HAUSDORFF DIMENSION OF GAUSS–CANTOR SETS AND TWO APPLICATIONS TO CLASSICAL LAGRANGE AND MARKOV SPECTRA

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ABSTRACT. This paper is dedicated to the study of two famous subsets of the real line, namely Lagrange spectrum L and Markov spectrum M. Our first result, Theorem 2.1, provides a rigorous estimate on the smallest value t_1 such that the portion of the Markov spectrum $(-\infty, t_1) \cap M$ has Hausdorff dimension 1. Our second result, Theorem 3.1, gives a new upper bound on the Hausdorff dimension of the set difference $M \setminus L$.

Our method combines new facts about the structure of the classical spectra together with finer estimates on the Hausdorff dimension of Gauss–Cantor sets of continued fraction expansions whose entries satisfy appropriate restrictions.

1. INTRODUCTION

In this paper we are concerned with Hausdorff dimension of certain subsets of the real numbers which play an important rôle in number theory, particularly in connection to Diophantine approximation. By using a dynamical re-interpretation we can reduce the problem to estimating the dimension of limit sets of iterated function systems. The construction of these limits sets is quite delicate and recent progress on rigorous bounds on their dimension provides the basis for our results.

In 1879—1880 Markov introduced two subsets of the positive real numbers called the Lagrange spectrum and the Markov spectrum [9], [10]. The Markov spectrum M was defined in terms of quadratic forms and the Lagrange spectrum L was defined in terms of properties of Diophantine approximation. In 1921 Perron [14] gave a simple characterization in terms of continued fractions. Following Perron, see also [12], we may consider a set of bi-infinite sequences $(\mathbb{N}^*)^{\mathbb{Z}}$ of natural numbers. To any sequence $\underline{a} = (a_n)_{n\in\mathbb{Z}} \in (\mathbb{N}^*)^{\mathbb{Z}}$ we associate a pair of bi-infinite sequences of real numbers defined in terms of continued fraction expansions

$$[a_n; a_{n+1}, a_{n+2}, \ldots] \quad \text{and} \quad [0; a_{n-1}, a_{n-2}, \ldots], \quad \text{for all } n \in \mathbb{Z},$$

and consider a map $\lambda_0(\underline{a}) := [a_0; a_1, a_2, \ldots] + [0; a_{-1}, a_{-2}, \ldots]$. Let us denote by σ the Bernoulli shift on $(\mathbb{N}^*)^{\mathbb{Z}}$ given by $\sigma((a_n)_{n\in\mathbb{Z}}) = (a_{n+1})_{n\in\mathbb{Z}}$. The Lagrange value of $\underline{a}$ is the limit superior of values of λ_0 along the σ-orbit of $\underline{a}$:

$$\ell(\underline{a}) := \limsup_{n\to\infty} \lambda_0(\sigma^n \underline{a}) = \limsup_{n\to\infty}([a_n; a_{n+1}, a_{n+2}, \ldots] + [0; a_{n-1}, a_{n-2}, \ldots])$$

and the Markov value of $\underline{a}$ is the supremum of values of λ_0 along the σ-orbit of α:

$$m(\underline{a}) := \sup_{n\in\mathbb{Z}} \lambda_0(\sigma^n \underline{a}) = \sup_{n\in\mathbb{Z}}([a_n; a_{n+1}, a_{n+2}, \ldots] + [0; a_{n-1}, a_{n-2}, \ldots]).$$

Date: June 14, 2021.

The third author is partly supported by ERC-Advanced Grant 833802-Resonances and EPSRC grant EP/T001674/1. The fourth author is partly supported by EPSRC grant EP/T001674/1.

The collection of Lagrange (Markov) values is called the Lagrange (Markov) spectrum, which we denote by L and M, respectively:

$$L := \left\{\ell(\alpha) \mid \alpha \in (\mathbb{N}^*)^{\mathbb{Z}}\right\} \quad \text{and} \quad M := \left\{m(\alpha) \mid \alpha \in (\mathbb{N}^*)^{\mathbb{Z}}\right\}. \tag{1.1}$$

It is known that $L \subset M$. Moreover, the structure of both sets in a neighbourhood of infinity is clear. The intersections $(-\infty, 3] \cap L = (-\infty, 3] \cap M$ agree and are countable. At the other extreme, there exists the smallest number c such that $[c, +\infty) \cap L = [c, +\infty) \cap M = [c, +\infty)$. It has been computed explicitly by Freiman in [3] to be[1] $c = 4.5278\ldots$ and is called Freiman's constant. The set $[c, +\infty)$ is sometimes called *Hall's ray*.

In contrast to this, the sets $L \cap [3, c]$ and $M \cap [3, c]$ have a complicated and mysterious structure. Nevertheless, some facts have been established. In particular, it was shown by Hall [4] that $M \cap [0, \sqrt{10}]$ has zero Lebesgue measure. A few years later this result was improved by Pavlova and Freiman [13] (cf. [2, Theorem 2, Chapter 6]), when they showed that $M \cap [0, \sqrt{689}/8]$ has zero Lebesgue measure[2].

More recently, it was shown by the second author in [12] that for any $t > 0$ the sets $(-\infty, t] \cap M$ and $(-\infty, t] \cap L$ have the same Hausdorff dimension:

$$\dim_H((-\infty, t] \cap M) = \dim_H((-\infty, t] \cap L)$$

and, moreover, the function

$$f(t) := \dim_H((-\infty, t] \cap M))$$

is a continuous non-decreasing function on the real line.

We now introduce the number which is the subject of our invetigations.

$$t_1 := \inf\{t \in \mathbb{R} \mid f(t) = 1\}. \tag{1.2}$$

In view of monotonicity of f the value t_1 is usually referred to as *the first transition point* of the classical Lagrange and Markov spectra. In 1982 Bumby [1] gave a heuristic estimate $3.33437 < t_1 < 3.33440$; while the results by Hall [4] and the second author [12] give the best *rigorous* lower and upper bounds on t_1 to date:

$$\sqrt{10} = 3.162277\ldots < t_1 < \sqrt{12} = 3.464101\ldots. \tag{1.3}$$

Our first result, Theorem 2.1 confirms Bumby's claim and gives a rigourous estimate of $t_1 = 3.334384\ldots$ The proof is built on ideas developed by Bumby and uses a connection between Markov values and Gauss–Cantor sets defined in terms of continued fractions of their elements. The argument is computer—assisted and the result could be refined further with the method we present, subject to more computer time and resourses.

Lima and the second author in [7] recently conjectured that $(t_1, t_1 + \delta) \cap L$ has non-empty interior for all $\delta > 0$. Together with our new result $t_1 = 3.334384\ldots$ this would imply, in particular, that $(3.334384, 3.334385) \cap L$ has non-empty interior and thus prove an open folklore conjecture that the interior of $(-\infty, \sqrt{12}) \cap L$ is non-empty.

The second part of our paper concerns the set difference of the Markov and Lagrange spectrum. It is known that $M \setminus L$ has zero Lebesgue measure. Furthermore, it was proved in [11] and [15] that the Hausdorff dimension of $M \setminus L$ satisfies

$$0.5312 < \dim_H(M \setminus L) < 0.8823.$$

Our second result, Theorem 3.1, shows that the Hausdorff dimension of $M \setminus L$ has sharper bounds

$$0.537152 < \dim_H(M \setminus L) < 0.796445.$$

[1]All numbers are truncated, not rounded.

[2]Note that $\sqrt{689}/8 = 3.2811\ldots$ and $\sqrt{10} = 3.162277\ldots$

The proof is also computer—assisted. Following the approach developed by the first two authors [11], we use fine-grained combinatorial analysis of continued fractions to construct a cover $M \setminus L$ by arithmetic sums of Gauss–Cantor sets and the so-called “Cantor sets of the gaps”. We then apply the new method for computing the Hausdorff dimension recently developed by the last two authors [15] to several Gauss–Cantor sets to obtain sharper upper bounds on $\dim_H(M \setminus L)$.

We organize this article as follows. In §2, we reduce the problem of computing t_1 to the problem of constructing two Gauss–Cantor sets X and Y such that

$$\dim_H X < 0.5 < \dim_H Y, \tag{1.4}$$

and the substrings of $\alpha \in (\mathbb{N}^*)^{\mathbb{Z}}$ with $m(\alpha)$ close to t_1 are “controlled” by X and Y. The conditions that X, Y and t_1 should jointly satisfy are slightly more subtle and we describe them in detail in §2.1.2. Next §3 is dedicated to the construction and analysis of the arithmetic sums of Gauss–Cantor sets and “Cantor sets of the gaps” which cover $M \setminus L$, and subsequently allows us to obtain an upper bound on $\dim_H(M \setminus L)$. The intricate character of the Gauss–Cantor sets involved in estimates in §2 and §3 means that the algorithm for computing the Hausdorff dimension developed in [15] has to be considerably adapted and improved. For completeness, in §4 we explain how the Hausdorff dimension of the complicated Gauss–Cantor sets can be computed and give some details of the numerical implementation.

Remark 1.1. On our way to establishing the results mentioned in the previous paragraphs, we encounter some other interesting facts about the structure of the classical spectra. For example, Lemma 3.9 below says that 4.5 is a non-trivial rational point in L in the sense that it occurs after 3 and before the beginning $c = 4.5278\ldots$ of Hall’s ray. Hence the value 4.5 is realised as the Markov value of two sequences which arise in our study of $(4.4984, \sqrt{21}) \cap (M \setminus L)$.

2. Phase transition in classical spectra

In this section, we give the theoretical basis for the proof of our first main result which gives rigorous bounds on the first transition point t_1 and construct explicitly the relevant Gauss–Cantor sets. The theoretical background for computer-assisted calculations which are used to obtain estimates on Hausdorff dimension of those Gauss–Cantor sets which are constructed here can be found in §4.

Theorem 2.1. $t_1 = \inf\{t \in \mathbb{R} : \dim_H((-\infty, t] \cap M) = 1\} = 3.334384\ldots$*, where this value is accurate to the* 6 *decimal places presented.*

2.1. **Preliminaries.** We begin by describing the basic strategy to deduce bounds on t_1 which generalizes the approach of Hall. The first Cantor set we introduce is relatively famous and consists of all real number whose continued fraction expansion has only digits 1 and 2:

$$E_2 := \{a = [0; \alpha_1, \alpha_2, \ldots] \mid \alpha_j \in \{1, 2\},\ j \geq 1\}.$$

Its Hausdorff dimension has been computed to high precision (see, for instance [5], and references therein), and for our purposes it is suffices that

$$\dim_H E_2 > 0.53128. \tag{2.1}$$

In what follows, we identify a subset $A \subseteq E_2$ with a set of one-sided sequences corresponding to the continued fraction expansions of its elements.

In the sequel we use a simple observation that the constant bi-infinite sequence $\beta_n \equiv 1$, $n \in \mathbb{Z}$, has the minimal Markov value among all bi-infinite sequences $\alpha \in \{1,2\}^{\mathbb{Z}}$, and, moreover, for any $\alpha \neq \beta$ we have $m(\beta) < m(\alpha)$. Straightforward computation gives

$$m(\beta) = \sqrt{5} \leq m(\alpha) \quad \text{for any } \alpha \in \{1,2\}^{\mathbb{Z}}.$$

2.1.1. *Approach to lower bound.* We fix some threshold T and attempt to construct a finite set of finite "forbidden" strings $\beta_{-k} \dots \beta_{-1}\beta_0\beta_1 \dots \beta_n$ so that *all* infinite extensions $\left\{\alpha \in \{1,2\}^{\mathbb{Z}} \mid \beta_j = \alpha_j,\ -k \leq j \leq n\right\}$ of these strings have Markov values $m(\alpha) > T$.

It is easy to see that after excluding from E_2 all irrational numbers whose continued fraction expansion contains a "forbidden" string, we obtain a Cantor set $K \subset E_2$ such that

$$M \cap \left(\sqrt{5}, T\right) \subset 2 + K + K. \tag{2.2}$$

Recall that $\dim_H(K+K) \leqslant \dim_H K + \overline{\dim_B} K$, where $\overline{\dim_B} K$ denotes the upper box dimension. It is known that $\overline{\dim_B} K = \dim_H K$ for these types of sets and hence $\dim_H K < 0.5$ implies that $t_1 \geqslant T$.

2.1.2. *Approach to upper bound.* Now let S be the maximal Markov value of strings which do not contain a forbidden string as a substring and let $K \subseteq E_2$ be as above. It was shown in [12, proof of Lemma 3] that

$$\min\{2 \cdot \dim_H K, 1\} \leqslant \dim_H((\sqrt{5}, S) \cap M), \tag{2.3}$$

Therefore we deduce that $\dim_H K \geq 0.5$ implies $t_1 \leqslant S$.

In order to illustrate this methodology we shall show the double inequality (1.3).

Example 2.2. To establish the lower bound, we can use a result by Hall [4] stating that if $\alpha \in \{1,2\}^{\mathbb{Z}}$ doesn't contain the string 121, then $m(\alpha) < \sqrt{10}$. So we choose

$$K := \{[0; \alpha_1, \alpha_2, \dots] \mid \alpha_j \in \{1,2\},\ \text{and } (\alpha_j\alpha_{j+1}\alpha_{j+2}) \neq (121) \text{ for all } j \geq 1\},$$

and apply the algorithm from §4 to show that $\dim_H K < 0.45$. Then (2.2) gives

$$\dim_H \left(\left\{\alpha \in \{1,2\}^{\mathbb{Z}} : \sqrt{5} < m(\alpha) \leqslant \sqrt{10}\right\}\right) \leq 2 \dim_H K \leq 0.9$$

and the lower bound $t_1 \geq \sqrt{10}$ follows.

To establish the upper bound, we recall a result by Perron [14] which states that $m(\alpha) \leqslant \sqrt{12}$ if and only if $\alpha \in \{1,2\}^{\mathbb{Z}}$. Therefore we may choose the empty set of forbidden strings and $K = E_2$. Combining (2.1) with (2.3) we get

$$\dim_H((\sqrt{5}, \sqrt{12}) \cap M) \geq \min\{2 \cdot \dim_H E_2, 1\} = \min(2 \cdot 0.54318, 1) = 1,$$

and conclude that $t_1 \leq \sqrt{12}$.

This approach has been used by Bumby to obtain heuristic estimates and we will review it in detail below. Since we already know that $m(\alpha) \leq \sqrt{12}$ if and only if $\alpha \in \{1,2\}^{\mathbb{Z}}$, until the end of §2 we study only sequences of 1s and 2s.

2.2. **Computation of the set of forbidden strings.** In this section we explain how to find a suitable set of forbidden strings which can be employed to define a set K to use in (2.2) or (2.3).

Recall the map λ introduced in §1

$$\lambda_0 \colon \mathbb{N}^{\mathbb{Z}} \to \mathbb{R}, \qquad \lambda(\alpha) = [\alpha_0; \alpha_1, \alpha_2, \dots] + [0; \alpha_{-1}, \alpha_{-2}, \dots].$$

On the one hand, it is clear from definition that $m(\alpha) \geq \lambda_0(\alpha)$. On the other hand, it is a well known fact that for any Markov value $m \in M$, there exists a sequence α such that $\lambda_0(\alpha) = m$; see for instance [2, Lemma 6, Chapter 1]. Therefore, one can attempt to construct a suitable set of forbidden strings by studying the function λ_0. This brings us to introducing a function J, which associates to a finite string a closed interval.

In the sequel we will use the following shorthand notation for certain finite substrings of a string α: $\alpha_{-k,j} = \alpha_{-k} \dots \alpha_{-1}\alpha_0\alpha_1 \dots \alpha_j$, where $j, k \geq 0$.

Definition 2.3. We denote by $J(\alpha_{-k,j})$ the interval given by the convex hull of the set of values $\lambda_0(\beta)$ for strings $\beta \in \{1,2\}^{\mathbb{Z}}$ such that $\beta_n = \alpha_n$ for all $-k \leqslant n \leqslant j$.

Let us denote by $\overline{\alpha}$ the periodic sequence obtained by infinite repetition of a given finite string α. The following technical Lemma allows one to compute the interval $J(\alpha_{-k,j})$ explicitly.

Lemma 2.4. *For any sequence $\alpha \in \{1,2\}^{\mathbb{Z}}$ we have an upper bound*

$$\lambda_0(\alpha) \leq \begin{cases} [\alpha_0; \alpha_1 \dots \alpha_j \overline{12}] + [0; \alpha_{-1} \dots \alpha_{-k} \overline{12}], & \text{if } k \text{ and } j \text{ are even,} \\ [\alpha_0; \alpha_1 \dots \alpha_j \overline{21}] + [0; \alpha_{-1} \dots \alpha_{-k} \overline{12}], & \text{if } k \text{ is even and } j \text{ is odd,} \\ [\alpha_0; \alpha_1 \dots \alpha_j \overline{12}] + [0; \alpha_{-1} \dots \alpha_{-k} \overline{21}], & \text{if } k \text{ is odd and } j \text{ is even,} \\ [\alpha_0; \alpha_1 \dots \alpha_j \overline{21}] + [0; \alpha_{-1} \dots \alpha_{-k} \overline{21}], & \text{if } k \text{ and } j \text{ are odd;} \end{cases}$$

and a lower bound

$$\lambda_0(\alpha) \geq \begin{cases} [\alpha_0; \alpha_1 \dots \alpha_j \overline{21}] + [0; \alpha_{-1} \dots \alpha_{-k} \overline{21}], & \text{if } k \text{ and } j \text{ are even,} \\ [\alpha_0; \alpha_1 \dots \alpha_j \overline{12}] + [0; \alpha_{-1} \dots \alpha_{-k} \overline{21}], & \text{if } k \text{ is even and } j \text{ is odd,} \\ [\alpha_0; \alpha_1 \dots \alpha_j \overline{21}] + [0; \alpha_{-1} \dots \alpha_{-k} \overline{12}], & \text{if } k \text{ is odd and } j \text{ is even,} \\ [\alpha_0; \alpha_1 \dots \alpha_j \overline{12}] + [0; \alpha_{-1} \dots \alpha_{-k} \overline{12}], & \text{if } k \text{ and } j \text{ are odd.} \end{cases}$$

Proof. This follows immediately from the fact that $\inf E_2 = [0; \overline{21}] = \frac{1}{2}(\sqrt{3} - 1)$ and $\sup E_2 = [0; \overline{12}] = \sqrt{3} - 1$, where $\overline{21}$ and $\overline{12}$ represent infinite sequences of alternating 1s and 2s. □

This Lemma also allows us to establish two more properites of the function J which will be useful for our analysis.

(1) The function J is invariant under reversal of the string (note that reversal keeps the 0'th place unchanged).
(2) Extensions of a string $\alpha_{-k,j}$ correspond to subintervals of $J(\alpha_{-k,j})$.

$$\begin{aligned} J(\alpha_{-k,j}) &= J(1\alpha_{-k} \dots \alpha_j) \cup J(2\alpha_{-k} \dots \alpha_j) \\ &= J(\alpha_{-k} \dots \alpha_j 1) \cup J(\alpha_{-k} \dots \alpha_j 2); \end{aligned}$$

Observe that unions need not to be disjoint.

The second property allows us to organise the intervals obtained from continuations of a given string in a binary tree, so that the union of children is equal to the parent.

