^{-2b}\right)^{2ite^b} + O(e^{-b}) = \\
&= 15x^2 - 6x^2e^{2it}\cdot\left(1+O(e^{-b})\right) + O(e^{-b}) = \\
&= x^2\left(15-6y^2\right) + O(e^{-b}) = x^2P_2(y) + O(e^{-b}),
\end{aligned}$$

where we have been using the fact that $\left(1+e^{-b}+2e^{-2b}\right)^{e^b} = e + O\left(e^{-b}\right)$.

□

Now we can prove the second part of Theorem 1.4.

Proof. We find that the matrix $B(e^{it})$ has exactly four different eigenvalues $\beta_k(t)$, $k = 1,\dots,4$:

$$\begin{aligned}
\beta_1(t) &= (e^{it}-1)^2 \\
\beta_2(t) &= (e^{it}+1)^2 \\
\beta_3(t) &= 1 - \frac{e^{2it}}{2} - e^{it}\frac{\sqrt{4-3e^{2it}}}{2} \\
\beta_4(t) &= 1 - \frac{e^{2it}}{2} - e^{it}\frac{\sqrt{4-3e^{2it}}}{2}
\end{aligned}$$

Therefore we deduce that the zero set of the determinant $\det\left(I - \exp(-2\sigma - 2itbe^b)B(e^{it})\right)$ belongs to the subset $\left\{(\sigma,t)\in\mathbb{R}^2 \mid \exists k\colon |\exp(2\sigma+2itbe^b)| = |\beta_k|\right\}$ The four equations $\exp(2\sigma) = |\beta_k(t)|$ give us four curves

$$\begin{aligned}
\mathscr{C}_1 &= \left\{\frac{1}{2}\ln|2-2\cos(t)| \mid t\in\mathbb{R}\right\}; \\
\mathscr{C}_2 &= \left\{\frac{1}{2}\ln|2+2\cos(t)| \mid t\in\mathbb{R}\right\}; \\
\mathscr{C}_3 &= \left\{\frac{1}{2}\ln\left|1-\frac{1}{2}e^{2it}-\frac{1}{2}e^{it}\sqrt{4-3e^{2it}}\right| \mid t\in\mathbb{R}\right\}; \\
\mathscr{C}_4 &= \left\{\frac{1}{2}\ln\left|1-\frac{1}{2}e^{2it}+\frac{1}{2}e^{it}\sqrt{4-3e^{2it}}\right| \mid t\in\mathbb{R}\right\}.
\end{aligned}$$

which correspond to the curves suggested in **O2**.

Since the curves $\mathscr{C}_j$ do not have horizontal tangencies $\sigma\equiv\text{const}$, without loss of generality we may define the neighbourhoods as follows:

$$B(\mathscr{C}_j,\varepsilon)\colon = \left\{(\sigma,t) \mid \left|2\sigma - \ln|\beta_j(t)|\right| > 2\varepsilon\right\}$$

To complete the argument we shall show that for all $\varepsilon > 0$ and $T > 0$ there exists $b_0 > 0$ such that for any $b > b_0$ the zeros of the function $Z\left(\frac{\sigma}{b} + ite^b\right)$ with $0 \le \sigma \le 1$ and $|t| \le e^{(2-\varkappa)b}$ belong to a neighbourhood $B(\cup_k \mathscr{C}_k, \varepsilon)$ of the union of the curves $\cup_k \mathscr{C}_k$.

Indeed, given $\varepsilon > 0$ and a point $z_0 = \sigma_0 + it_0$ outside of ε-neighbourhood of $\cup_{j=1}^4 \mathscr{C}_j$ we see that the determinant $\det(I - \exp(-2\sigma_0 - it_0 be^b)B(\exp(it_0))) > \exp(-6\varepsilon)(\exp\varepsilon - 1)^6 > 0$ is bounded away from zero and the bound is independent of b. Therefore for b sufficiently large all zeros of the function $Z_X(\sigma/b + ite^b)$ belong to the ε-neighbourhood of $\cup_{j=1}^4 \mathscr{C}_j$. □

We have concentrated on the particular case of the symmetric pair of pants (whose boundary curves have the same lengths). However, the same method works in the case that the boundary curves have different length as well as in the case of symmetric punctured torus, and allows to explain the nature of the patterns of zeros described in the sections 5.1 and 5.2 of [5].

7 *L*-functions and Covering surfaces

Our results have concentrated on a special class of surfaces, but can be easily adapted to cover a large class of geometrically finite surfaces of infinite area.

There is a fairly simple method for constructing quite complicated surfaces using the basic pair of pants V. We can write $V = \mathbb{H}^2/\Gamma$. We then write a cover $V_0 = \mathbb{H}^2/\Gamma_0$ for V in terms of a normal subgroup $\Gamma_0 < \Gamma$.

Let us denote by $G = \Gamma/\Gamma_0$ the finite quotient group. Let γ be a closed geodesic on V and then this is covered by the union of closed geodesics $\gamma_1, \cdots, \gamma_n$ on $\widehat{V}$.