Now we can describe a recursive process for the construction of sets of forbidden strings. A basic idea is that we fix a threshold T, close to a conjectured lower bound on t_1 and look for finite strings $\alpha_{-k,j}$ such that the corresponding intervals $J(\alpha_{-k,j})$ lie to the *right* of T. We call these finite strings "forbidden" and obtain the Cantor set K by removing from E_2 all numbers whose continued fraction expansion contains a forbidden substring. If an interval $J(\alpha_{-k,j})$ lies to the *left* of T, we make a record of its right end point as a possible upper bound on t_1. If $T \in J(\alpha_{-k,j})$ then we subdivide the interval into two by adding an extra symbol to $\alpha_{-k,j}$ either in the beginning or at the end and study these two new intervals at the next step of the recursive process. When we find a new forbidden substring, we recompute the Hausdorff dimension of the updated set K. We may need to lower the original threshold, if $\dim_H K > 0.5$ *and* the right end points of the intervals which lie to the left of T_1 is too large; on the other hand, we may need to increase the original threshold if $\dim_H K < 0.5$ and we *look to improve* an existing lower bound. We terminate the recursion when we find two sets which are suitable to confirm lower and upper bounds on t_1 using (2.2) and (2.3) respectively.

2.2.1. *Analysis of Bumby's cuts near* 3.33438. In preparation for our estimate for t_1 we will first rigorously confirm bounds close to the heuristic values of Bumby [1]. This analysis will be an integral part of our subsequent improved estimates.

Following Bumby, keeping in mind the heuristic estimate $3.33437 < t_1 < 3.33440$ which we would like to confirm rigorously, let us fix the threshold

$$T_1 = 3.334369.$$

We are ready to start the recursive process of computing the set of forbidden strings. We use the asterisk to mark the zeroth place in the string, i.e. 22^*1 corresponds to $\alpha_{-1} = 2$, $\alpha_0 = 2$, $\alpha_1 = 1$.

We begin with a simple observation that a sequence $\alpha \in \{1,2\}^{\mathbb{Z}}$ with $3 < \lambda_0(\alpha) < \sqrt{12}$ satisfies $\alpha_0 = 2$. (Since by Lemma 2.4 we get $J(1^*) = [\sqrt{3}, \sqrt{12}-1]$ and $J(2^*) = [\sqrt{3}+1, \sqrt{12}]$.) We may now consider two continuations 2^*2 and 2^*1. Applying Lemma 2.4 again we compute

$$1+\sqrt{3} = [2; 2\overline{12}] + [0; \overline{21}] \leqslant \lambda_0(2^*2) \leqslant [2; 2\overline{21}] + [0; \overline{12}] = 2 + \frac{2}{\sqrt{3}}.$$

We conclude that $J(22^*) = J(2^*2) \subset [1+\sqrt{3}, 3.155] < T_1$ and proceed to analyse continuations of $x_{-1}x_0^*x_1 = 12^*1$.

The process of constructing the set of forbidden strings is depicted in Figure 1. We begin at the root marked 2^* and follow two edges up adding letters marked by the bar symbol in the beginning, as a prefix, and letters marked by the hat symbol at the end, as a suffix. Thus every vertex corresponds to a finite string and we compute the interval corresponding to this string in order to decide how to proceed further. Starting from the root, after two steps we arrive at a vertex which corresponds to 12^*1. Taking two steps further we obtain 212^*12 which is our first excluded string because by Lemma 2.4, the corresponding interval is $J(212^*12) \subset [3.4, \sqrt{12}] > T_1$.

The intervals corresponding to the vertices of the tree which are crucial to our analysis are recorded in Table 1. For completeness in §§2.2.2–2.2.10 we give details of our analysis. All intervals are computed using Lemma 2.4.

2.2.2. *Exclusion of* 21212. Note that

$$J(112^*11) \subset [3.1547, 3.268], \quad J(112^*12) \subset [3.28, 3.3661], \quad J(212^*12) \subset [3.4, \sqrt{12}].$$

Since $3.268 < T_1 < 3.4$, we exclude 212^*12 and we analyse the continuation of 212^*11. For this purpose, we decompose $J(112^*12)$ into $J(1112^*12)$ and $J(2112^*12)$.

Note that

$$J(2112^*12) \subset [3.2802, 3.3193] \quad \text{and} \quad J(1112^*12) \subset [3.3149, 3.3661].$$

Thus, it suffices to study the continuations $x = 1112^*12$ (as $T_1 > 3.3193$).

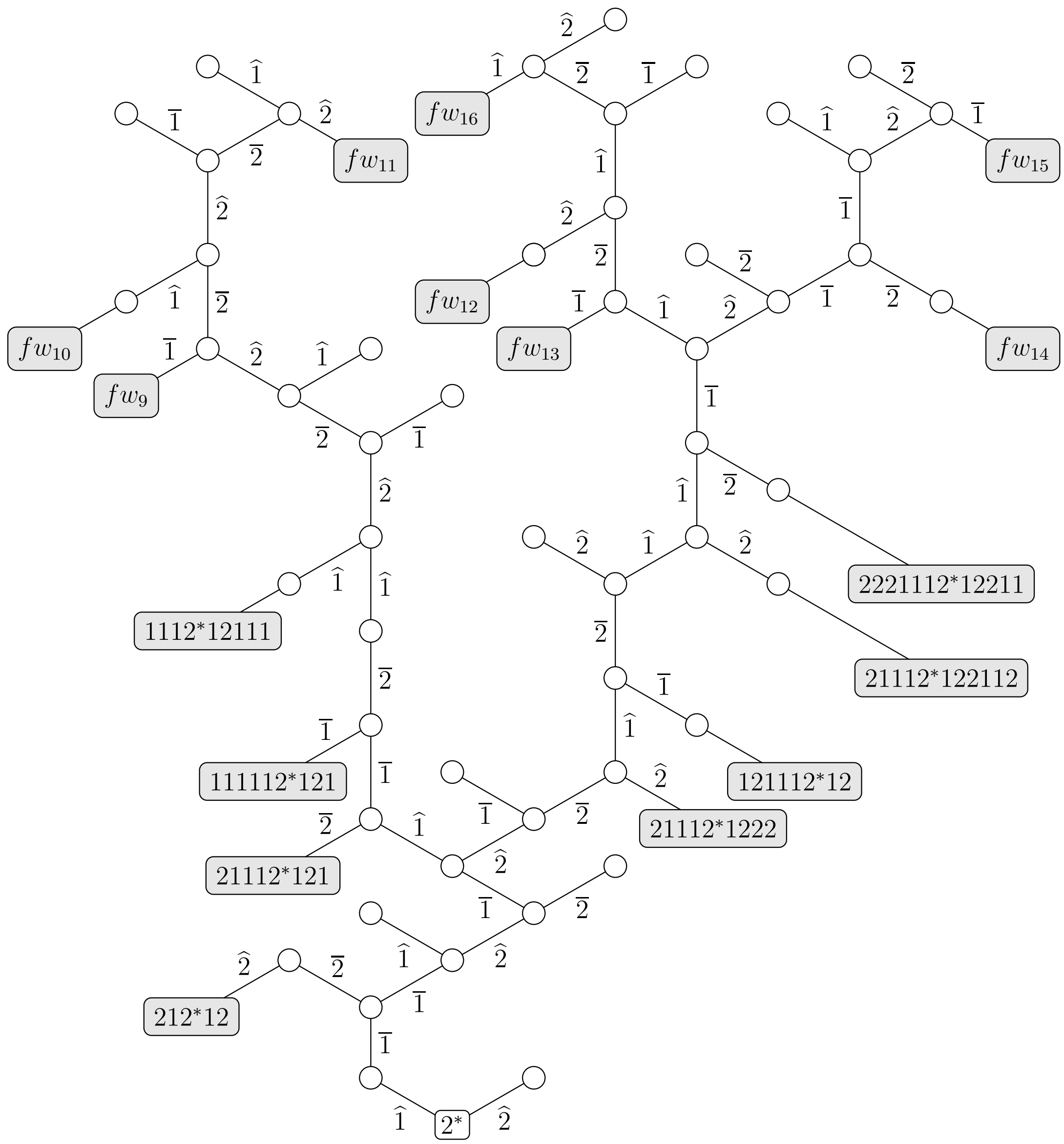

FIGURE 1. The tree depicting the process of constructing forbidden strings. Each vertex corresponds to a finite string of 1s and 2s, which can be recovered by going down to the root writing the labels along the path marked by bars and going up to the vertex writing the labels marked by hats. The short forbidden words are written explicitly, the longer ones abbreviated as fw_j, $j = 9, \dots, 14$. A dashed edge without a label leading to a forbidden word means that the forbidden word corresponds to a longer interval than the one which corresponds to the vertex it is connected to. The vertices R_1, R_2, R_3 will be used as the roots for new trees which we build in Section 2.3 in order to prove our first main result, Theorem 2.1.

Set	Vertex	String $\alpha_{-k,j}$	Interval $J_0 \supset J(\alpha_{-k,j})$	Action
B_1		212*12	$[3.4, \sqrt{12}]$	E
	1	112*11	$[3.1547, 3.268]$	A
	2	112*12	$[3.28, 3.3661]$	S
	3	2112*12	$[3.2802, 3.3193]$	A
	4	1112*12	$[3.3149, 3.3661]$	S
	5	11112*121	$[3.3324, 3.3524]$	S
		21112*121	$[3.35, 3.3661]$	E
	6	11112*122	$[3.3149, 3.3282]$	A
		21112*1222	$[3.337, 3.3419]$	E
	7	21112*1221	$[3.3329, 3.3389]$	S
		111112*121	$[3.3376, 3.3524]$	E
	8	211112*121	$[3.3324, 3.3456]$	S
		121112*12	$[3.3353, 3.3661]$	E
	9	221112*1221	$[3.3329, 3.3356]$	S
		1112*12111	$[3.3351, 3.3588]$	E
	10	221112*12211	$[3.3341, 3.3356]$	S
	11	221112*12212	$[3.33294, 3.33397]$	A
	12	211112*12112	$[3.3324, 3.3348]$	S
	13	1211112*12112	$[3.3324, 3.3339]$	A
		21112*122112	$[3.3347, 3.3389]$	E
	14	2211112*12112	$[3.33369, 3.33426]$	S
		2221112*12211	$[3.33469, 3.335541]$	E
	fw_9	12211112*121122	$[3.33448, 3.33472]$	E
	fw_{10}	2211112*1211221	$[3.33441, 3.33472]$	E
	fw_{13}	11221112*1221111	$[3.33447, 3.334684]$	E
	15	21221112*1221112	$[3.33414, 3.33424]$	A
	16	111221112*1221112	$[3.3343, 3.334393]$	S
	fw_{14}	211221112*12211	$[3.3343894, 3.3352]$	E
	17	21221112*12211111	$[3.3343, 3.334402]$	S
	fw_{12}	21112*12211112	$[3.3344009, 3.3384]$	E
	18	122211112*1211222	$[3.334335, 3.334375]$	S
	R_1	222211112*12112221	$[3.334371, 3.3343876]$	E
	fw_{11}	222211112*12112222	$[3.3343899, 3.33441]$	E
B_2	19	111221112*12211121	$[3.334304, 3.334363]$	A
	20	2111221112*12211122	$[3.33434, 3.334362]$	A
	R_2	1111221112*12211122	$[3.3343695, 3.334393]$	E
	21	1121221112*12211111	$[3.334325, 3.334373]$	S
	22	221221112*122111112	$[3.33435, 3.3343683]$	A
	R_3	221221112*122111111	$[3.334378, 3.33441]$	E

TABLE 1. Strings and intervals crucial to our analysis. The action column indicates how to proceed with the tree construction further: E — Exclude the string (the case $J_0 > T_1 = 3.334369$), the corresponding vertex is a leaf; A — Abandon the branch (the case $J_0 < T_1$) the corresponding vertex is a leaf; S — Subdivide the interval into two parts (the case $T_1 \in J_0$), the vertex is a branching point. An estimate for $J(211221112^*12211)$ used the fact that 21112121 is excluded, in addition to Lemma 2.4. There is only one branch out of the 8th vertex because 2^*1212 is already excluded, so the only possible suffix is $\widehat{1}$.

2.2.3. *Exclusion of* 21112121 *and* 211121222. Let's consider the decompositions of the intervals $J(1112^*121)$ and $J(1112^*122)$ where the string 21212 doesn't appear. For the first interval, it amounts to studying $J(11112^*121)$, $J(21112^*121)$. Note that

$$J(11112^*121) \subset [3.3324, 3.3524] \quad \text{and} \quad J(21112^*121) \subset [3.35, 3.3661].$$

For the second interval, we have

$$J(21112^*1222) \subset [3.337, 3.3419], \qquad J(11112^*1222) \subset [3.3189, 3.3282]$$

$$J(21112^*1221) \subset [3.3329, 3.3389], \qquad J(11112^*1221) \subset [3.3149, 3.3252].$$

Since $3.3282 < T_1 < 3.337$, we exclude 21112121 and 211121222, and we shall consider the decompositions of the intervals $J(11112^*121)$ and $J(21112^*1221)$.

2.2.4. *Exclusion of* 111112121 *and* 12111212. Note that

$$J(111112^*121) \subset [3.3376, 3.3524], \quad J(211112^*121) \subset [3.3324, 3.3456],$$

and

$$J(121112^*1221) \subset J(121112^*12) \subset [3.3353, 3.3661], \quad J(221112^*1221) \subset [3.3329, 3.3356].$$

Because $T_1 < 3.3353$, we exclude 111112121 and 12111212, and we consider the decompositions of $J(211112^*121)$ and $J(221112^*1221)$. Actually, given that the string 21212 is already excluded, our task is to study the decompositions of $J(211112^*1211)$ and $J(221112^*1221)$.

2.2.5. *Exclusion of* 1111212111. Observe that

$$J(211112^*12111) \subset J(1112^*12111) \subset [3.3351, 3.3588], \quad J(211112^*12112) \subset [3.3324, 3.3348],$$

and

$$J(221112^*12212) \subset [3.33294, 3.33397], \quad J(221112^*12211) \subset [3.3341, 3.3356].$$

Because $3.33397 < T_1 < 3.3351$, we exclude 111212111 and we decompose $J(211112^*12112)$ and $J(221112^*12211)$.

2.2.6. *Exclusion of* 21112122112. Note that $J(211112^*12112)$ decomposes into $J(2211112^*12112)$ and

$$J(1211112^*12112) \subset [3.3324, 3.3339].$$

Similarly, $J(221112^*12211)$ decomposes into $J(221112^*122111)$ and

$$J(221112^*122112) \subset J(21112^*122112) \subset [3.3347, 3.3389].$$

Since $3.3339 < T_1 < 3.3347$, we exclude 21112122112, and we decompose $J(2211112^*12112)$ and $J(221112^*122111)$.

2.2.7. *Exclusion of* 222111212211. Note that $J(2211112^*12112)$ breaks into $J(2211112^*121122)$ and

$$J(2211112^*12112) \subset [3.33369, 3.33426]$$

Analogously, $J(221112^*122111)$ decomposes into $J(1221112^*122111)$ and

$$J(2221112^*122111) \subset J(2221112^*12211) \subset [3.33469, 3.335541].$$

Given that $3.33426 < T_1 < 3.33469$, we exclude 222111212211, and we proceed to analyse the decompositions of $J(2211112^*121122)$ and $J(1221112^*122111)$.

2.2.8. *Exclusion of* 12211112121122. We break the previous intervals into $J(12211112^*121122)$, $J(22211112^*121122)$ and $J(1221112^*1221111)$, $J(1221112^*1221112)$, and we observe that

$$J(12211112^*121122) \subset [3.33448, 3.33472].$$

Because $T_1 < 3.33448$, we exclude 12211112121122, and we decompose $J(22211112^*121122)$, $J(1221112^*1221111)$ and $J(1221112^*1221112)$.

2.2.9. *Exclusion of two extra strings.* We decompose the interval $J(22211112^*121122)$ into $J(22211112^*1211222)$ and

$$J(22211112^*1211221) \subset J(2211112^*1211221) \subset [3.33441, 3.33472].$$

Similarly, $J(1221112^*1221111)$ subdivides into $J(21221112^*1221111)$ and

$$J(11221112^*1221111) \subset [3.33447, 3.334684].$$

Analogously, $J(1221112^*1221112)$ breaks into $J(11221112^*1221112)$ and

$$J(21221112^*1221112) \subset [3.33414, 3.33424].$$

Because $3.33424 < T_1 < 3.33441$, we exclude 112211121221111 and 22111121211221, and we analyse $J(22211112^*1211222)$, $J(21221112^*1221111)$ and $J(11221112^*1221112)$.

2.2.10. *Exclusion of three extra strings.* We decompose the interval $J(11221112^*1221112)$ into

$$J(111221112^*1221112) \subset [3.3343, 3.334393],$$

$$J(211221112^*1221112) \subset J(211221112^*12211) \subset [3.3343894, 3.3352].$$

(Here, we estimated the second interval using the fact that 21112121 is excluded.)

Similarly, we break $J(21221112^*1221111)$ into

$$J(21221112^*12211111) \subset [3.3343, 3.334402],$$

$$J(21221112^*12211112) \subset J(21112^*12211112) \subset [3.3344009, 3.3384].$$

Finally, we observe that $J(22211112^*1211222)$ subdivides into $J(122211112^*1211222)$, $J(222211112^*12112221)$, $J(222211112^*12112222)$ with

$$J(122211112^*1211222) \subset [3.334335, 3.334375],$$

$$J(222211112^*12112221) \subset [3.334371, 3.3343876],$$

and

$$J(222211112^*12112222) \subset [3.3343899, 3.33441].$$

Since $T_1 < 3.3343894$, we exclude 21122111212211, 2111212211112 and 22221111212112222.

2.2.11. *Upper bound on t_1 revisited.* Using numerical data from the top part of Table 1 we are now in a position to get an upper bound on t_1 in line with the heuristic estimate of 3.33440 suggested by Bumby. Denote by B_1 the Cantor set of numbers whose continued fraction expansions in $\{1, 2\}^{\mathbb{N}}$ which do not contain the following fourteen strings (nor their transposes) taken from the lines of Table 1 marked for exclusion:

- 21212, 21112121, 211121222, 111112121, 12111212, 111212111, 21112122112,
- 222111212211, 12211112121122, 112211121221111, 22111121211221,
- 21122111212211, 2111212211112 and 22221111212112222.

The algorithm described in Section 4 provides us lower and upper bounds (see Subsection 4.6.1 for numerical data and implementation notes)

$$0.50001 < \dim B_1 < 0.50005 \tag{2.4}$$

which confirms Bumby's heuristics in [1]. Consequently, applying (2.3) we get that t_1 is bounded from above by the maximum of the right endpoints of the non-excluded intervals that appeared in the process of construction of the set B_1 (both abandonned and marked for subdivision). This turns out to be the right end point of the interval corresponding to the vertex 17. In particular, we have that

$$t_1 \leqslant S_1 = 3.334402.$$

2.2.12. *Lower bound on t_1 revisited.* With a little more work we can get a lower bound on t_1 which supports Bumby's lower bound on t_1 of 3.33437. Continuing to follow Bumby [1], let us further analyse the intervals

$$J(21221112^*12211111) \text{ and } J(111221112^*1221112),$$

which correspond to the 16th and 17th vertices of the tree and marked for subdivision in Table 1. Our computations are presented in the bottom part of Table 1. In particular, we see that one can also exclude 111122111212211122 and 221221112122111111 in order to obtain a smaller Cantor set $B_2 \subsetneq B_1$. Applying the algorithm for computing Hausdorff dimension described in §4 we obtain estimates on dimension (see §4.6.2 for implementation notes):

$$0.499975 < \dim_H B_2 < 0.49999 \tag{2.5}$$

This is quite close to Bumby's heuristic claim that $\dim_H(B_2) < 0.499974$ and we conclude that

$$t_1 \geqslant T_1 = 3.334369.$$

Summing up, we have rigorously confirmed that the heuristic argument by Bumby in favour of looking for t_1 inside the interval $(3.33437, 3.33440)$ was correct.

After the above review of Bumby's work [1], we now turn to the proof of our main result Theorem 2.1.

2.3. **Proof of Theorem 2.1.** Recall that our goal is to show that the first transition point $t_1 = 3.334384\ldots$. It is sufficient to prove that

$$3.3343840 < t_1 < 3.33438495.$$

For this purpose, let us fix the thresholds

$$T_2 := 3.334384009 \quad \text{and} \quad S_2 := 3.3343849341. \tag{2.6}$$

Our goal now is to modify the Cantor sets B_1 and B_2 defined above to obtain two Cantor sets X and Y such that the intervals corresponding to forbidden strings used to define X lie to the right of T_2 and S_2 is the right end point of the intervals corresponding to the non-excluded strings which appear in the construction of Y. Furthermore, we also require that the double inequality $\dim_H X < 0.5 < \dim_H Y$ holds.

In this direction we consider the intervals listed in Table 1 and choose the smallest (by inclusion) intervals which contain both T_2 and S_2 in order to subdivide them further and to identify forbidden strings exclusion of which will result in Cantor sets with dimension closer to 0.5 than $\dim_H B_1$ and $\dim_H B_2$. These turn out to be the intervals corresponding to the vertices R_1, R_2, and R_3. We list the corresponding strings: $R_1 = 222211112^*12112221$, $R_2 = 1111221112^*12211122$, and $R_3 = 221221112^*122111111$.

String $\alpha_{-k,j}$	Interval $J_0 \supset J(\alpha_{-k,j})$	Action
$R_1 2$	$[3.334371, 3.334381]$	A
$2 R_1 1$	$[3.334376, 3.33438141]$	A
$1 R_1 12$	$[3.334384049, 3.3343876]$	E
$11 R_1 11$	$[3.334381, 3.3343837]$	A
$121 R_1 1$	$[3.334384009, 3.3343876]$	E
$221 R_1 112$	$[3.33438368, 3.33438401]$	S
$21 R_1 111$	$[3.3343844, 3.33438551]$	E

TABLE 2. Numerical data for the subdivision of the interval $J(R_1) = J(222211112^*12112221)$. The corresponding tree is shown in Figure 2. Strings corresponding to the intervals to the right of $T_2 = 3.334384009$ marked for exclusion.

FIGURE 2. Continuation of the string $R_1 = 222211112^*12112221$.

String $\alpha_{-k,j}$	Interval $J_0 \supset J(\alpha_{-k,j})$	Action
$1 R_2$	$[3.334369, 3.33438361]$	A
$2 R_2 1$	$[3.33438668, 3.33439261]$	E
$2 R_2 21$	$[3.3343815, 3.3343847]$	S
$12 R_2 22$	$[3.33438429, 3.3343856]$	E

TABLE 3. Numerical data for the subdivision of the interval $J(R_2) = J(1111221112^*12211122)$. The subdivision tree is shown in Figure 3. Strings corresponding to the intervals to the right of $T_2 = 3.334384009$ marked for exclusion.

We subdivide each of the intervals $J(R_1)$, $J(R_2)$, and $J(R_3)$ following the same process as before, with a separate decision tree in each case.

2.3.1. *Refinement of* $J(222211112^*12112221)$. The tree depicting continuation of the string $R_1 = 222211112^*12112221$ is shown in Figure 2 and the numerical data for the key intervals is given in Table 2 (obtained using Lemma 2.4). Three extra strings are marked for exclusion, namely $1R_112$, $121R_11$, and $21R_1111$.

2.3.2. *Refinement of* $J(1111221112^*12211122)$. The tree depicting continuation of the string $R_2 = 1111221112^*12211122$ is shown in Figure 3 and the numerical data for the key intervals is given in Table 3. Based on the threshold T_2 we exclude $2R_21 = 21111221112^*122111221$ and $12R_222 = 121111221112^*1221112222$.

2.3.3. *Refinement of* $J(221221112^*122111111)$. The tree depicting continuation of the string $R_3 = 221221112^*122111111$ is shown in Figure 4 and the numerical data for the

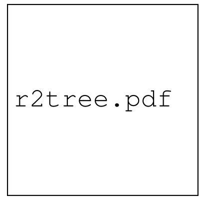

FIGURE 3. Continuation of the string $R_2 = 1111221112^*12211122$.

String $\alpha_{-k,j}$	Interval $J_0 \supset J(\alpha_{-k,j})$	Action
$R_3 2$	$[3.33439, 3.334402]$	E
$21\,R_3$	$[3.3343856, 3.334402]$	E
$1\,R_3 11$	$[3.3343866, 3.3343922]$	E
$211\,R_3 12$	$[3.334383, 3.33438429]$	S
$111\,R_3 121$	$[3.33438375, 3.334384636]$	S
$111\,R_3 122$	$[3.3343846357, 3.3343853]$	E
$12\,R_3 1$	$[3.334378, 3.33438459]$	S
$22\,R_3 12$	$[3.334379, 3.3343806]$	S
$22\,R_3 111$	$[3.3343829, 3.33438403]$	S
$22\,R_3 112$	$[3.3343847, 3.3343855]$	E

TABLE 4. Numerical data for the subdivision of the interval $J(R_3) = J(221221112^*122111111)$. The corresponding tree is shown in Figure 4. Strings corresponding to the intervals to the right of $T_2 = 3.334384009$ marked for exclusion.

FIGURE 4. Continuation of the string $R_3 = 1111221112^*12211122$.

key intervals is shown in Table 4. Based on the threshold T_2 five additional strings are marked for exclusion: $R_3 2$, $21R_3$, $1R_3 11$, $111R_3 122$, and $22R_3 112$

2.3.4. *Lower bound on t_1.* In order to confirm the lower bound stated in Theorem 2.1, we collect together numerical data from calculations in §§2.3.1–2.3.3. Consider the Cantor set $X \subset E_2$ of numbers which continued fraction expansions do not contain neither any of the following 24 strings nor their transposes:

- The 14 words proposed by Bumby, listed in §2.2.11, cf. Table 1: 21212, 21112121, 211121222, 111112121, 12111212, 111212111, 21112122112, 222111212211, 12211112121122, 112211121221111, 22111121211221, 21122111212211, 2111212211112, 22221111212112222 ;
- The 3 words obtained in §2.3.1 as continuations of R_1: 12222111121211222112, 121222211112121122211, and 2122221111212112221111;

- The 2 words obtained in §2.3.2 as continuations of R_2: 211112211121221112 21, 12111122111212211 12222;
- The 5 words obtained in §2.3.3 as continuations of R_3: 2212211121221111112, 21221221112122111111, 12212211121221111111 1, 1112212211121221111111122, 2222122111212211111111 2

In Subsection 4.6.3 the algorithm described in [15] will be implemented to rigorously establish the bound $\dim_H X < 0.5 - 10^{-8}$. Summing up, we get the desired lower bound

$$3.334384009 = T_2 \le t_1. \tag{2.7}$$

2.3.5. *Upper bound on* t_1. We are now ready to justify the upper bound $t_1 \le S_2 = 3.3343849341$ proposed in Theorem 2.1. Following the method explained in §2.1.2, we need to modify the set X, increasing its dimension, so that the right end point of a non-excluded interval is no smaller than S_2. Therefore from the intervals marked for exclusion in Tables 2, 3, 4 we choose the shortest ones which contain S_2. These turn out to be the intervals corresponding to the strings

$$1R_112 = 1222211112^*1211222112, \qquad 221R_1111 = 221222211112^*12112221111,$$
$$12R_222 = 121111221112^*1221112222, \quad 121R_111 = 121222211112^*1211222111.$$

We proceed to study their subintervals applying Lemma 2.4 while excluding all intervals to the right of the value

$$T_3 := 3.3343846357 \in (T_2, S_2).$$

The analysis of the first interval $J(1R_112)$ is relatively simple. More precisely, it breaks into

$$J(2\,1R_112) \subset [3.334386, 3.3343876] > T_3, \text{ and} \tag{2.8}$$
$$J(11\,1R_112) \subset [3.33438473, 3.3343858] > T_3 \tag{2.9}$$
$$J(21\,1R_112) \subset [3.334384049, 3.33438484] < T_3.$$

Following the approach explained in the beginning of §2.2, we exclude the string $21R_1112$ corresponding to the first of them, since every element is larger than T_3.

Similarly, $J(12R_222)$ breaks into

$$J(1\,12R_222) \subset [3.33438429, 3.3343849341] < T_3, \text{ and} \tag{2.10}$$
$$J(2\,12R_222) \subset [3.3343851, 3.3343856] > T_3. \tag{2.11}$$

We exclude the string $212R_222$ corresponding to the second interval, since it lies to the right of $T_3 = 3.3343846357$.

The third interval $J(221R_1111)$ subdivides into

$$J(221R_1111\,1) \subset [3.33438448, 3.334384762] \ni T_3, \text{ and}$$
$$J(221R_1111\,2) \subset J(21R_11112) \subset [3.33438488, 3.33438551] > T_3. \tag{2.12}$$

Finally, $J(121R_111)$ decomposes as

$$J(121R_111\,1) \subset [3.3343848, 3.3343856] > T_3 \tag{2.13}$$
$$J(121R_111\,2) \subset [3.334384009, 3.33438445] < T_3.$$

We may now define Y to be the Cantor set of continued fraction expansions in $\{1,2\}^{\mathbb{N}}$ which do not contain the following 25 strings (nor their transposes):

- The 14 words composed by Bumby, listed in §2.2 cf. Table 1: 21212, 21112121, 211121222, 111112121, 12111212, 111212111, 21112122112, 222111212211, 122111121211 22, 112211121221111, 22111121211221, 21122111212211, 2111212211112, 22221111212112222;
- The 2 words constructed as continuations of R_2: $2R_21 = 21111221112122111221$ (cf. Table 3) and $212R_222 = 21211112211121221112222$ (see (2.11) above),
- The 5 words obtained in §2.3.3 as continuations of R_3 : 2212211121221111112, 21221221112122111111, 122122111212211111111, 11122122111212211111122, 2222122111212211111112 (cf. Table 4 for numerical data on the intervals);
- The 4 words composed as continuations of R_1: $21R_112 = 212222111121211222112$ (2.8), $111R_112 = 1112222111121211222112$ (by (2.9)), $121R_1111 = 1212222111121211222111 1$ (by (2.13)), and $21R_11112 = 2122222111121211222111 12$ (by (2.12)).

then the fact that a rigorous estimate in Subsection 4.6.4 gives that $\dim_H Y > 0.5 + 10^{-8}$ allows to conclude that

$$t_1 \leq S_2 = 3.3343849341, \tag{2.14}$$

which is the right end point of the non-excluded interval corresponding to $112R_222 = 1121111221112^*1221112222$ (see (2.10)).

The inequalities (2.7) and (2.14) complete the proof of Theorem 2.1.

3. Bounds on $\dim_H(M \setminus L)$

In this section we establish our second main result

Theorem 3.1. *The Hausdorff dimension of the difference of Markov and Lagrange spectra satisfies*

$$0.537152 < \dim_H(M \setminus L) < 0.796445$$

3.1. **Lower bounds.** It was shown in [8, §2.5.4] that $\dim_H((M \setminus L) \cap (3.7, 3.71))$ coincides with the dimension of a certain Gauss–Cantor set Ω with complicated structure. Implementing the algorithm described in §4, we obtain an estimate $\dim_H \Omega = 0.537152\dots$ (see § 4.6.5 for computation notes). A combination of these two results gives the best lower bound on $M \setminus L$ so far:

$$\dim_H((M \setminus L) \cap (\sqrt{13}, 3.84)) \geqslant \dim_H \Omega > 0.537152.$$

3.2. **Upper bounds.** Recall that Freiman and Schecker independently showed circa 1973 that see, e.g. [2]

$$[\sqrt{21}, +\infty) = L \cap [\sqrt{21}, +\infty) = M \cap [\sqrt{21}, +\infty).$$

More recently, it was shown in [11] and [15] that $\dim_H((M \setminus L) \cap (\sqrt{5}, \sqrt{13})) < 0.73$. Hence in order to establish an upper bound of 0.796455, it suffices to study $M \setminus L$ within the interval $(\sqrt{13}, \sqrt{21})$.

Let us now set out the strategy which we will employ for the rest of this section. We consider a partition of $(\sqrt{13}, \sqrt{21})$ into several small intervals (x, y) and study the intersections $(M \setminus L) \cap (x, y)$. To find an upper bound for the Hausdorff dimension of $(M \setminus L) \cap (x, y)$, we continue to develop the ideas from [11].

Very roughly speaking, we select two transitive subshifts of finite type $B \subset C \subset (\mathbb{N}^*)^{\mathbb{Z}}$ with $m(\alpha) < x$ for all $\alpha \in B$ and any $\beta \in (\mathbb{N}^*)^{\mathbb{Z}}$ with $m(\beta) < y$ belongs to C. We require that B and C are symmetric in the sense that $K(B) = K^-(B)$ and $K(C) = K^-(C)$, where $K(A) := \{[0; \alpha_1, \alpha_2, \dots] \mid (\alpha_n)_{n \in \mathbb{Z}} \in A\}$ and $K^-(A) := \{[0; \alpha_{-1}, \alpha_{-2}, \dots] \mid$

$(\alpha_n)_{n\in\mathbb{Z}} \in A\}$ stand for the unstable and stable Gauss–Cantor sets associated to a given subshift of finite type $A \subset (\mathbb{N}^*)^{\mathbb{Z}}$.

At this stage, we want to employ a shadowing lemma type argument to get that, up to transposition, any sequence ζ with $m(\zeta) \in (M \setminus L) \cap (x, y)$ has the property that if N is large, $n \geqslant N$, τ is a finite string and α, α' are infinite strings with distinct first elements such that the two sequences $\dots\zeta_{-N}\dots\zeta_n\tau\alpha$ and $\dots\zeta_{-N}\dots\zeta_n\tau\alpha'$ have Markov values in $(M \setminus L) \cap (x, y)$, then the unstable Cantor set $K(B) = \{[0; \theta_1, \theta_2, \dots] : (\theta_n)_{n\in\mathbb{Z}} \in B\}$ of B doesn't intersect the interval $[[0; \alpha], [0; \alpha']]$. In particular, by taking $\tau = \varnothing$, the allowed continuations of ζ with $m(\zeta) \in (M \setminus L) \cap (x, y)$ live in a small "Cantor set" K_{gap} in the gaps of $K(B)$, so that $\dim_H((M \setminus L) \cap (x, y)) \leqslant \dim_H(K(C)) + \dim_H(K_{gap})$.

As it turns out, the rest of this section relies on the formalisation of the idea of the previous paragraph based on a version of Lemma 6.1 of [11].

Definition 3.2. Consider two transitive and symmetric subshifts of finite type $\Sigma(B) \subset \Sigma(C)$. Let $\alpha \in \Sigma(C)$ be a sequence with $m(\alpha) = \lambda_0(\alpha) = m \in M$. We say that α *connects positively* to B, if for every $k \in \mathbb{N}$ there exist a finite sequence τ and an infinite sequence $\upsilon \in \Sigma^+(B)$ such that for $\tilde{\alpha} := \dots\alpha_{-2}\alpha_{-1}\alpha_0\dots\alpha_k\tau\upsilon$ we have

$$m(\tilde{\alpha}) < m(\alpha) + 2^{-k}. \tag{3.1}$$

We say that α *connects negatively* to B if the reversed sequence α^t connects positively to B.

Remark 3.3. Observe that we can replace 2^{-k} in (3.1) by any sequence converging to zero, or, in other words, the inequality (3.1) can be replaced by

$$\lim_{k\to\infty} \inf_{\substack{\tau \text{ finite word in } C,\\ \upsilon\in\Sigma^+(B)}} m(\dots a_{-2}a_{-1}a_0^*\dots a_k\tau\upsilon) = m. \tag{3.2}$$

The following equivalent definition is slightly more elaborate, but more useful for our purposes.

Definition 3.2′. Let $\Sigma(B) \subset \Sigma(C)$ be two transitive and symmetric subshifts of finite type. We say that $\alpha \in \Sigma(C)$ connects positively to B if for every $k \in \mathbb{N}$ there exist a finite sequence τ and a pair of infinite sequences $\upsilon_C \in \Sigma^+(C)$ and $\upsilon_B \in \Sigma^+(B)$ such that the concatenation $\tilde{\alpha} := \upsilon_C^t\alpha_{-k,k}\tau\upsilon_B$ satisfies $m(\tilde{\alpha}) < m(\alpha) + 2^{-k}$.

The advantage of this more complicated alternative definition is that for each k the hypothesis is formulated in terms of the finite subsequence $\alpha_{-k,k}$. Notice that if α *does not* connect to B, then there exists a fixed positive value of k for which the condition above fails. In the sequel, instead of Lemma 6.1 in [11], we shall use the following statement.

Lemma 3.4. *Consider two transitive and symmetric subshifts of finite type $\Sigma(B) \subset \Sigma(C)$. Let x be such that $m(\beta) \leq x$ for all $\beta \in \Sigma(B)$. Suppose that a sequence $\gamma \in \Sigma(C)$ satisfies $m(\gamma) = \lambda_0(\gamma) = m > x$ and connects positively and negatively to B. Then $m \in L$.*

Proof. By Theorem 2 in Chapter 3 of Cusick–Flahive book [2], it is sufficient to show that $m = \lim\limits_{k\to\infty} m(P_k)$ where P_k is a sequence of periodic points in $\Sigma(C)$.

Since γ connects positively and negatively to B, there exist finite sequences $\tau, \widetilde{\tau}$ and infinite sequences $\upsilon, \widetilde{\upsilon} \in \Sigma^+(B)$ such that

$$m(\dots\gamma_{-2}\gamma_{-1}\gamma_0\dots\gamma_k\tau\upsilon) < m + 2^{-k} \text{ and } m(\widetilde{\upsilon}^t\widetilde{\tau}\gamma_{-k}\dots\gamma_0\gamma_1\gamma_2\dots) < m + 2^{-k}.$$

Let $\upsilon^k := \upsilon_1 \dots \upsilon_k$ and $\widetilde{\upsilon}^k := \widetilde{\upsilon}_k \dots \widetilde{\upsilon}_1$ be the segments of υ and $\widetilde{\upsilon}^t$ respectively. By transitivity of $\Sigma(B)$, there exists $\beta \in \Sigma(B)$ which contains non-overlapping occurencies of the strings υ^k and $\widetilde{\upsilon}^k$ in this order. Let us denote by $(\upsilon^k * \widetilde{\upsilon}^k)$ a finite substring of β which begins with υ^k and terminates with $\widetilde{\upsilon}^k$.

We next want to consider the periodic point $P_k \in \Sigma$ obtained by infinite concatenation of the finite block

$$\gamma_0 \dots \gamma_k \tau(\upsilon^k * \widetilde{\upsilon}^k)\gamma_{-k} \dots \gamma_{-1}.$$

Recall that for any finite sequence $\xi = \xi_1 \dots \xi_k$ of positive integers and for any pair of sequences $\alpha', \alpha'' \in (\mathbb{N}^*)^{\mathbb{N}}$ we have $|[0;\xi,\alpha'] - [0;\xi,\alpha'']| < 2^{1-k}$. Therefore for any $j \in \mathbb{Z}$ we get $\lambda_0(\sigma^j(P_k)) \leqslant m + 2^{2-k}$ and $\lambda_0(P_k) > \lambda_0(\gamma) - 2^{2-k} = m - 2^{2-k}$.