Let R_χ be an irreducible representation for G of degree d_χ with character $\chi = \mathrm{tr}(R_\chi)$. The regular representation of G can be written $R = \oplus_\chi d_\chi R_\chi$. where $|G| = \sum_\chi d_\chi^2$

Definition 7.1. *Given $s \in \mathbb{C}$ we define*

$$L(z,s,\chi) = \prod_\gamma \det\left(I - z^{|g|} e^{-(s+n)\lambda(g)} R(g\Gamma_0)\right)$$

where $g\Gamma_0$ is a coset in G.

Lemma 7.2. *For characters χ_1 and χ_2 we can write*

$$L(z,s,\chi_1 + \chi_2) = L(z,s,\chi_1)L(z,s,\chi_2).$$

If $H < G$ is a subgroup and χ is a character of H the we can write $G = \cup_{i=1}^m H\alpha_i$ and define the induced character χ^* of G by

$$\chi^*(g) = \sum_{\alpha_i g \alpha_i^{-1} \in H} \chi(\alpha_i g \alpha_i^{-1})$$

for $g \in G$.

Lemma 7.3 (Brauer–Frobenius)**.** *Each non-trivial character χ is a rational combination of characters χ_i^* of G induced from non-trivial characters χ_i of cyclic subgroups H_i.*

There exist integers $n_1, \cdots, n_k$ with

$$n\chi = \sum_{i=1}^{k} n_i \chi_i^*$$

and thus

$$L(s,z,\chi)^n = \sum_{\alpha_i g \alpha_i^{-1} \in H} \chi(\alpha_i g \alpha_i^{-1}))$$

Since it is easier to deal with cyclic covering groups. We need the following.

Lemma 7.4. *Let χ be a character of the subgroup $H < G$ and let $L(s,\chi)$ be the L-function with respect to the covering $\widetilde{V}$ of $\widetilde{V}/H$. Then $L(s,z,\chi) = L(s,z,\chi^*)$.*

The proof is analogous to that of the proof of Proposition 2 in [12].

This leads to the following.

Lemma 7.5. *If χ is an irreducible non-trivial character of G then $L(s,\chi)^n$ is a product of integer powers of L-functions defined with respect to non-trivial characters of cyclic subgroups of G.*

Finally this means that we can write the zeta function $Z_{\widetilde{V}}(s,z)$ in terms of the L-functions $L_V(s,z,\chi)$ for V.

Lemma 7.6. *We can write*

$$Z_{\widetilde{V}}(s,z) = \prod_{\chi \text{ irreducible}} L_V(s,z,\chi)^{d_\chi}.$$

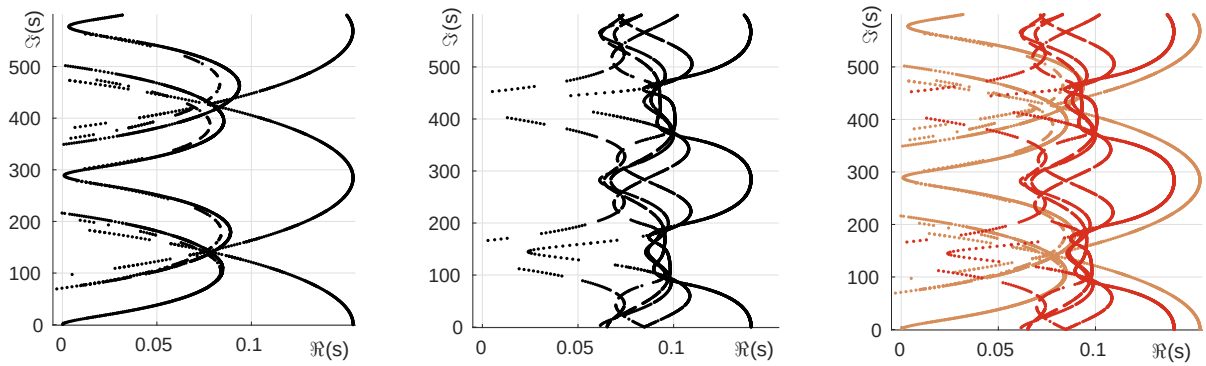
where the product is over all irreducible representations of G.

In particular, the zeros for $Z_{\widetilde{V}}(s,z)$ will be the union of the zeros for the L-functions $L_V(s,z,\chi)^{d_\chi}$.

Example 7.7. *We can take a double cover $\widetilde{V}$ for a pair of pants V, which corresponds to a sphere with four holes. The corresponding covering group is simply $\mathbb{Z}_2$ and the zeta function $Z_{\widetilde{V}}(s)$ is then the product of:*

1. *the zeta function $Z_V(s)$ for the original pair of pants;*
2. *the L-function $L_V(s,z,\chi)$ corresponding to the representation $\chi : \pi_1(V) \to \mathbb{Z}_2$ where $\chi(g) = (-1)^{n(g)}$ where $n(g)$ counts the number of times the generator a, say, occurs in g.*

In particular the zeros for $Z_{\widetilde{V}}(s)$ are a union of the figures for these two functions.