In particular, $m = \lim\limits_{k\to\infty} m(P_k)$. This completes the argument. □

Remark 3.5. Assume that $\Sigma(B) \subset \Sigma(C)$ are two transitive symmetric subshifts and let x be such that $m(\beta) \le x$ for all $\beta \in \Sigma(B)$. Consider a sequence $\gamma \in \Sigma(C)$ with $m(\gamma) = \lambda_0(\gamma) = m > x$. Then for any finite sequence τ and half-infinite sequence $\upsilon \in \Sigma^+(B)$ directly from definition of Lagrange and Markov numbers we get

$$\limsup_{j\to+\infty} \lambda_0(\sigma^j(\dots\gamma_{-2}\gamma_{-1}\gamma_0\dots\gamma_k\tau\upsilon)) < m,$$

where σ is the Bernoulli shift. Thus, if we want to get that $m(\dots\gamma_{-2}\gamma_{-1}\gamma_0\dots\gamma_k\tau\upsilon) < m + 2^{-k}$, then it suffices to check that

$$\lambda_0(\sigma^j(\dots\gamma_{-2}\gamma_{-1}\gamma_0\dots\gamma_k\tau\upsilon)) < m + 2^{-k}$$

for finitely many values of j, namely, for all $0 \le j \le k + |\tau| + l$ where l is sufficiently large (so that $2^{1-l} \le m - x + 2^{-k}$).

The following elementary fact is quite useful to us.

Lemma 3.6. *Let $\Sigma(B) \subset \Sigma(C)$ be two transitive symmetric subshifts. Assume that three half-infinite sequences $\beta^1, \beta^2, \beta^3 \in \Sigma^+(B)$ are such that $[0;\beta^1] < [0;\beta^2] < [0;\beta^3]$. Then for all $\alpha \in \Sigma(C)$ and for all $j \le n+1$*

$$\lambda_0\left(\sigma^j(\dots\alpha_{-2}\alpha_{-1}\alpha_0\dots\alpha_n\beta^2)\right) \le \max\left(m(\dots\alpha_{-2}\alpha_{-1}\alpha_0\dots\alpha_n\beta^1), m(\dots\alpha_{-2}\alpha_{-1}\alpha_0\dots\alpha_n\beta^3)\right).$$

We will use Lemmas 3.4 and 3.6 and Remark 3.5 in order to estimate Hausdorff dimensions of $(M \setminus L) \cap (x, y)$ in the following way. Recall that [2, Lemma 6, Chapter 1] for any $m \in M$ there exists a sequence α such that $\lambda_0(\alpha) = m(\alpha)$. Therefore to study $(M \setminus L) \cap (\sqrt{13}, \sqrt{21})$ we may consider

$$Y := \left\{\alpha \in \{1,2,3,4\}^{\mathbb{Z}} \mid m(\alpha) = \lambda_0(\alpha) \in M \setminus L\right\}.$$

In order to prove that $\dim_H(M \setminus L) \leqslant d$, it suffices to consider the cylinder sets $V_n(\alpha) := \left\{\tilde\alpha \in \{1,2,3,4\}^{\mathbb{Z}} \mid \tilde\alpha_j = \alpha_j,\ -n \le j \le n\right\}$ and to show that for every $\alpha \in Y$, there is $n \in \mathbb{N}$ such that

$$\dim_H(m(V_n(\alpha) \cap Y)) \le d.$$

In this direction, we will associate (see Tables 5 and 6) to an interval (x, y) two symmetric transitive subshifts of finite type $\Sigma(B) = \Sigma(B_x) \subset \Sigma(C) = \Sigma(C_y) \subset (\mathbb{N}^*)^{\mathbb{Z}}$ such that

- $m(\beta) < x$ for all $\beta \in \Sigma(B)$; and
- for all $\gamma \in (\mathbb{N}^*)^{\mathbb{Z}}$ such that $m(\gamma) < y$ we have $\gamma \in \Sigma(C)$.

If $m(\alpha) = \lambda_0(\alpha) = m \in M \setminus L$, then by Lemma 3.4, α doesn't connect neither positively nor negatively to B. Suppose without loss of generality that it doesn't connect positively to B. Then by Definition 3.2′ there exists $k \in \mathbb{N}$ such that, for any $N \geqslant k+2$, any finite sequence τ and infinite sequences $\upsilon_C \in \Sigma^+(C)$ and $\upsilon_B \in \Sigma^+(B)$ the concatenation $\widetilde{\alpha} = \upsilon_C^t \alpha_{-N,N} \tau \upsilon_B$ satisfies $m(\widetilde{\alpha}) \geq m + 2^{-k} \geqslant m + 2^{-N+2}$.

At this point, we will proceed as follows. In the remainder of this section, for each interval (x, y) introduced below, we will construct[3] a finite collection $X_1, \dots, X_r$ of finite sets of finite sequences over $\{1, 2, 3, 4\}$ with the following property: if $\mu < m < m + 2^{-N+2} < \nu$ and, for some $n \geq N$, a sequence $\upsilon_C^t \alpha_{-N,n}$ has continuations $\upsilon^1, \upsilon^2 \in \{1, 2, 3, 4\}^{\mathbb{N}}$ with *different* subsequent term (of index $n+1$) leading to Markov values which are smaller than $m + 2^{-N+2}$, then there is X_j (depending only on $\alpha_{-N,n}$) such that the initial segment of *any* $\upsilon \in \{1, 2, 3, 4\}^{\mathbb{N}}$ with the property that $m(\upsilon_C^t \alpha_{-N,n} \upsilon) < m + 2^{-N+2}$ belongs to X_j (these elements tend to live on gaps of $K(B)$).

Notice now that $V_N(\alpha) \cap Y$ is contained in the set of sequences $\beta = \upsilon_1^t \alpha_{-N,N} \upsilon_2$ such that $m(\beta) < m + 1/2^{N-2}$. Hence, if $s > 0$ is such that, for all $X_1, \dots, X_r$ and all positive integers $b_1, \dots, b_n$, we have

$$\sum_{\tau \in X_j} |I(b_1, \dots, b_n, \tau)|^s \leqslant |I(b_1, \dots, b_n)|^s,$$

where $I(a_1, \dots, a_k) = \{[0; a_1, \cdots, a_k, \rho] : \rho > 1\}$, then Markov values in $m(V_N(\alpha) \cap Y)$ belong to the arithmetic sum of $K(C)$ with a set K_{gap} whose Hausdorff dimension is at most s, and thus its Hausdorff dimension is at most $d = \dim_H(K(C)) + s$ (by a classical mass transference principle, see e.g. [8, Proposition E.1]).

We can now make a first choice of disjoint subintervals of $(\sqrt{13}, \sqrt{21})$ with their corresponding subshifts $\Sigma(B)$, which we will subdivide further in the next subsections: cf. Table 5 below. Note that these choices of $\Sigma(B)$ are *simpler* than the original choices in [11] (and this is possible because Lemma 3.4 is more flexible than [11, Lemma 6.1]).

n	Interval R_n	B_n	$\mathcal{F}_n$
1	$(\sqrt{13}, 3.92)$	$1, 2$	
2	$(3.92, 4.32372)$	$1, 2, 3$	13, 31
3	$(4.32372, 4.4984)$	$1, 2, 3$	131
4	$(4.4984, \sqrt{21})$	$1, 2, 3$	1313, 3131

TABLE 5. Subshifts $\Sigma(B_n) = \{\beta \in B_n^{\mathbb{Z}} \mid \alpha \text{ has no substring from } \mathcal{F}_n\}$.

Also, we collect together in Table 6 the subshifts C_y and the rigorous upper bounds on $\dim_H K(C_y)$ (derived from the same method as before, described in §4) we need for the sequel.

We are now ready to proceed to the detailed analysis of the sets K_{gap} constructed below to analyse different parts of $M \setminus L$. However, for the sake of completeness, let us briefly postpone this to the next subsections while closing the current discussion with an illustration of the method for the region $(M \setminus L) \cap (\sqrt{5}, \sqrt{13})$.

[3]In most cases below, X_j is a *pair* of finite sequences (e.g., $X_1 = \{23, 1133\}$ in §3.4), but sometimes we use larger finite sets (e.g., one of the X_j in §3.14 is $X_j = \{34313131, 344434, 213131\}$). In principle, we could explicitly list all X_j appearing below, but, for the sake of simplicity of exposition, we will refrain from doing so: in other words, the relevant sets X_j will always be *implicit* in our subsequent discussions.

n	Interval S_n	$\mathcal{A}_n$	$\mathcal{F}_n$	$\dim_H K(C_n)$
1	$(\sqrt{5}, 3.042)$	$1, 2$	121, 212, 2111222, 2221112	0.346453
2	$(\sqrt{13}, 3.84)$	$1, 2, 3$	13, 31	0.573961
3	$(3.84, 3.92)$	$1, 2, 3$	131, 313, 231, 132, 312, 213	0.594179
4	$(3.92, 4.01)$	$1, 2, 3$	131, 313, 2312, 2132	0.643354
5	$(4.01, 4.1165)$	$1, 2, 3$	131	0.666993
6	$(4.1165, 4.1673)$	$1, 2, 3$	1313, 3131, 1312, 2131, 13111, 11131	0.6694154
7	$(4.1673, 4.2527275)$	$1, 2, 3$	1313, 3131, 1312, 2131	0.677846
8	$(4.2527275, 4.32372)$	$1, 2, 3$	1313, 3131, 21312	0.691289
9	$(4.32372, 4.385)$	$1, 2, 3$	31313, 21313, 31312, 21312, 1113131, 1313111, 3131112, 2111313, 3131113, 3111313	0.694718
10	$(4.385, \sqrt{20})$	$1, 2, 3$	31313, 31312, 21313, 121312, 213121	0.697493
11	$(\sqrt{20}, 4.4984)$	$1, 2, 3$	31313	0.704213
12	$(4.4984, 4.513)$	$1, 2, 3, 4$	14, 41, 24, 42, 343, 31313	0.704700
13	$(4.527, 4.55)$	$1, 2, 3, 4$	14, 41, 24, 42, 3433, 3343, 3434, 4343	0.708245
14	$(4.4984, \sqrt{21})$	$1, 2, 3, 4$	14, 41, 24, 42	0.709394

TABLE 6. Subshifts $\Sigma(C_n) = \{\alpha \in \mathcal{A}_n^{\mathbb{Z}} \mid \alpha$ has no substring from $\mathcal{F}_n\}$ used in our analysis, and dimension of $K(C_n)$ calculated using the method from §4.

Take $C_1 \subset \{1,2\}^{\mathbb{Z}}$ where 121, 212, 2111222 and 2221112 are forbidden. Notice that $\lambda_0(12^*1) > 3.15$, in the sense that if $\underline{a} = (a_n)_{n\in\mathbb{Z}} \in \Sigma(\mathcal{A}_1)$ and $(a_{-1}, a_0, a_1) = (1,2,1)$ then $\lambda_0(\underline{a}) > 3.15$. Indeed, in this case, we have $\lambda_0(\underline{a}) \geq [2;1,\overline{1,2}] + [0;1,\overline{1,2}] > 3.15$. We also have $\lambda_0(21^*2) \geq [2;1,2,\overline{2,1}] + [0;\overline{2,1}] > 3.06$ and $\lambda_0(222^*1112) \geq [2;2,2,\overline{2,1}] + [0;1,1,1,2,\overline{2,1}] > 3.042$ (and so, by symmetry, $\lambda_0(21112^*22) > 3.042$). These inequalities imply that $M \cap (\sqrt{5}, 3.042] \subset 2 + K(C_1) + K(C_1)$, and thus

$$\dim_H((M \setminus L) \cap (\sqrt{5}, 3.042]) \leqslant \dim_H(M \cap (\sqrt{5}, 3.042]) \leq 2\dim_H(K(C_1)) < 0.693,$$

since $\dim_H(K(C_1)) < 0.3465$ (as it can be checked with the method from §4).

Now let $(\mu, \nu) = (3.042, \sqrt{13})$. Here we take $C = \{1,2\}^{\mathbb{Z}}$ and $B \subset \{1,2\}^{\mathbb{Z}}$ where 121, 212 and 21112 are forbidden. Note that if $\underline{b} \in \Sigma(B)$, then $m(\underline{b}) \leq [2;2,\overline{1,1,1,2,2,2}] + [0;1,1,1,1,1,\overline{2,2,2,1,1,1}] < 3.041$. Given $\underline{a} \in Y$ with $m = m(\underline{a}) \in (\mu,\nu)$, we observe that if $N \geq 5$, then $m + 1/2^{N-2} \leq \sqrt{12} + 1/2^3 < \nu$. By the previous discussion (cf. Lemma 3.4), there is an integer k (which we may assume to be at least 3) such that, for any $N \geq k+2$, any finite sequence τ and infinite sequences $\gamma \in \Sigma^+(C)$, $\theta \in \Sigma^+(B)$, if $\underline{\hat{a}} = \gamma^t a_{-N} \dots a_0^* \dots a_N \tau\theta$, then $m(\underline{\hat{a}}) \geq m + 1/2^k \geq m + 1/2^{N-2}$.

Suppose that for some $n \geq N$, a sequence $\gamma^t a_{-N} \dots a_0^* \dots a_n$ has continuations with different subsequent term (of index $n+1$) whose Markov value are smaller than $m + 1/2^{N-2}$ - in this case this means that this sequence has such a continuation with $a_{n+1} = 1$ and another one with $a_{n+1} = 2$. We *claim* that these continuations should be of the type $a_n\alpha_n = a_n 112\alpha_{n+3}$ and $a_n\beta_n = a_n 221\beta_{n+3}$ thanks to the presence of the continuations $\overline{1122}$ and $\overline{2211}$. Indeed, we have two cases:

• If $a_n 1$ can be continued with $a_{n+2}a_{n+3} \neq 12$, since $[0;1,a_{n+2},a_{n+3}] > [0;1,1,2]$, it follows from Lemma 3.6 that the Markov values centered at a_k, with $k \leq n$ and at $a_{n+1} = 1$ are smaller than $m + 1/2^{N-2}$; by Remark 3.5, it is enough to verify that $[2;2,\overline{1,1,2,2}] + [0;1,1,a_n,\dots] \leq [2;2,\overline{1,1,2,2}] + [0;1,1,\overline{1,2}] < 3.021 < m$ in order to conclude that the Markov value of $\gamma^t a_{-N}\dots a_0^*\dots a_n\overline{1122}$ is smaller than $m+1/2^{N-2}$ and get the desired contradiction.

• If $a_n 2$ can be continued with $a_{n+2}a_{n+3} \neq 21$, since $[0;2,a_{n+2},a_{n+3}] < [0;2,2,1]$, it follows from Lemma 3.6 that the Markov values centered at a_k, with $k \leq n$ and at $a_{n+1} = 2$ are smaller than $m + 1/2^{N-2}$; by Remark 3.5, it is enough to verify that $[2;\overline{1,1,2,2}] + [0;2,a_n,\dots] \leq [2;\overline{1,1,2,2}] + [0;2,\overline{2,1}] < 3.01 < m$ in order to conclude that the Markov value of $\gamma^t a_{-N}\dots a_0^*\dots a_n\overline{1122}$ is smaller than $m + 1/2^{N-2}$ and derive again a contradiction.

At this point, we recall that it was shown in [11] that, for $s = 0.174813$, and all positive integers $b_1,\dots,b_n$, we have[4]

$$|I(b_1,\dots,b_n,1,1,2)|^s + |I(b_1,\dots,b_n,2,2,1)|^s \leqslant |I(b_1,\dots,b_n)|^s.$$

Thus, $\dim_H((M\setminus L)\cap(3.042,\sqrt{13})) \leq 0.174813+\dim_H(E_2) < 0.174813+0.531281 = 0.706094$.

Hence,

$$\begin{aligned} & \dim_H((M\setminus L)\cap(-\infty,\sqrt{13})) = \\ = \;& \max\{\dim_H((M\setminus L)\cap(-\infty,3.042],\dim_H((M\setminus L)\cap(3.042,\sqrt{13})\} \\ \leq \;& \max\{\dim_H(M\cap(-\infty,3.042],\dim_H((M\setminus L)\cap(3.042,\sqrt{13})\} \\ \leq \;& \max\{0.693, 0.706094\} = 0.706094 \end{aligned}$$

3.3. **Improvement of the upper bounds in the region** $(\sqrt{13},3.84)$**.** As specified in Table 6, we choose $\Sigma(C_2) = \{\alpha \in \{1,2,3\}^{\mathbb{Z}} \mid 13 \text{ and } 31 \text{ are not substrings of } \alpha\}$ and $\Sigma(B_1) = \{1,2\}^{\mathbb{Z}}$ to show that if $\alpha \in Y$ and $m(\alpha) \in M\setminus L$ then there are two possibilities for the sequences $\alpha_n = (\upsilon_n^1,\upsilon_{n+1}^1,\dots)$ and $\beta_n = (\upsilon_n^2,\upsilon_{n+1}^2,\dots)$ with $\upsilon_n^1 \neq \upsilon_n^2$ corresponding Markov values in $(M\setminus L)\cap(\sqrt{13},3.84)$:

(A) $\alpha_n = 3\alpha_{n+1}$ and $\beta_n = 2\beta_{n+1}$ (i.e., $\upsilon_n^1 = 3$, $\upsilon_n^2 = 2$)
(B) $\alpha_n = 2\alpha_{n+1}$ and $\beta_n = 1\beta_{n+1}$ (i.e., $\upsilon_n^1 = 2$, $\upsilon_n^2 = 1$)

where $\alpha_{n+k} := (\upsilon_{n+k}^1,\upsilon_{n+k+1}^1,\dots)$ and $\beta_{n+k} := (\upsilon_{n+k}^2,\upsilon_{n+k+1}^2,\dots)$.

Let us first look at (A). It continues with $\beta_n = 21\beta_{n+2}$ and, in fact, we see that $21\beta_{n+2} = \overline{21}$ because 2 appears in odd positions, 1 appears in even positions and 13 is forbidden. Thus $[[0;\alpha_n],[0;\beta_n]]\cap K(B) \neq \varnothing$.

Let us now look at (B). It continues with $\beta_n = 11\beta_{n+2}$. Since 13 is forbidden, $11\beta_{n+2} \to \dots \to 11\overline{21} \in K(B_1)$. Thus $[[0;\alpha_n],[0;\beta_n]]\cap K(B_1) \neq \varnothing$. Therefore it is not possible to have two different continuations which do not connect to B. Hence $\dim_H K_{gap} = 0$.

In particular, $\dim_H((M\setminus L)\cap(\sqrt{13},3.84)) \leqslant \dim_H(K(C_1)) \leq 0.574$.

Remark 3.7. This estimate should be compared with the inequality $\dim((M\setminus L)\cap(\sqrt{13},3.84)) \geqslant \dim(\Omega) > 0.537109$ from §3.1.

[4] Here and in the sequel, we use the well-known formula $|I(a_1,\dots,a_k)| = \frac{1}{q_k(q_k+q_{k-1})}$ where q_j stands for the denominator of $[0;a_1,\dots,a_j]$.

3.4. **Improvement of the upper bounds in the region** $(3.84, 3.92)$**.** Similarly to [11], we can use $C_3 \subset \{1,2,3\}^{\mathbb{Z}}$ where $131, 313, 231, 132$ are forbidden and a certain block B to show that the continuations of words with values in $(M \setminus L) \cap (3.84, 3.92)$ are

- 33 and 21
- 23 and 113 or 1121

We affirm that the first Cantor set of gaps is trivial. Indeed, if the continuation 21 is not $\overline{21}$, it must be $(21)^n 3$ for some $n \in \mathbb{N}$, a contradiction because 213 is *forbidden* in this region as

$$[3, 1, 2, \overline{3, 1}] + [0, \overline{3, 1}] > 3.95$$

Also, a similar argument shows that the option 1121 is trivial. Thus, the Cantor set of the gaps in this region consist of the options 23 and 1133 (since 131 and 132 are forbidden in C). It follows that the Cantor set of the gaps has dimension $\dim_H K_{gap} < 0.133$ because

$$0.016134^{0.133} + (1/690)^{0.133} < 1.$$

Since $\dim_H K(C_3) < 0.5942$, we deduce that $\dim_H((M \setminus L) \cap (3.84, 3.92)) < 0.5942 + 0.133 = 0.7272$.

3.5. **Refinement of the control in the region** $(3.92, 4.01)$**.**

3.5.1. *Refinement of the control in the region* $(3.92, 3.9623)$. Similarly to [11], we can use $C \subset \{1,2,3\}^{\mathbb{Z}}$ where $131, 313, 2312, 2132$ are forbidden and a certain block B to show that the continuations of words with values in $(M \setminus L) \cap (3.92, 3.9623)$ are

- 331 and 21
- 23 and 113

Note that in this regime we have

$$\lambda_0(3^*12) > [3; 1, 2, 3, 1, 1, 1, \overline{3, 1}] + [0; 3, 1, 2, 1, 3, 3, \overline{3, 1}] > 3.96238$$

so that the strings 312 and 213 are forbidden. Similarly, the strings 3231, 1323, 2231, 1322 are also forbidden.

Thus, the continuations 331 and 21 are not possible in this regime: indeed, given that 213 is forbidden, the smallest continuation of 21 would be $\overline{21}$, so that we would be able to connect to the block B, a contradiction.

Next, we affirm that the continuation 23 and 113 leads to 231 and 113: otherwise, if 232 or 233 is an allowed continuation, then we could use the largest continuation $\overline{23}$ to connect to an adequate block B, a contradiction. Since

$$\begin{aligned} & \left(\frac{|I(a_1, \dots, a_n, 2, 3, 1)|}{|I(a_1, \dots, a_n)|}\right)^{0.153} + \left(\frac{|I(a_1, \dots, a_n, 1, 1, 3)|}{|I(a_1, \dots, a_n)|}\right)^{0.153} \\ \leqslant \quad & (0.00254)^{0.153} + (1/63)^{0.153} < 1, \end{aligned}$$

we deduce that $\dim_H((M \setminus L) \cap (3.92, 3.9623)) < 0.643355 + 0.153 = 0.796355$.

3.5.2. *Refinement of the control in the region* $(3.9623, 3.9845)$. Similarly to [11], we can use $C \subset \{1,2,3\}^{\mathbb{Z}}$ where $131, 313, 2312, 2132$ are forbidden and a certain block B to show that the continuations of words with values in $(M \setminus L) \cap (3.9623, 3.9845)$ are

- 331 and 21
- 23 and 113

Since the strings 3231 and 1323 are forbidden in this regime (as $\lambda_0(323^*1) > 3.99$), the same argument of the previous subsection says that 23 and 113 actually must be 231 and 113 where

$$\begin{aligned}&\left(\frac{|I(a_1,\dots,a_n,2,3,1)|}{|I(a_1,\dots,a_n)|}\right)^{0.153}+\left(\frac{|I(a_1,\dots,a_n,1,1,3)|}{|I(a_1,\dots,a_n)|}\right)^{0.153}\\ \leqslant\ &(0.00254)^{0.153}+(1/63)^{0.153}<1,\end{aligned}$$

Thus, it suffices to analyse the case $\alpha_n = 331\alpha_{n+3}$ and $\beta_n = 21\beta_{n+2}$. For this sake, note that, in the current region, the strings 1213 and 3121 are forbidden because $\lambda_0(3^*121) > [3;1,2,1,\overline{1,3}] + [0;3,1,2,1,\overline{3,1}] > 3.9866$ (as 313 is forbidden). Also, the strings 23312 and 33312 are forbidden because $\lambda_0(333^*12) > \lambda_0(233^*12) > [3;1,2,\overline{3,1}] + [0;3,2,3,2,\overline{3,1}] > 3.98459$ (as 3231 is forbidden).

We claim that the nth digit a_n (before α_n and β_n) is 2 or 3: otherwise, we would have a continuation $1\beta_n = 121\beta_{n+2}$ connecting to the block B (as the smallest continuation would be $\overline{21}$). In view of the fact that $a_n \in \{2,3\}$, we have that $a_n\alpha_n = a_n331\alpha_{n+3} = a_n3311\alpha_{n+4}$ (as 313, 23312 and 33312 are forbidden) and, *a fortiori*, $a_n\alpha_n = a_n33111\alpha_{n+5}$ thanks to the presence of the continuation $33\overline{1}$. Indeed, if a_n3311 can be continued with $r > 1$, since $[0;3,3,1,1,r] < [0;3,3,1,1,1] < [0;2,1]$, by Remark 3.5, it follows that the Markov values centered at a_k, with $k \leq n$ and at $a_{n+1} = 3$ are smaller than m, and it is enough to verify that $[3;\overline{1}] + [0;3,a_n,\dots] \leq [3;\overline{1}] + [0;3,\overline{3,1}] < 3.93 < m$. We will use implicitly this kind of argument in several forthcoming cases. Since

$$\begin{aligned}&\left(\frac{|I(a_1,\dots,a_n,3,3,1,1,1)|}{|I(a_1,\dots,a_n)|}\right)^{0.15}+\left(\frac{|I(a_1,\dots,a_n,2,1)|}{|I(a_1,\dots,a_n)|}\right)^{0.15}\\ \leqslant\ &(2/3619)^{0.15}+(0.0718)^{0.15}<1,\end{aligned}$$

we derive that $\dim_H((M\setminus L)\cap(3.9623,3.9845)) < 0.643355 + 0.153 = 0.796355$.

3.5.3. *Refinement of the control in the region* $(3.9845, 4.01)$. Similarly to [11], we can use $C \subset \{1,2,3\}^{\mathbb{Z}}$ where $131, 313, 2312, 2132$ are forbidden and a certain block B to show that the continuations of words with values in $(M\setminus L)\cap(3.9845,4.01)$ are

- $\alpha_n = 331\alpha_{n+3}$ and $\beta_n = 21\beta_{n+2}$;
- $\alpha_n = 23\alpha_{n+2}$ and $\beta_n = 113\beta_{n+3}$.

Let us analyse the first possibility depending on the nth digit a_n appearing before $331\alpha_{n+3}$ and $21\beta_{n+2}$:

- if $a_n = 1$, then $\alpha_n = 3312\alpha_{n+4}$ thanks to the presence of the continuation $3312\overline{21}$ (which is valid as $\lambda_0(133^*12) < 3.984$);
- if $a_n \in \{2,3\}$, then $\beta_n = 213\beta_{n+3}$ thanks to the continuation $21331\overline{2}$.

Similarly, we can decompose the second possibility into two subcases depending on the digit appearing before $23\alpha_{n+2}$ and $113\beta_{n+3}$:

- if $a_n = 1$, then $\alpha_n = 231\alpha_{n+3}$ in view of $231\overline{12}$;
- if $a_n \in \{2,3\}$, then $\beta_n = 1132\beta_{n+4}$ in view of $1132\overline{12}$.

Since

$$\begin{aligned}&\left(\frac{|I(a_1,\dots,a_n,3,3,1,2)|}{|I(a_1,\dots,a_n)|}\right)^{0.152}+\left(\frac{|I(a_1,\dots,a_n,2,1)|}{|I(a_1,\dots,a_n)|}\right)^{0.152}\\ \leqslant\ &(1/1504)^{0.152}+(0.0718)^{0.152}<1,\end{aligned}$$

$$\begin{aligned} & \left(\frac{|I(a_1,\ldots,a_n,3,3,1)|}{|I(a_1,\ldots,a_n)|}\right)^{0.133} + \left(\frac{|I(a_1,\ldots,a_n,2,1,3)|}{|I(a_1,\ldots,a_n)|}\right)^{0.133} \\ \leqslant \quad & (1/255)^{0.133} + (0.0071)^{0.133} < 1, \end{aligned}$$

$$\begin{aligned} & \left(\frac{|I(a_1,\ldots,a_n,2,3,1)|}{|I(a_1,\ldots,a_n)|}\right)^{0.153} + \left(\frac{|I(a_1,\ldots,a_n,1,1,3)|}{|I(a_1,\ldots,a_n)|}\right)^{0.153} \\ \leqslant \quad & (0.0071)^{0.153} + (1/63)^{0.153} < 1, \end{aligned}$$

$$\begin{aligned} & \left(\frac{|I(a_1,\ldots,a_n,2,3)|}{|I(a_1,\ldots,a_n)|}\right)^{0.14} + \left(\frac{|I(a_1,\ldots,a_n,1,1,3,2)|}{|I(a_1,\ldots,a_n)|}\right)^{0.14} \\ \leqslant \quad & (0.0162)^{0.14} + (1/368)^{0.14} < 1, \end{aligned}$$

we derive that $\dim_H((M \setminus L) \cap (3.9845, 4.01)) < 0.643355 + 0.153 = 0.796355$.

3.6. **Refinement of the control in the region** $(4.01, 4.1165)$**.**

3.6.1. *Refinement of the control in the region* $(4.01, 4.054)$. Similarly to [11], we can use $C = \{1,2,3\}^{\mathbb{Z}}$ and a certain block B to show that the continuations of words with values in $(M \setminus L) \cap (4.01, \sqrt{20})$ are

- $\alpha_n = 331\alpha_{n+3}$ and $\beta_n = 213\beta_{n+3}$;
- $\alpha_n = 23\alpha_{n+2}$ and $\beta_n = 113\beta_{n+3}$.

Since $\lambda_0(13^*1) > 4.1165$, the string 131 is forbidden in our current regime. Also, $\lambda_0(213^*2) \geqslant [3;2,\overline{1,3,2,3}] + [0;1,2,\overline{3,1,3,2}] > 4.054$ when 13^*1 is forbidden. Hence, the first transition extends as $331\alpha_{n+3}$ and $2133\beta_{n+4}$.

Next, since 131 is forbidden and $\lambda_0(113^*2\overline{12}) < 4.0078$, the second transition above is actually $\alpha_n = 23\alpha_{n+2}$ and $\beta_n = 1132\beta_{n+4}$. This transition extends in two possible ways:

- if the digit appearing before is $a_n \in \{1,2\}$, we have $\lambda_0(a_n 23^*1\overline{12}) < 4.0014$ and, hence, the transition becomes $\alpha_n = 231\alpha_{n+3}$ and $\beta_n = 1132\beta_{n+4}$;
- if the digit appearing before is $a_n = 3$, we have $\lambda_0(a_n 113^*23\overline{2}) < 4.0026$ and, thus, the transition becomes $\alpha_n = 23\alpha_{n+2}$ and $\beta_n = 11323\beta_{n+5}$.

Now, we recall that $\frac{|I(a_1,\ldots,a_n,3,3,1)|}{|I(a_1,\ldots,a_n)|} \leqslant \frac{1}{255}$, $\frac{|I(a_1,\ldots,a_n,2,1,3,3)|}{|I(a_1,\ldots,a_n)|} \leqslant 0.000641$, $\frac{|I(a_1,\ldots,a_n,2,3,1)|}{|I(a_1,\ldots,a_n)|} \leqslant 0.0071$, $\frac{|I(a_1,\ldots,a_n,1,1,3,2)|}{|I(a_1,\ldots,a_n)|} \leqslant \frac{1}{368}$, $\frac{|I(a_1,\ldots,a_n,2,3)|}{|I(a_1,\ldots,a_n)|} \leqslant 0.0162$, $\frac{|I(a_1,\ldots,a_n,1,1,3,2,3)|}{|I(a_1,\ldots,a_n)|} \leqslant \frac{1}{3905}$, and

$$\max\{(\tfrac{1}{255})^{0.11} + (0.000641)^{0.11}, (\tfrac{1}{368})^{0.129} + (0.0071)^{0.129}, (\tfrac{1}{3905})^{0.117} + (0.0162)^{0.117}\} < 1.$$

Therefore, $\dim_H((M \setminus L) \cap (4.01, 4.054)) < 0.667 + 0.129 = 0.796$ (because the Cantor set of continued fraction expansions in $\{1,2,3\}^{\mathbb{N}}$ which avoid 131 has dimension < 0.667).

3.6.2. *Refinement of the control in the region* $(4.054, 4.06326)$. The string 131 is still forbidden in our current regime and the same argument of the previous subsection can be employed to treat the second transition. Thus, it remains only to analyse the first transition $\alpha_n = 331\alpha_{n+3}$ and $\beta_n = 213\beta_{n+3}$.

If the digit appearing before the first transition is $a_n = 1$, we get a valid continuation $\lambda_0(a_n 33^*133\overline{12}) < 4.0468$, so that the first transition becomes $\alpha_n = 3313\alpha_{n+4}$ and $\beta_n = 213\beta_{n+3}$.

If the digit appearing before the first transition is $a_n = 2$, we claim that $a_n 2132$ is forbidden: indeed, $\lambda_0(213^*2b_m) > 4.1$ when $b_m \in \{2,3\}$, $\lambda_0(213^*211) > 4.072$, $\lambda_0(a_n 213^*212) > 4.067$, $\lambda_0(a_n 213^*2133) > 4.06352$, and

$$\lambda_0(a_n 213^*21321) \geqslant [3;2,1,3,2,1,2,\overline{3,1}] + [0;1,2,2,\overline{1,3,2,3}] > 4.06326$$

(here we used that 213211 and 131 are forbidden), so that all continuations of $a_n 2132$ are large. Thus, the first transition becomes $\alpha_n = 331\alpha_{n+3}$ and $\beta_n = 2133\beta_{n+4}$.

If the digit appearing before the first transition is $a_n = 3$, we have that $a_n 3313$ is also forbidden (as $\lambda_0(a_n 33^*13) > 4.0679$) and $a_n 3312\overline{12}$ is a valid continuation (as $\lambda_0(a_n 33^*12\overline{12}) < 4.03845$), so that the first transition becomes $\alpha_n = 33121\alpha_{n+3}$ and $\beta_n = 213\beta_{n+3}$.

Since $\frac{|I(a_1,\dots,a_n,3,3,1,3)|}{|I(a_1,\dots,a_n)|} \leqslant \frac{1}{2592}$, $\frac{|I(a_1,\dots,a_n,2,3,1)|}{|I(a_1,\dots,a_n)|} \leqslant 0.0071$, $\frac{|I(a_1,\dots,a_n,3,3,1)|}{|I(a_1,\dots,a_n)|} \leqslant \frac{1}{255}$, $\frac{|I(a_1,\dots,a_n,2,1,3,3)|}{|I(a_1,\dots,a_n)|} \leqslant 0.000641$, $\frac{|I(a_1,\dots,a_n,3,3,1,2,1)|}{|I(a_1,\dots,a_n)|} \leqslant \frac{1}{3552}$, $\frac{|I(a_1,\dots,a_n,2,1,3)|}{|I(a_1,\dots,a_n)|} \leqslant 0.0071$ and

$$\max\{(\tfrac{1}{2592})^{0.111}+(0.0071)^{0.111}, (\tfrac{1}{255})^{0.11}+(0.000641)^{0.11}, (\tfrac{1}{3552})^{0.11}+(0.0071)^{0.11}\} < 1,$$

we conclude that $\dim_H((M\setminus L)\cap(4.054, 4.06326)) < 0.667 + 0.129 = 0.796$.

3.6.3. *Refinement of the control in the region* $(4.06326, 4.0679)$. Once again, 131 is still forbidden in our current regime, so that we can focus on the first transition $\alpha_n = 331\alpha_{n+3}$ and $\beta_n = 213\beta_{n+3}$. Actually, the fact that we are looking at Markov values below 4.0679 makes that the same argument above can still be employed to treat the first transition in the case of the digits appearing before are $a_n \in \{1,3\}$.

Finally, if the digit appearing before is $a_n = 2$, then $a_n 33133\overline{12}$ is a valid continuation because the fact that 131 and $2132b_m$, $b_m \in \{2,3\}$ are forbidden says that

$$\lambda_0(a_n 33^*133\overline{12}) \leqslant [3;1,3,3,\overline{1,2}] + [0;3,2,1,3,2,1,\overline{1,3}] < 4.063251.$$

In particular, the first transition becomes $3313\alpha_{n+4}$ and $213\beta_{n+3}$ and we derive that $\dim_H((M\setminus L)\cap(4.06326, 4.0679)) < 0.667 + 0.129 = 0.796$.

3.6.4. *Refinement of the control in the region* $(4.0679, 4.1)$. Since 131 is forbidden here, it suffices to analyse the first transition $\alpha_n = 331\alpha_{n+3}$ and $\beta_n = 213\beta_{n+3}$. Moreover, the same argument above can still be employed to treat the first transition in the case of the digit appearing before is $a_n = 1$. Furthermore, $2132b_m$, $b_m \in \{2,3\}$ is also forbidden here, so that the same argument above also treats the case of the first transition when the digit appearing before is $a_n = 2$. Finally, if the digit appearing before is $a_n = 3$, we see that the first transition becomes $331\alpha_{n+3}$ and $2132\beta_{n+4}$ as $a_n 213\overline{21}$ is a valid continuation (since $\lambda_0(a_n 213^*\overline{21}) < 4.063582$). Given that $\frac{|I(a_1,\dots,a_n,3,3,1)|}{|I(a_1,\dots,a_n)|} \leqslant \frac{1}{255}$, $\frac{|I(a_1,\dots,a_n,2,1,3,2)|}{|I(a_1,\dots,a_n)|} \leqslant 0.00121$, and $(\frac{1}{255})^{0.114} + (0.00121)^{0.114} < 1$, we obtain that $\dim_H((M\setminus L)\cap(4.06326, 4.0679)) < 0.667 + 0.129 = 0.796$.

3.6.5. *Refinement of the control in the region* $(4.1, 4.1165)$. Using for the last time that 131 is forbidden, we will again concentrate only on the first transition $\alpha_n = 331\alpha_{n+3}$ and $\beta_n = 213\beta_{n+3}$. Here, we observe that $33133\overline{12}$ is a valid continuation (as $\lambda_0(33^*133\overline{12}) < 4.0721$), so the first transition becomes $3313\alpha_{n+4}$ and $213\beta_{n+3}$ and we get $\dim((M\setminus L)\cap(4.1, 4.1165)) < 0.667 + 0.129 = 0.796$.

3.7. **Refinement of the control in the region $(4.1165, 4.1673)$.**

3.7.1. *Refinement of the control in the region* $(4.1165, 4.1271)$. Recall that in the region $(M \setminus L) \cap (4.01, \sqrt{20})$ our task is to analyse the transitions

- $\alpha_n = 331\alpha_{n+3}$ and $\beta_n = 213\beta_{n+3}$;
- $\alpha_n = 23\alpha_{n+2}$ and $\beta_n = 113\beta_{n+3}$.

We begin by observing that $\lambda_0(313^*1) > 4.32372$, $\lambda_0(213^*1) > 4.2527275$ (when 3131 is forbidden), and the dimension of $C = 1, 2, 3$ with 3131, 2131, 13111 and their transposes forbidden is < 0.66942. Recalling that $3313\overline{312}$ is a valid continuation (as $\lambda_0(33^*133\overline{12}) < 4.0721$), the first transition always becomes $3313\alpha_{n+4}$ and $213\beta_{n+3}$ in the region between 4.1165 and 4.2527275. Since $\frac{|I(a_1,\dots,a_n,3,3,1,3)|}{|I(a_1,\dots,a_n)|} \leqslant \frac{1}{2592}$, $\frac{|I(a_1,\dots,a_n,2,1,3)|}{|I(a_1,\dots,a_n)|} \leqslant 0.0071$, $(1/2592)^{0.111} + (0.0071)^{0.111} < 1$, and $0.66942 + 0.111 = 0.78042$, the first transition is completely treated in the region between 4.1165 and 4.2527275.