(a) zeros for $Z_V(s)$ (b) zeros for $L_V(s,z,\chi)$ (c) superposition

Figure 8: Zeros of the zeta function and zeros of the L-function in the case $\ell(\gamma_{1,2,3}) = 9$.

A Examples of the coefficients

In this Appendix we present the asymptotic formulae for the hyperbolic length of the first closed geodesics, which then lead to the asymptotic expressions for the first few non-zero coefficients a_2, a_4, a_6.

Using the identity

$$\ell(\gamma_{j_1,j_2,\dots j_{2n}}) = 2\,\mathrm{Arcosh}\left(\frac{1}{2}\,\mathrm{tr}(R_{j_1}R_{j_2}\dots R_{j_{2n}})\right),$$

relating the length of the closed geodesic corresponding to the cutting sequence of period $2n$ to the matrices defining the reflections, we compute the lengths of the closed geodesics for $n = 1,2,3,4$.

The case $n = 2$. It has been established in Lemmas 2.5, and 3.1, there are exactly 6 geodesics of the length $\ell(\gamma_{j_1 j_2}) = 2b$.

The case $n = 4$. There are 6 geodesics of the length $4b$ and 12 geodesics of the length

$$\ell(\gamma_{j_1 j_2 j_3 j_2}) = 2\,\mathrm{Arcosh}\big(\cosh(b) + 2\cosh^2(b)\big).$$

The case $n = 6$. There are $4^3 + 2 = 66$ homotopy classes of closed geodesics; among which there are 6 geodesics of the length $6b$ and of the length

$$\ell(\gamma_{j_1 j_2 j_3 j_1 j_2 j_3}) = 2\,\mathrm{Arcosh}\big(4\cosh^3(b) + 6\cosh^2(b) - 1\big) = 6b + 6e^{-b} + O(e^{-2b}).$$

There are 18 geodesics of the length

$$\ell(\gamma_{j_1 j_2 j_1 j_3 j_2 j_3}) = 2\,\mathrm{Arcosh}\left(8\cosh^2\left(\frac{b}{2}\right)\cdot\cosh^2(b) - 1\right) = 6b + 4e^{-b} + O(e^{-2b}).$$

Finally, there are 36 geodesics of the length

$$\ell(\gamma_{j_3j_2j_1j_2j_1j_2}) = 2\,\mathrm{Arcosh}\big(4\cosh^3(b)+2\cosh^2(b)-\cosh(b)\big) = 6b+2e^{-b}+O(e^{-2b}).$$

The case $n=8$. There are $4^4+2=258$ homotopy classes of closed geodesics; among which there are 6 geodesics of the length $8b$. Moreover, there are 24 geodesics of the length

$$\ell(\gamma_{j_3j_1j_3j_1j_2j_1j_2j_1}) = 2\,\mathrm{Arcosh}(-4\cosh^2(b)+4\cosh^3(b)+8\cosh^4(b)+1) = 8b+2e^{-b}+O(e^{-2b});$$

and another 48 geodesics of the length

$$\ell(\gamma_{j_3j_1j_2j_1j_2j_1j_2j_1}) = 2\,\mathrm{Arcosh}(-\cosh(b)-4\cosh^2(b)+4\cosh^3(b)+8\cosh^4(b)) = 8b+2e^{-b}+O(e^{-2b}).$$

In addition, we have 12 geodesics of the length

$$\ell(\gamma_{j_3j_1j_2j_1j_3j_1j_2j_1}) = 2\,\mathrm{Arcosh}(2\cosh^2(b)+8\cosh^3(b)+8\cosh^4(b)-1) = 8b+4e^{-b}+O(e^{-2b});$$

and 48 geodesics of the length

$$\ell(\gamma_{j_3j_1j_2j_3j_1j_2j_1j_2}) = 2\,\mathrm{Arcosh}(\cosh(b)\cdot(4\cosh(2b)+2\cosh(3b)+4\cosh(b)+1)) = 8b+4e^{-b}+O(e^{-2b});$$

and another 48 geodesics of the length

$$\ell(\gamma_{j_3j_2j_1j_3j_1j_2j_1j_2}) = 2\,\mathrm{Arcosh}(-\cosh(b)+8\cosh^3(b)+8\cosh^4(b)) = 8b+4e^{-b}+O(e^{-2b}).$$

Finally, there are 24 geodesics of the length

$$\ell(\gamma_{j_3j_2j_1j_2j_3j_1j_2j_1}) = 2\,\mathrm{Arcosh}(\cosh(b)\cdot(6\cosh(2b)+2\cosh(3b)+8\cosh(b)+3)) = 8b+6e^{-b}+O(e^{-2b});$$

and further more 48 geodesics of the length

$$\ell(\gamma_{j_3j_1j_2j_3j_1j_2j_1j_2}) = 2\,\mathrm{Arcosh}(-3\cosh(b)+12\cosh^3(b)+8\cosh^4(b)) = 8b+6e^{-b}+O(e^{-2b}).$$

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