Let us now focus on the second transition. Since $23\overline{1}$ is a valid continuation (as $\lambda_0(23^*\overline{1}) < 4.06$), the second transition becomes $231\alpha_{n+3}$ and $113\beta_{n+3}$.

If the digit appearing before is $a_n \in \{1, 2\}$, since $\lambda_0(a_n113^*1) > 4.134215$ and $113\overline{23}$ is a valid continuation (as $\lambda_0(113^*\overline{23}) < 4.079$), the second transition becomes $231\alpha_{n+3}$ and $11323\beta_{n+5}$. Given that $\frac{|I(a_1,\dots,a_n,1,1,3,2,3)|}{|I(a_1,\dots,a_n)|} \leqslant \frac{1}{3905}$, $\frac{|I(a_1,\dots,a_n,2,3,1)|}{|I(a_1,\dots,a_n)|} \leqslant 0.0071$, $(1/3905)^{0.11} + (0.0071)^{0.11} < 1$, and $0.66942 + 0.11 = 0.77942$, we are done.

If the digit appearing before is $a_n = 3$, we observe that $\lambda_0(a_n23^*1b_m) > 4.1271$ for $b_m \in \{2, 3\}$ and $\lambda_0(23^*111\overline{3}) < 4.081$, so that the second transition becomes $231113\alpha_{n+6}$ and $113\beta_{n+3}$. Given that $\frac{|I(a_1,\dots,a_n,2,3,1,1,1,3)|}{|I(a_1,\dots,a_n)|} \leqslant 0.0001$, $\frac{|I(a_1,\dots,a_n,1,1,3)|}{|I(a_1,\dots,a_n)|} \leqslant 1/63$, $(0.0001)^{0.11} + (1/63)^{0.11} < 1$, and $0.66942 + 0.11 = 0.77942$, we are done. In summary, we showed that $\dim((M \setminus L) \cap (4.1165, 4.1271)) < 0.78042$.

3.7.2. *Refinement of the control in the region* $(4.1271, 4.12733)$. In view of the arguments of the previous subsection, our task is reduced to discuss the second transition $231\alpha_{n+3}$ and $113\beta_{n+3}$ when the digit appearing before is $a_n = 3$.

Note that $\lambda_0(a_n23^*13) = \lambda_0(323^*13) > 4.199$. Moreover, we claim that all continuations of a_n2312 are large. Indeed, $\lambda_0(a_n23^*12b_m) > 4.1358$ for $b_m \in \{1, 2\}$, $\lambda_0(a_n23^*123c_m) > 4.1296$ for $c_m \in \{2, 3\}$, $\lambda_0(a_n23^*1231d_m) > 4.1275$ for $d_m \in \{1, 2\}$. Since $\lambda_0(a_n23^*123133) > 4.12733$, $\lambda_0(23^{**}132) > 4.1288$, and 3131 is forbidden, we conclude that a_n2312 has no short continuation. In view of the valid continuation $23111\overline{3}$ (with $\lambda_0(23^*111\overline{3}) < 4.081$), we see that the second transition becomes $231113\alpha_{n+5}$ and $113\beta_{n+3}$. Hence, we can apply again the argument from the previous subsection to derive that $\dim((M \setminus L) \cap (4.1271, 4.12733)) < 0.78042$.

3.7.3. *Refinement of the control in the region* $(4.12733, 4.12762)$. In view of the arguments of the previous subsection, our task is again reduced to discuss the second transition $231\alpha_{n+3}$ and $113\beta_{n+3}$ when the digit appearing before is $a_n = 3$.

If the digit appearing before a_n is $a_{n-1} \in \{1, 2\}$, we have $\lambda_0(a_{n-1}a_n113^*\overline{113}) < 4.1264$, so that the second transition becomes $231\alpha_{n+3}$ and $113113\beta_{n+6}$ (since 1313, 1312, 13111 and 13112 are forbidden for any sequence with Markov value < 4.134215). Given that $\frac{|I(a_1,\dots,a_n,2,3,1)|}{|I(a_1,\dots,a_n)|} \leqslant 0.0071$, $\frac{|I(a_1,\dots,a_n,1,1,3,1,1,3)|}{|I(a_1,\dots,a_n)|} \leqslant 0.000241$, $(0.0071)^{0.11} + (0.000241)^{0.11} < 1$, and $0.66942 + 0.11 = 0.77942$, we are done.

If the digit appearing before a_n is $a_{n-1} = 3$, we have three possibilities. If the digit before a_{n-1} is $a_{n-2} = 3$, we have $\lambda_0(a_{n-2}a_{n-1}a_n113^*\overline{113}) < 4.1272999969$ and, hence, the argument of the previous paragraph can be repeated. If $a_{n-2} = 2$, we recall that $\lambda_0(323^*13) > 4.199$, $\lambda_0(23323^*12) > 4.1277$ (when 3131 is forbidden) and $\lambda_0(23111\overline{3}) < 4.081$ to get that the second transition becomes $231113\alpha_{n+6}$ and

$113\beta_{n+3}$. Finally, if $a_{n-2}=1$, then we have three subcases: if $a_{n-3}\in\{2,3\}$, we note that $\lambda_0(a_{n-3}a_{n-2}a_{n-1}a_n113^*1)\geqslant\lambda_0(2133113^*1131133\overline{132})>4.1277$ (when 1313, 1312, 13111 and 13112 are forbidden) and $\lambda_0(113^*\overline{23})<4.079$, so that the second transition becomes $231\alpha_{n+3}$ and $11323\beta_{n+5}$; if $a_{n-3}=1$ and $a_{n-4}\in\{1,2\}$, we get $\lambda_0(a_{n-4}a_{n-3}a_{n-2}a_{n-1}a_n113^*1)\geqslant\lambda_0(21133113^*1131133\overline{132})>4.12762$, so that the second transition still is $231\alpha_{n+3}$ and $11323\beta_{n+5}$; if $a_{n-3}=1$ and $a_{n-4}=3$, we get $\lambda_0(a_{n-4}a_{n-3}a_{n-2}a_{n-1}a_n23^*12)\geqslant\lambda_0(3113323^*123133\overline{31})>4.12762$ (because $\lambda_0(23^{**}132)>4.1288$), so that the second transition becomes $231113\alpha_{n+6}$ and $113\beta_{n+3}$. In any event, we conclude that $\dim((M\setminus L)\cap(4.12733,4.12762))<0.78042$.

3.7.4. *Refinement of the control in the region* $(4.12762, 4.134215)$. In view of the arguments of the previous subsection, our task is reduced to discuss the second transition $231\alpha_{n+3}$ and $113\beta_{n+3}$ when the digits appearing before are $a_n=3$, $a_{n-1}=3$ and $a_{n-2}\in\{1,2\}$.

If $a_{n-2}=2$, we have $\lambda_0(233113^*\overline{113})<4.12751$, and 1313, 1312, 13111 and 13112 are forbidden on any sequence with Markov value <4.134215, so that the second transition becomes $231\alpha_{n+3}$ and $113113\beta_{n+6}$ and we are done.

If $a_{n-2}=1$ and $a_{n-3}\in\{2,3\}$, we recall that $\lambda_0(323^*13)>4.199$, $\lambda_0(323^*12b_m)>4.1358$ for $b_m\in\{1,2\}$ and $\lambda_0(a_{n-3}a_{n-2}a_{n-1}a_n23^*123133\overline{1})<4.127471$, so that the second transition becomes $23123\alpha_{n+5}$ and $113\beta_{n+3}$. Given that $\frac{|I(a_1,\dots,a_n,2,3,1,2,3)|}{|I(a_1,\dots,a_n)|}\leqslant 0.000111$, $\frac{|I(a_1,\dots,a_n,1,1,3)|}{|I(a_1,\dots,a_n)|}\leqslant 1/63$, $(0.000111)^{0.111}+(1/63)^{0.111}<1$, and $0.66942+0.111=0.78042$, we are done in this case. If $a_{n-2}=1=a_{n-3}$ and $a_{n-4}\in\{1,2\}$, we get $\lambda_0(a_{n-4}a_{n-3}a_{n-2}a_{n-1}a_n23^*123133211\overline{3})<4.12761982$ and the second transition still is $23123\alpha_{n+5}$ and $113\beta_{n+3}$, and we are done. If $a_{n-2}=1=a_{n-3}$ and $a_{n-4}=3$, we have $\lambda_0(a_{n-4}a_{n-3}a_{n-2}a_{n-1}a_n113^*113113\overline{32})<4.127618$, so that the second transition becomes $231\alpha_{n+3}$ and $113113\beta_{n+6}$, and we are done.

In any case, we get that $\dim((M\setminus L)\cap(4.127672,4.134215))<0.78042$.

3.7.5. *Refinement of the control in the region* $(4.134215, 4.137519)$. In view of the arguments of the previous subsections, our task is to discuss the second transition $231\alpha_{n+3}$ and $113\beta_{n+3}$ when the digits appearing before are $a_n=2$ and $a_n=3$. For later reference, we remark that the Cantor set $C=1,2,3$ where 1313, 1312, 13111 and their transposes are forbidden has dimension <0.6694155.

If $a_n=2$, we have two possibilities. If the digit appearing before $a_n=2$ is $a_{n-1}\in\{2,3\}$, then $\lambda_0(a_{n-1}a_n113^*1)>4.143241$ and $\lambda_0(113^*\overline{23})<4.079$, so that the second transition becomes $231\alpha_{n+3}$ and $11323\beta_{n+3}$ and we are done. If the digit before $a_n=2$ is $a_{n-1}=1$, we have $\lambda_0(a_{n-1}a_n23^*13)>4.1837$, $\lambda_0(a_{n-1}a_n23^*121)>4.137519$ (as 1313, 1312 and 13111 are forbidden), and $\lambda_0(1223^*122\overline{12})<4.127$, so that the second transition becomes $23122\alpha_{n+5}$ and $113\beta_{n+3}$. Given that $\frac{|I(a_1,\dots,a_n,2,3,1,2,2)|}{|I(a_1,\dots,a_n)|}\leqslant 0.00021$, $\frac{|I(a_1,\dots,a_n,1,1,3)|}{|I(a_1,\dots,a_n)|}\leqslant 1/63$, $(0.00021)^{0.115}+(1/63)^{0.115}<1$, and $0.67+0.115=0.785$, we are done.

If $a_n=3$, since 1313, 1312 are forbidden and $\lambda_0(3113^*\overline{113})<4.128$, the second transition becomes $231\alpha_{n+3}$ and $11311\beta_{n+5}$. Since $\frac{|I(a_1,\dots,a_n,2,3,1)|}{|I(a_1,\dots,a_n)|}\leqslant 0.007042603$, $\frac{|I(a_1,\dots,a_n,1,1,3,1,1)|}{|I(a_1,\dots,a_n)|}\leqslant 1/400$, $(0.007042603)^{0.1270292}+(1/400)^{0.1270292}<1$, and $0.6694155+0.1270292=0.7964447<0.796445$, we are done.

In any event, we get that $\dim((M\setminus L)\cap(4.134215,4.137519))<0.796445$.

3.7.6. *Refinement of the control in the region* $(4.137519, 4.1407)$. In view of the arguments of the previous subsections, our task is reduced to discuss the second transition $231\alpha_{n+3}$ and $113\beta_{n+3}$ when the digits appearing before are $a_n = 2$ and $a_{n-1} = 1$.

If the digit before a_{n-1} is $a_{n-2} \in \{2,3\}$, we have $\lambda_0(1223^*13) > 4.1837$ and

$$\lambda_0(a_{n-2}a_{n-1}a_n 23^*121) > 4.1409$$

(thanks to the fact that 1313, 1312, 13111 are forbidden), so that we are back to the situation in the previous subsection.

If the digit before a_{n-1} is $a_{n-2} = 1$, we have $\lambda_0(a_{n-2}a_{n-1}a_n 113^*1) > 4.1407$ (as 1313, 1312, 13111 are forbidden) and $\lambda_0(2113^*\overline{23}) < 4.027$, so that the second transition becomes $231\alpha_{n+3}$ and $11323\beta_{n+5}$ and we are done.

In summary, we get that $\dim((M \setminus L) \cap (4.137519, 4.1407)) < 0.796445$.

3.7.7. *Refinement of the control in the region* $(4.1407, 4.1673)$. In view of the arguments of the previous subsections, our task is reduced to discuss the second transition $231\alpha_{n+3}$ and $113\beta_{n+3}$ when the digit appearing before is $a_n = 2$ (since 11131 is forbidden because $\lambda_0(1113^*1) > 4.1673$).

If the digit before a_n is $a_{n-1} \in \{2,3\}$, we have $\lambda_0(a_{n-1}a_n 23^*12\overline{1}) < 4.1387$ and $\lambda_0(a_{n-1}a_n 23^*13) > 4.175$, so that the second transition becomes $23121\alpha_{n+5}$ and $113\beta_{n+3}$. Since $\frac{|I(a_1,\dots,a_n,2,3,1,2,1)|}{|I(a_1,\dots,a_n)|} \leqslant 0.00051$, $\frac{|I(a_1,\dots,a_n,1,1,3)|}{|I(a_1,\dots,a_n)|} \leqslant \frac{1}{63}$, $(0.00051)^{0.122} + (\frac{1}{63})^{0.122} < 1$, and $0.67 + 122 = 0.792$, we are done.

If the digit before a_n is $a_{n-1} = 1$, we have two possibilities. If the digit before a_{n-1} is $a_{n-2} = 1$, then $\lambda_0(1223^*13) > 4.1837$ and $\lambda_0(a_{n-2}a_{n-1}a_n 23^*121133\overline{1}) < 4.13997$ (as 1313, 1312, 13111 are forbidden), so that the second transition becomes $23121\alpha_{n+5}$ and $113\beta_{n+3}$ and we are back to the situation of the previous paragraph. If the digit before a_{n-1} is $a_{n-2} \in \{2,3\}$, then the facts that 1313, 1312 are forbidden, and $\lambda_0(a_{n-2}a_{n-1}a_n 113^*\overline{113}) < 4.13984$ imply that the second transition is $231\alpha_{n+3}$ and $11311\beta_{n+5}$ and we are done.

In summary, we get that $\dim((M \setminus L) \cap (4.1407, 4.1673)) < 0.796445$.

3.8. **Refinement of the control in the region** $(4.1673, 4.2527275)$**.** In view of the arguments of the previous subsections, our task is reduced to discuss the second transition $231\alpha_{n+3}$ and $113\beta_{n+3}$. We shall describe the possible extensions of this transition in terms of the digits appearing before and/or the Markov values of the words.

If the digit appearing before is $a_n = 1$, the second transition becomes $23132\alpha_{n+5}$ and $113\beta_{n+3}$ because 3131 is forbidden and $\lambda_0(a_n 23132\overline{12}) < 4.1619$.

If the digit appearing before is $a_n = 3$, the facts that 3131 and 2131 are forbidden, $\lambda_0(a_n 23^*13) > 4.1991$, $\lambda_0(a_n 23^*1\overline{2}) < 4.149$, and $\lambda_0(a_n 113^*\overline{113}) < 4.128$ can be used to say that the second transition becomes $2312\alpha_{n+4}$ and $11311\beta_{n+5}$ in the region $(4.1673, 4.199)$. Furthermore, the fact that $\lambda_0(a_n 113^*111\overline{12}) < 4.1785$ allows to conclude that the second transition becomes $231\alpha_{n+3}$ and $113111\beta_{n+6}$ in the region $(4.199, 4.2527275)$.

If the digit appearing before is $a_n = 2$, the facts that 3131 and 2131 are forbidden, $\lambda_0(a_n 23^*13) > 4.175$, $\lambda_0(a_n 23^*1\overline{2}) < 4.132$, and $\lambda_0(a_n 113^*\overline{113}) < 4.1521$ can be used to say that the second transition becomes $2312\alpha_{n+4}$ and $11311\beta_{n+5}$ in the region $(4.1673, 4.175)$. Moreover, the fact that $\lambda_0(a_n 113^*111) > 4.1857$ and $\lambda_0(a_n 113^*11\overline{21}) < 4.16781$ allows to conclude that the second transition becomes $231\alpha_{n+3}$ and $113112\beta_{n+6}$ in the region $(4.175, 4.1857)$. Hence, it remains to analyse the region $(4.1857, 4.2527275)$ when $a_n = 2$.

If the digit appearing before $a_n = 2$ is $a_{n-1} \in \{2,3\}$, the facts that 3131 and 2131 are forbidden, $\lambda_0(a_{n-1}223^*1\overline{3}) < 4.18261$, and $\lambda_0(a_n113^*\overline{113}) < 4.1521$ imply that the second transition becomes $2313\alpha_{n+4}$ and $11311\beta_{n+5}$ in the region $(4.1857, 4.2527275)$. Thus, it suffices to treat the case $a_n = 2$ and $a_{n-1} = 1$ in the region $(4.1857, 4.2527275)$.

If the digit appearing before $a_{n-1}a_n = 12$ is $a_{n-2} = 1$, the facts that 3131 and 2131 are forbidden, $\lambda_0(a_{n-2}12113^*111) > 4.189$, and $\lambda_0(a_n113^*11\overline{21}) < 4.16781$ imply that the second transition becomes $231\alpha_{n+3}$ and $113112\beta_{n+6}$ in the region $(4.1857, 4.189)$. Also, the second transition becomes $2313\alpha_{n+4}$ and $11311\beta_{n+5}$ in the region $(4.189, 4.2527275)$ because $\lambda_0(a_{n-2}1223^*13\overline{3}) < 4.1881$.

If the digit appearing before $a_{n-1}a_n = 12$ is $a_{n-2} = 3$, the facts that 3131 and 2131 are forbidden, $\lambda_0(a_{n-2}1223^*13) > 4.1889$, $\lambda_0(a_n23^*1\overline{2}) < 4.132$ and $\lambda_0(a_n113^*\overline{113}) < 4.1521$ imply that the second transition becomes $2312\alpha_{n+4}$ and $11311\beta_{n+5}$ in the region $(4.1857, 4.1889)$. Also, the second transition becomes $231\alpha_{n+3}$ and $113111\beta_{n+6}$ in the region $(4.1889, 4.2527275)$ because $\lambda_0(a_{n-2}12113^*1111\overline{32}) < 4.1865$.

At this point, it remains only to investigate the region $(4.1857, 4.2527275)$ when the digit appearing before $a_{n-1}a_n = 12$ is $a_{n-2} = 2$. For this sake, we shall distinguish three subcases.

3.8.1. *The subcase* $a_{n-3} = 3$ *and* $a_{n-2}a_{n-1}a_n = 212$. Since 3131 and 2131 are forbidden, $\lambda_0(a_{n-3}212113^*111) \geqslant [3; 111\overline{1311}] + [0; 11212a_{n-3}\overline{31}] > 4.1876$. Because $\lambda_0(a_n113^*11\overline{21}) < 4.16781$, we conclude that the second transition becomes $231\alpha_{n+3}$ and $113112\beta_{n+6}$ in the region $(4.1857, 4.1876)$. Moreover, $\lambda_0(a_{n-3}21223^*133132\overline{1}) < 4.1874$ and $\lambda_0(a_n113^*\overline{113}) < 4.1521$, so that the second transition becomes $2313\alpha_{n+4}$ and $11311\beta_{n+5}$ in the region $(4.1876, 4.2527275)$.

3.8.2. *The subcase* $a_{n-3} = 1$ *and* $a_{n-2}a_{n-1}a_n = 212$. Since 3131 and 2131 are forbidden, $\lambda_0(a_{n-3}21223^*13) > 4.1878$. Because $\lambda_0(a_n23^*1\overline{2}) < 4.132$ and $\lambda_0(a_n113^*\overline{113}) < 4.1521$, we conclude that the second transition becomes $2312\alpha_{n+4}$ and $11311\beta_{n+5}$ in the region $(4.1857, 4.1878)$. Also, $\lambda_0(a_{n-3}212113^*1111311\overline{32}) < 4.1873$, so that the second transition becomes $231\alpha_{n+3}$ and $113111\beta_{n+6}$ in the region $(4.1878, 4.2527275)$.

3.8.3. *The subcase* $a_{n-3} = 2$ *and* $a_{n-2}a_{n-1}a_n = 212$. If the digit appearing before a_{n-3} is $a_{n-4} \in \{2,3\}$, we have $\lambda_0(a_{n-4}221223^*13) > 4.187566$, $\lambda_0(a_n23^*1\overline{2}) < 4.132$ and $\lambda_0(a_n113^*\overline{113}) < 4.1521$, so that the second transition becomes $2312\alpha_{n+4}$ and $11311\beta_{n+5}$ in the region $(4.1857, 4.187566)$. Moreover, $\lambda_0(a_{n-4}2212113^*1111311\overline{32}) < 4.187564$, so that the second transition becomes $231\alpha_{n+3}$ and $113111\beta_{n+5}$ in the region $(4.187566, 4.2527275)$.

If the digit appearing before a_{n-3} is $a_{n-4} = 1$, we have $\lambda_0(a_{n-4}2212113^*111) \geqslant [3; 11113112\overline{31311}] + [0; 1121222\overline{113}] > 4.187546$ in the region $(4.1857, 4.199)$ because the strings 3131, 2131, 1113111 are forbidden. Thus, the second transition becomes $231\alpha_{n+3}$ and $113112\beta_{n+6}$ in the region $(4.1857, 4.187546)$ since $\lambda_0(a_n113^*11\overline{21}) < 4.16781$. Moreover, $\lambda_0(a_{n-4}221223^*13313\overline{21}) < 4.187543$, so that the second transition becomes $2313\alpha_{n+4}$ and $11311\beta_{n+4}$ in the region $(4.187546, 4.2527275)$.

In summary, we showed that the possibilities for the second transition in the region $(4.1673, 4.2527275)$ are $23132\alpha_{n+5} - 113\beta_{n+3}$, $2312\alpha_{n+4} - 11311\beta_{n+5}$, $231\alpha_{n+3} - 113111\beta_{n+6}$, $231\alpha_{n+4} - 113112\beta_{n+6}$ and $2313\alpha_{n+4} - 11311\beta_{n+5}$. By combining this information with the facts that the Cantor set $C = 1, 2, 3$ with 3131 and 2131 forbidden has dimension < 0.67785, and $0.67785 + 0.118 = 0.79585$, we derive that $\dim((M \setminus L) \cap (4.1673, 4.2527275)) < 0.79585$.

3.9. **Refinement of the control in the region** $(4.2527275, 4.32372)$**.** Recall that in the region $(4.01, \sqrt{20})$, we have to investigate the transitions:

- $\alpha_n = 331\alpha_{n+3}$ and $\beta_n = 213\beta_{n+3}$;
- $\alpha_n = 23\alpha_{n+2}$ and $\beta_n = 113\beta_{n+3}$.

Since 3131 is forbidden here, $\lambda_0(33^*13\overline{23}) < 4.081$, $\lambda_0(3313^{**}\overline{23}) < 4.21$, $\lambda_0(23^{***}2) < 4$, the first transition becomes $331323\alpha_{n+6}$ and $213\beta_{n+3}$.

Similarly, since 3131 is forbidden, $\lambda_0(23^*132\overline{12}) < 4.21271$, $\lambda_0(113^*113\overline{1}) < 4.2022$ and $\lambda_0(113113^*\overline{1}) < 4.18$, so the second transition becomes $23132\alpha_{n+5}$ and $1131\beta_{n+4}$.

Given that $\frac{|I(a_1,\dots,a_n,3,3,1,3,2,3)|}{|I(a_1,\dots,a_n)|} < \frac{1}{160678}$, $\frac{|I(a_1,\dots,a_n,2,1,3)|}{|I(a_1,\dots,a_n)|} < 0.0071$, $\frac{|I(a_1,\dots,a_n,2,3,1,3,2)|}{|I(a_1,\dots,a_n)|} < 0.00012$, $\frac{|I(a_1,\dots,a_n,1,1,3,1)|}{|I(a_1,\dots,a_n)|} < \frac{1}{144}$, and

$$(\tfrac{1}{160678})^{0.09} + (0.0071)^{0.09} < 1, \quad (0.00012)^{0.1021} + (\tfrac{1}{144})^{0.1021} < 1,$$

we conclude that $\dim((M \setminus L) \cap (4.2527275, 4.32372)) < 0.6913 + 0.1021 = 0.7934$ thanks to the fact that $C = 1, 2, 3$ with 1313, 3131 and 21312 forbidden has dimension < 0.6913.

3.10. **Refinement of the control in the region** $(4.32372, 4.385)$**.** Similarly to [11], we can use $C = \{1, 2, 3\}^{\mathbb{Z}}$ and a block B to show that the continuations of words with values in $(M \setminus L) \cap (4.01, \sqrt{20})$ are

- $\alpha_n = 331\alpha_{n+3}$ and $\beta_n = 213\beta_{n+3}$;
- $\alpha_n = 23\alpha_{n+2}$ and $\beta_n = 113\beta_{n+3}$.

For later use, we note that 31313, 21313, 31312, 21312, 1113131, 1313111, 3131112, 2111313, 3131113, 3111313 are forbidden here, and the corresponding Cantor set has dimension < 0.6948.

Let us discuss the first transition. The continuation $213113\overline{231}$ is valid in the region $(4.3, 4.385)$, so that the first transition becomes $3313\alpha_{n+4}$ and $21311\beta_{n+5}$.

Let us now investigate the second transition. The validity of the continuations $2313\overline{3}$ and $113\overline{113}$ say that the second transition becomes $2313\alpha_{n+4}$ and $1131\beta_{n+4}$.

In the region $(4.3353, 4.385)$, the second transition is $231311\alpha_{n+4}$ and $1131\beta_{n+3}$ thanks to the valid continuation $2313113111\overline{1323}$ and the fact that 31313 and 31312 are forbidden.

Next, we observe that 313112, 211313, 313111 and 111313 are forbidden in the region $(4.32372, 4.3353)$. We will analyse the second transition in this region depending on the digits appearing before it.

If the digit before is $a_n = 3$, the second transition becomes $2313\alpha_{n+4}$ and $11313\beta_{n+5}$ in the region $(4.332, 4.3353)$ because the continuation $a_n113133\overline{132}$ is valid, and it becomes $23132313\alpha_{n+8}$ and $1131\beta_{n+4}$ in the region $(4.32372, 4.332)$ because the string 23131 is forbidden and the continuation $a_n2313231\overline{3}$ is valid.

If the digit before is $a_n = 2$, the second transition is $2313\alpha_{n+4}$ and $113121\beta_{n+5}$ in the region $(4.32372, 4.3353)$ because the continuation $a_n1131211\overline{323}$ is valid and 211313 is forbidden.

If the digits before are $a_{n-1}a_n = 11$, the second transition is $2313\alpha_{n+4}$ and $11312\beta_{n+5}$ in the region $(4.32372, 4.3353)$ because 111313 is forbidden and $a_{n-1}a_n113123\overline{132}$ is a valid continuation.

If the digits before are $a_{n-1}a_n = 21$, the second transition is $2313\alpha_{n+4}$ and $11312\beta_{n+5}$ in the region $(4.329, 4.3353)$ because 111313 is forbidden and $a_{n-1}a_n113123\overline{132}$ is valid, and it becomes $23132313\alpha_{n+8}$ and $1131\beta_{n+4}$ in the region $(4.32372, 4.329)$ since 23131 is forbidden and $2313231\overline{3}$ is valid.

If the digits before are $a_{n-1}a_n = 31$, the second transition is $23131\alpha_{n+5}$ and $1131\beta_{n+4}$ in the region $(4.332, 4.3353)$ because $a_{n-1}a_n231311311\overline{1323}$ is valid, and it becomes $2313\alpha_{n+4}$ and $1131113\beta_{n+7}$ in the region $(4.32372, 4.332)$ because $a_{n-1}a_n11312$ and $a_{n-1}a_n11313$ are forbidden and $a_{n-1}a_n11311133\overline{132}$ is valid.

In summary, we established that the possible transitions are $3313\alpha_{n+4} - 21311\beta_{n+5}$, $231311\alpha_{n+6} - 1131\beta_{n+4}$, $2313\alpha_{n+4} - 11313\beta_{n+5}$, $23132313\alpha_{n+8} - 1131\beta_{n+4}$, $2313\alpha_{n+4} - 11312\beta_{n+5}$, $2313\alpha_{n+4} - 1131113\beta_{n+7}$. Since $0.6948 + 0.10155 = 0.79635$, we conclude that $\dim((M \setminus L) \cap (4.32372, 4.385)) < 0.79635$.

3.11. **Refinement of the control in the region** $(4.385, 4.41)$**.** Recall that in the region between 4.01 and $\sqrt{20}$ our goal is to study the transitions:

- $\alpha_n = 331\alpha_{n+3}$ and $\beta_n = 213\beta_{n+3}$;
- $\alpha_n = 23\alpha_{n+2}$ and $\beta_n = 113\beta_{n+3}$.

Let us discuss the second transition. It extends as $2313111\alpha_{n+7}$ and $1131\beta_{n+4}$ in this region because the strings 31313, 31312 and 21313 are forbidden and the continuations $2313111\overline{1323}$ and $113\overline{113}$ are valid.

We observe that the Cantor set $C = 1, 2, 3$ with 31313, 31312, 21313, 121312 and 213121 forbidden (as they are in this region) has Hausdorff dimension < 0.6975.

The first transition becomes $3313111\alpha_{n+7}$ and $2131\beta_{n+4}$ in this region due to the valid continuations $3313111\overline{1323}$ and $2131\overline{1323}$ and to the fact that 331312 and 31313 are forbidden.

In summary, we showed that the possible transitions in our region are $3313111\alpha_{n+7} - 2131\beta_{n+4}$, and $2313111\alpha_{n+7} - 1131\beta_{n+4}$.

Since $0.6975 + 0.0963 = 0.7938$, we conclude that $\dim((M \setminus L) \cap (4.385, 4.41)) < 0.7938$.

3.12. **Refinement of the control in the region** $(4.41, \sqrt{20})$**.** The second transition extends as $23131\alpha_{n+5}$ and $11313\beta_{n+5}$ in this region due to the valid continuations $231311\overline{1323}$ and $11313\overline{3132}$.

Let us now discuss the first transition. Since $331311\overline{1323}$ and $21311\overline{323}$ are valid continuations when the Markov value is > 4.332, the first transition extends $33131\alpha_{n+5}$ and $2131\beta_{n+4}$ in our region.

In the region $(4.46151, \sqrt{20})$, we have a valid continuation $331312\overline{3132}$, so that the first transition becomes $331312\alpha_{n+6}$ and $2131\beta_{n+4}$ (as 31313 is forbidden). Thus, it remains only to treat the region $(4.41, 4.46151)$.

If the digits appearing before the first transition are $a_{n-1}a_n = 13$, the first transition becomes:

- $33131\alpha_{n+5}$ and $21313\beta_{n+5}$ in the region $(4.4608, 4.46151)$ thanks to the valid continuation $21313\overline{3132}$;
- $3313111\alpha_{n+7}$ and $2131\beta_{n+4}$ in the region $(4.41, 4.461)$ due to the valid continuation $3313111\overline{1323}$ and the fact that $a_{n-1}a_n331312$ and 31313 are forbidden.

If the digits appearing before the first transition are $a_{n-1}a_n \neq 13$, the first transition becomes:

- $33131\alpha_{n+5}$ and $213121\beta_{n+5}$ in the region $(4.456, 4.46151)$ thanks to the fact that $a_{n-1}a_n21313$ is forbidden and the validity of the continuation $213121\overline{1323}$;
- $3313111\alpha_{n+7}$ and $2131\beta_{n+4}$ in the region $(4.41, 4.459)$ due to the valid continuation $3313111\overline{1323}$ and the fact that $a_{n-1}a_n331312$ and 31313 are forbidden.

In summary, we showed that the possible transitions in our region are $331312\alpha_{n+6} - 2131\beta_{n+4}$, $33131\alpha_{n+5} - 213121\beta_{n+6}$, $3313111\alpha_{n+7} - 2131\beta_{n+4}$, $33131\alpha_{n+5} - 21313\beta_{n+5}$, and $23131\alpha_{n+5} - 11313\beta_{n+5}$. Since $0.7057 + 0.0903 = 0.796$, we conclude that $\dim((M \setminus L) \cap (4.41, \sqrt{20})) < 0.796$.

3.13. **Refinement of the control in the region** $(\sqrt{20}, 4.4984)$**.** Similarly to [11], we can use $C \subset \{1,2,3,4\}^{\mathbb{Z}}$ where $14, 41, 24, 42$ are forbidden and a certain block B to show that the continuations of words with values in $(M \setminus L) \cap (\sqrt{20}, 4.4984)$ are

- $\alpha_n = 4\alpha_{n+1}$ and $\beta_n = 3131\beta_{n+4}$, or
- $\alpha_n \in \{33131\alpha_{n+5}, 34\alpha_{n+3}\}$ and $\beta_n = 2131\beta_{n+4}$, or
- $\alpha_n = 23\alpha_{n+2}$ and $\beta_n = 1131\beta_{n+4}$.

Note that a sequence containing the strings 343 or 31313 has Markov value > 4.52. In particular, we can refine C into $C = 1, 2, 3, 4$ where $14, 41, 24, 42, 343, 31313$ are forbidden. Note that $\dim(C) < 0.705$.

If the sequence θ in C contains 4 and it is not $\overline{4}$ (whose Markov value is $\sqrt{20}$), then it contains 43 and, *a fortiori*, $\theta = \dots 443 \dots$.

Suppose that $\lambda_0(\dots 44^*3 \dots) \geqslant \lambda_0(\dots 4^*43 \dots)$, i.e.,

$$4 + \alpha + \frac{1}{4+\beta} \geqslant 4 + \beta + \frac{1}{4+\alpha}$$

where $\alpha = [0; 3, \dots]$. This would imply that $\alpha \geqslant \beta$, so that

$$\lambda_0(\dots 44^*3 \dots) \geqslant 4 + \alpha + \frac{1}{4+\alpha}$$

for $\alpha = [0; 3, \dots]$. Because the minimal value of α extracted from a sequence $\theta \in C$ is

$$\alpha \geqslant [0; \overline{3, 1, 3, 1, 2, 1}],$$

we would have that

$$\lambda_0(\dots 44^*3 \dots) \geqslant [4; \overline{3, 1, 3, 1, 2, 1}] + [0; 4, \overline{3, 1, 3, 1, 2, 1}] > 4.4984.$$

Therefore, we can assume that 4 doesn't appear in sequences θ producing Markov values in the interval $(\sqrt{20}, 4.4984)$. In particular, the continuations of words with values in $(M \setminus L) \cap (\sqrt{20}, 4.4984)$ are actually

(i) $\alpha_n = 33131\alpha_{n+5}$ and $\beta_n = 2131\beta_{n+4}$, or
(ii) $\alpha_n = 23\alpha_{n+2}$ and $\beta_n = 1131\beta_{n+4}$.

We affirm that (i) has $\alpha_n = 331312\alpha_{n+6}$: indeed, this happens because of the presence of the continuation $3313123\overline{1}$ (which is valid as $\lambda_0(\dots 3313^*123\overline{1}) \leqslant 4.463 < \sqrt{20}$). Similarly, we affirm that (ii) has $\alpha_n = 23131\alpha_{n+5}$ and $\beta_n = 113131\beta_{n+6}$: in fact, this happens because of the presence of the continuations $2313\overline{1}$ and $11313\overline{1}$ (which are valid as $\lambda_0(\dots 2313^*\overline{1}) < 4.394 < \sqrt{20}$ and $\lambda_0(\dots 113^*13\overline{1}) < 4.42521 < \sqrt{20}$).

Since

$$\begin{aligned} & \left(\frac{|I(a_1,\dots,a_n,3,3,1,3,1,2)|}{|I(a_1,\dots,a_n)|}\right)^{0.09} + \left(\frac{|I(a_1,\dots,a_n,2,1,3,1)|}{|I(a_1,\dots,a_n)|}\right)^{0.09} \\ = & \left(\frac{(r+1)}{(53r+173)(72r+235)}\right)^{0.09} + \left(\frac{(r+1)}{(5r+14)(9r+25)}\right)^{0.09} \\ \leqslant & \ (1/34691)^{0.09} + (0.003106)^{0.09} < 0.985 < 1 \end{aligned}$$

and

$$\begin{aligned}
&\left(\frac{|I(a_1,\dots,a_n,2,3,1,3,1)|}{|I(a_1,\dots,a_n)|}\right)^{0.09}+\left(\frac{|I(a_1,\dots,a_n,1,1,3,1,3,1)|}{|I(a_1,\dots,a_n)|}\right)^{0.09}\\
=\ &\left(\frac{(r+1)}{(19r+43)(34r+77)}\right)^{0.09}+\left(\frac{(r+1)}{(24r+43)(43r+77)}\right)^{0.09}\\
\leqslant\ &(0.00031)^{0.09}+(1/3311)^{0.09}<0.966<1
\end{aligned}$$

for $0<r<1$, we derive that $\dim((M\setminus L)\cap(\sqrt{20},4.4984))<0.705+0.09=0.795$.

Remark 3.8. Even though this fact will not be used here, we note that 4.4984 is somewhat close to the point $\alpha+\frac{1}{\alpha}=4.49846195\dots$, where

$$\alpha=[4;31312133\overline{1131312231312111233131212112}],$$

which is the smallest element of the Lagrange spectrum accumulated by Lagrange values of sequences containing the letter 4 infinitely often: cf. [16].

3.14. **Refinement of the control in the region** $(4.4984,\sqrt{21})$**.** Similarly to [11], we can use $C\subset\{1,2,3,4\}^{\mathbb{Z}}$ where $14,41,24,42$ are forbidden and a certain block B to show that the continuations of words γ with values in $(M\setminus L)\cap(4.4984,\sqrt{21})$ are

(1) $\alpha_n=4\alpha_{n+1}$ and $\beta_n=3131\beta_{n+4}$, or
(2) $\alpha_n\in\{33131\alpha_{n+5},34\alpha_{n+3}\}$ and $\beta_n=2131\beta_{n+4}$, or
(3) $\alpha_n=23\alpha_{n+2}$ and $\beta_n=1131\beta_{n+4}$.

In the sequel, we shall significantly refine the analysis of these continuations.

3.14.1. *The case* (1) *of* $\alpha_n=4\alpha_{n+1}$ *and* $\beta_n=3131\beta_{n+4}$. We affirm that $\alpha_n=44\alpha_{n+2}$ in this situation. In fact, given the nature of C, our task is to rule out the other possibility that $\alpha_n=43\alpha_{n+2}$. In this direction, the following lemma (obtained from a direct calculation) will be helpful:

Lemma 3.9. $[4;\overline{4}]+[0;3,\overline{1,3,1,2}]=\frac{9}{2}=[3;1,3,\overline{4}]+[0;\overline{1,2,1,3}]$. *In particular,*

$$9/2=m(\overline{4}3\overline{1312})=\lim_{n\to\infty}m(\overline{4^n3(1312)^n1313})\in L\cap\mathbb{Q}.$$

If γ has allowed continuations $\alpha_n=43\alpha_{n+2}$ and $\beta_n=3131\beta_{n+4}$, then its Markov value is $<9/2$. Otherwise, its Markov value $m(\gamma)$ would be $>9/2$ (as $9/2\in L$ and γ is assumed to give rise to an element of $M\setminus L$), and the previous lemma would permit us to connect γ with an adequate block B via $\gamma4^n3\overline{1312}$, a contradiction.

Now, if γ has allowed continuations $\alpha_n=43\alpha_{n+2}$ and $\beta_n=3131\beta_{n+4}$, and its Markov value is $<9/2$, then let us write $\gamma=\theta a_{n+1}\dots$, select $\beta\in\{\theta,\theta4\}$ with $[\beta^t]\geqslant[4;\overline{4}]$ and let us consider the Markov value $m(\beta3\rho)<9/2$ of an allowed continuation $\beta3\rho$. If $[0;3\rho]\geqslant[0;3,\overline{1,3,1,2}]$, then $m(\beta3\rho)\geqslant[\beta^t]+[0;3,\rho]\geqslant[\overline{4}]+[0;3,\overline{1,3,1,2}]=9/2$, a contradiction. If $[0;3,\rho]<[0;3,\overline{1,3,1,2}]$, then $\rho=13\mu$ with $[0;\mu]>[0;\overline{1,2,1,3}]$, but $m(\beta3\rho)<9/2$ would force $[3;\mu]+[0;1,3,\beta^t]<9/2$, so that

$$9/2>m(\beta3\rho)\geqslant[\beta^t]+[0;3,1,3,\mu]>[\beta^t]+[0;3,1,9/2-[0;1,3,\beta^t]].$$

This is a contradiction because the right-hand side is an increasing function of $[\beta^t]\geqslant[\overline{4}]$ whose value at $[\beta^t]=[\overline{4}]$ is $9/2$ after the previous lemma.

In summary, we proved that, in any scenario, the case (1) is actually

$(1')$ $\alpha_n=44\alpha_{n+2}$ and $\beta_n=3131\beta_{n+4}$.

In what follows, we shall analyse the natural subdivision $(1')$ into two scenarios:

$(1i)$ $\alpha_n = 444\alpha_{n+3}$ is an allowed continuation,
$(1ii)$ $\alpha_n = 443\alpha_{n+3}$ and $\beta_n = 3131\beta_{n+4}$.

3.14.2. *The subcase* $(1i)$. We affirm that $\beta_n = 3131213\beta_{n+7}$ in this situation. In fact, let us begin by noticing that the Markov value of $\gamma = \rho a_{n+1}\dots$ with allowed continuations of type $(1i)$ is < 4.513: otherwise, we would be able to connect to an adequate block B by continuing with $\alpha_n' = 443\overline{1}$ (since $[4;3,\overline{1}]+[0;4,\rho^t] \leqslant [4;3,\overline{1}]+[0;4,\overline{4,3}] < 4.513$). Next, we observe that the strings 31313 and 343 are forbidden for any sequence with Markov value < 4.513.

Now, let us study the possible extensions of $\beta_n = 3131\beta_{n+4}$. We have that $\gamma = \rho a_{n+1}\dots$ where ρ ends with 3 or 4 (because $\alpha_n = 4\dots$ is allowed and $14, 24$ are forbidden strings). If ρ ends with 4, we observe that the estimate

$$[3;1,3,\rho^t]+[0;1,2,1,\overline{3,3,1,3}] \leqslant [3;1,3,\overline{4,4,3,4}]+[0;1,2,1,\overline{3,3,1,3}] < 4.49838$$

would allow to connect γ to an adequate block B unless $\beta_n = 3131213\beta_{n+7}$. Similarly, if ρ ends with 3, say $[\rho^t] = 3+x$ with $0 < x < [0;1,3,1,2,\overline{1,3}]$, then the continuation $\alpha_n = 444\alpha_{n+3}$ would lead to an estimate

$$\begin{aligned} m(\rho 444\alpha_{n+3}) &\geqslant [4;3+x]+[0;4,4,\alpha_{n+3}] \geqslant [4;3+x]+[0;4,4,\overline{4,3,4,4}] \\ &\geqslant [3;1,3,3+x]+[0;1,2,1,\overline{3,3,1,3}]+0.000076 \end{aligned}$$

allowing to connect γ to an adequate block B unless $\beta_n = 3131213\beta_{n+7}$.

In other terms, we showed that $(1i)$ actually is

$(1i')$ $\alpha_n \in \{444\alpha_{n+3}, 443\alpha_{n+3}\}$ and $\beta_n = 3131213\beta_{n+4}$.

Here, note that the relevant Cantor set $C = 1,2,3,4$ with where $14, 41, 24, 42, 343, 31313$ are forbidden has $\dim(C) < 0.705$, and

$$\begin{aligned} &\frac{|I(a_1,\dots,a_n,4,4)|^{0.086}+|I(a_1,\dots,a_n,3,1,3,1,2,1,3)|^{0.086}}{|I(a_1,\dots,a_n)|^{0.086}} \\ \leqslant\ & \left(\frac{r+1}{(4r+17)(5r+21)}\right)^{0.086}+\left(\frac{r+1}{(71r+269)(90r+341)}\right)^{0.086} \\ \leqslant\ & \left(\frac{1}{273}\right)^{0.086}+\left(\frac{1}{73270}\right)^{0.086} < 1. \end{aligned}$$

3.14.3. *The subcase* $(1ii)$. If the Markov value of γ is $m(\gamma) \leqslant 4.5274$, then the string 31313 is forbidden. In particular, $\beta_n = 313121\beta_{n+6}$ (by comparison with $31312\overline{1}$). Here,

$$\begin{aligned} &\frac{|I(a_1,\dots,a_n,4,4,3)|^{0.087}+|I(a_1,\dots,a_n,3,1,3,1,2,1)|^{0.087}}{|I(a_1,\dots,a_n)|^{0.087}} \\ \leqslant\ & \left(\frac{r+1}{(13r+55)(17r+72)}\right)^{0.087}+\left(\frac{r+1}{(14r+53)(19r+72)}\right)^{0.087} \\ \leqslant\ & \left(\frac{1}{3026}\right)^{0.087}+\left(\frac{2}{6097}\right)^{0.087} < 1. \end{aligned}$$

If the Markov value of γ is $m(\gamma) > 4.5274$, then we affirm that it can not continued as $\alpha_n = 4431\alpha_{n+4}$: otherwise, we would have a continuation $44323\overline{1}$ connecting to an adequate block B, a contradiction. This leaves us with two possibilities:

$(1ii')$ $m(\gamma) \leq 4.53422$, so $\alpha_n = 443\alpha_{n+3}$ cannot extend as 4433 nor 4434, and thus $\alpha_n = 443\alpha_{n+3}$ extends only as $\alpha_n = 4432\alpha_{n+4}$;
$(1ii'')$ $m(\gamma) > 4.53422$.

In the subcase $(1ii')$, we observe that

$$\begin{aligned}
&\frac{|I(a_1,\dots,a_n,4,4,3,2)|^{0.0881}+|I(a_1,\dots,a_n,3,1,3,1)|^{0.0881}}{|I(a_1,\dots,a_n)|^{0.0881}}\\
\leqslant\ &\left(\frac{r+1}{(30r+127)(43r+182)}\right)^{0.0881}+\left(\frac{r+1}{(5r+19)(9r+34)}\right)^{0.0881}\\
\leqslant\ &\left(\frac{2}{35325}\right)^{0.0881}+\left(\frac{1}{516}\right)^{0.0881}<1.
\end{aligned}$$

In this case we will use the fact that the Cantor set $C=1,2,3,4$ where 41, 42, 434, 433 and their transposes are forbidden has dimension <0.7081. Notice that $0.7081+0.0881=0.7962$.

In the subcase $(1ii'')$, we note that $\beta_n=31313\beta_{n+5}$: otherwise, a continuation $31313443\overline{2}$ would allow to connect to an adequate block B, a contradiction. Here, we observe for later use that

$$\begin{aligned}
&\frac{|I(a_1,\dots,a_n,4,4,3)|^{0.084}+|I(a_1,\dots,a_n,3,1,3,1,3)|^{0.084}}{|I(a_1,\dots,a_n)|^{0.084}}\\
\leqslant\ &\left(\frac{r+1}{(13r+55)(17r+72)}\right)^{0.084}+\left(\frac{r+1}{(19r+72)(24r+91)}\right)^{0.084}\\
\leqslant\ &\left(\frac{1}{3026}\right)^{0.084}+\left(\frac{2}{10465}\right)^{0.084}<1.
\end{aligned}$$

3.14.4. *The case* (3) *of* $\alpha_n=23\alpha_{n+2}$ *and* $\beta_n=1131\beta_{n+4}$. Analogously to the analysis of this situation in the region $(\sqrt{20},4.4984)$, we have that $\alpha_n=23131\alpha_{n+5}$ and $\beta_n=113131\beta_{n+6}$ together with the estimate

$$\begin{aligned}
&\left(\frac{|I(a_1,\dots,a_n,2,3,1,3,1)|}{|I(a_1,\dots,a_n)|}\right)^{0.0857}+\left(\frac{|I(a_1,\dots,a_n,1,1,3,1,3,1)|}{|I(a_1,\dots,a_n)|}\right)^{0.0857}\\
=\ &\left(\frac{(r+1)}{(19r+43)(34r+77)}\right)^{0.0857}+\left(\frac{(r+1)}{(24r+43)(43r+77)}\right)^{0.0857}\\
\leqslant\ &(0.00031)^{0.0857}+(1/3311)^{0.0857}<0.9997<1.
\end{aligned}$$

3.14.5. *The case* (2) *of* $\alpha_n\in\{33131\alpha_{n+5},34\alpha_{n+3}\}$ *and* $\beta_n=2131\beta_{n+4}$. Suppose that *both* continuations $33131\alpha_{n+5}$ and $34\alpha_{n+3}$ are allowed. In this context, *any* extension of $34\alpha_{n+3}$ which does *not* increase Markov values would be valid. Among them, we see from the discussion in the beginning of subsection about the region $(\sqrt{20},4.4984)$ that such a minimal extension has the form $\rho3443\rho^t$. Thus, ρ and ρ^t can not connect on an adequate block B and, hence, we could use Proposition 7.8 in [11] to get that the set of the Markov values associated to such $\gamma=\rho a_{n+1}\dots$ has Hausdorff dimension $<2\cdot0.173<0.35$.

Therefore, there is no loss of generality in assuming that only one of the continuations $33131\alpha_{n+5}$ and $34\alpha_{n+3}$ is allowed.

If the continuation $34\alpha_{n+3}$ is not allowed, we have two possibilities:

- if $m(\gamma)<4.52$, then the strings 31313 and 343 are forbidden and we get $\alpha_n=331312\alpha_{n+6}$ (by comparison with $33131\overline{2}$) and $\beta_n=2131\beta_{n+4}$;
- if $m(\gamma)\geqslant4.52$, then we get $\alpha_n=33131\alpha_{n+5}$ and $\beta_n=213131\beta_{n+6}$ (thanks to the continuation $21313\overline{1}$ with $\lambda_0(\dots213^*13\overline{1})<4.5197$).

In the first case, since $\frac{|I(a_1,\dots,a_n,3,3,1,3,1,2)|}{|I(a_1,\dots,a_n)|} \leqslant \frac{r+1}{(53r+173)(72r+235)} \leqslant \frac{1}{34691}$, $\frac{|I(a_1,\dots,a_n,2,1,3,1)|}{|I(a_1,\dots,a_n)|} \leqslant \frac{r+1}{(5r+14)(9r+25)} \leqslant 0.00311$, and

$$\left(\frac{1}{34691}\right)^{0.09} + (0.00311)^{0.09} < 0.9851,$$

recalling that, if $C = 1,2,3,4$ where $14, 41, 24, 42, 343, 31313$ are forbidden, then $\dim(C) < 0.705$ we will get the upper estimate $0.705 + 0.09 = 0.795$. In the second case, since $\frac{|I(a_1,\dots,a_n,3,3,1,3,1)|}{|I(a_1,\dots,a_n)|} \leqslant \frac{r+1}{(19r+62)(34r+111)} \leqslant \frac{2}{11745}$, $\frac{|I(a_1,\dots,a_n,2,1,3,1,3,1)|}{|I(a_1,\dots,a_n)|} \leqslant \frac{r+1}{(24r+67)(43r+120)} \leqslant 0.000136$, and

$$\left(\frac{2}{11745}\right)^{0.08} + (0.000136)^{0.08} < 0.991,$$

it remains only to treat the possibility of $\alpha_n = 34\alpha_{n+3}$ and $\beta_n = 2131\beta_{n+4}$ being the unique allowed continuation.

In the case of $\alpha_n = 34\alpha_{n+3}$ and $\beta_n = 2131\beta_{n+4}$, if $m(\gamma) < 4.527$, then the strings 31313 and 343 are forbidden and $\alpha_n = 344\alpha_{n+4}$. If *both* continuations $\alpha_n = 3443\alpha_{n+5}$ and $\alpha_n = 3444\alpha_{n+5}$ are allowed, then any continuation $3444\dots$ which does not increase Markov values would be allowed and the same analysis of the first paragraph of this subsection (considering sequences of the type $\gamma 344443\gamma^t$) implies that the corresponding set of Markov values $m(\gamma) < 4.527$ has Hausdorff dimension < 0.35. In other words, there is no loss of generality in assuming that only one of the continuations $\alpha_n = 3443\alpha_{n+5}$ or $\alpha_n = 3444\alpha_{n+5}$ when $m(\gamma) < 4.527$. Since $\frac{|I(a_1,\dots,a_n,3,4,4,4)|}{|I(a_1,\dots,a_n)|}$, $\frac{|I(a_1,\dots,a_n,3,4,4,3)|}{|I(a_1,\dots,a_n)|} \leqslant \frac{2}{71065}$, $\frac{|I(a_1,\dots,a_n,2,1,3,1)|}{|I(a_1,\dots,a_n)|} \leqslant 0.00311$ and

$$\left(\frac{2}{71065}\right)^{0.09} + (0.00311)^{0.09} < 0.985 < 1,$$

our task is reduced to discuss the case of $\alpha_n = 34\alpha_{n+3}$, $\beta_n = 2131\beta_{n+4}$, and $m(\gamma) \geqslant 4.527$. In this regime, the continuation $21313\overline{1}$ is allowed, so that $\beta_n = 213131\beta_{n+7}$. If $4.527 \leqslant m(\gamma) \leqslant 4.55$, the strings 3433, 3434 and 2131313 are forbidden (as $\lambda_0(34^*33) > 4.56593$ and $\lambda_0(21313^*13) > 4.55065$) so that $\beta_n = 21313121\beta_{n+9}$ (thanks to the continuation $2131312\overline{1}$) in this context. Since

$$\left(\frac{|I(a_1,\dots,a_n,3,4)|}{|I(a_1,\dots,a_n)|}\right)^{0.08745} + \left(\frac{|I(a_1,\dots,a_n,2,1,3,1,3,1,2,1)|}{|I(a_1,\dots,a_n)|}\right)^{0.08745} \leqslant$$
$$\left(\frac{2}{357}\right)^{0.08745} + (9.71\times 10^{-6})^{0.08745} < 0.99992 < 1,$$

In this case we will use the fact that the Cantor set $C = 1,2,3,4$ where 41, 42, 3433, 3434 and their transposes are forbidden has dimension < 0.7083. Notice that $0.7083 + 0.08745 = 0.79575$.

It remains to treat the case $\alpha_n = 34\alpha_{n+3}$, $\beta_n = 213131\beta_{n+7}$, and $m(\gamma) > 4.55$. If α_n can *not* be extended as *both* 343 or 344, we can use the estimates $\frac{|I(a_1,\dots,a_n,3,4,4)|}{|I(a_1,\dots,a_n)|}$, $\frac{|I(a_1,\dots,a_n,3,4,3)|}{|I(a_1,\dots,a_n)|} \leqslant \frac{1}{1980}$, $\frac{|I(a_1,\dots,a_n,2,1,3,1,3,1)|}{|I(a_1,\dots,a_n)|} \leqslant 0.000136$ and $(1/1980)^{0.085} + (0.000136)^{0.085} < 0.994$ to reduce our task to the study of the situation where *both* extensions 343 and 344 *are* allowed. Here, we can use the case (1) above to see that 344 extends as 34443 (as $\lambda_0(34^*444) < 4.546$ would permit to continue as $344443\overline{1}$ and so to connect to an adequate block B). Furthermore, 34443 must continue as 344434 (in view of the allowed continuation $3444344\overline{3}$). Also, 343 extends as 34313131 (thanks to the continuation $3431313\overline{1}$

- notice that $[3;1,3,\overline{1}]+[0;1,3,4,3,\dots]\le[3;1,3,\overline{1}]+[0;1,3,4,3,\overline{1,3}]<4.55$ and, if 343 can be followed by some word $a_{n+4}a_{n+5}a_{n+6}a_{n+7}a_{n+8}$ with $[0;a_{n+4},a_{n+5},a_{n+6},a_{n+7},a_{n+8}]<[0;1,3,1,3,1]$, then $[4;3,1,3,1,3,\overline{1}]<[4;3,a_{n+4},a_{n+5},a_{n+6},a_{n+7},a_{n+8},\dots]$). In summary, if $m(\gamma)>4.55$ and both 343 and 344 are permitted, then $\alpha_n\in\{34313131\alpha_{n+9},344434\alpha_{n+7}\}$ and $\beta_n=213131\beta_{n+7}$. Since

$$\frac{|I(a_1,\dots,a_n,3,4,3,1,3,1,3,1)|}{|I(a_1,\dots,a_n)|}\leqslant\frac{2}{4835349},\frac{|I(a_1,\dots,a_n,3,4,4,4,3,4)|}{|I(a_1,\dots,a_n)|}\leqslant\frac{1}{11142860},$$

$$\frac{|I(a_1,\dots,a_n,2,1,3,1,3,1)|}{|I(a_1,\dots,a_n)|}\leqslant 0.000136$$

and $(\frac{2}{4835349})^{0.0857}+(\frac{1}{11142860})^{0.0857}+(0.000136)^{0.0857}<0.999<1$, the analysis of the case (2) is now complete.

3.14.6. *End of the study of the region* $(4.4984,\sqrt{21})$. Our discussion of the cases (1), (2) and (3) above shows that $\dim((M\backslash L)\cap(4.4984,\sqrt{21}))<\max\{0.7094+0.0857,0.7962\}=0.7962$.

4. Algorithm for computing the Hausdorff dimension

It remains to get rigorous bounds for $\dim_H B_1$, $\dim_H B_2$, $\dim_H X$, $\dim_H Y$ and $\dim_H\Omega$ claimed in the previous sections. In this section we would like to explain how to adapt the method developed in [15] to the present setting, to make the computation practical.

4.1. **The setting.** We begin with a general setting.

Definition 4.1. Let $\mathcal{A}=\{1,2\}$ be an alphabet. Let $\bar r=(r_1,\dots,r_k)\in\mathbb{N}^k$ be the vector of lengths. Consider a set of *forbidden words*

$$F:=F(\bar r)=\left\{d_1^{(1)}d_2^{(1)}\dots d_{r_1}^{(1)}\in\mathcal{A}^{r_1},\ d_1^{(2)}d_2^{(2)}\dots d_{r_2}^{(2)}\in\mathcal{A}^{r_2},\dots,d_1^{(k)}d_2^{(k)}\dots d_{r_k}^{(k)}\in\mathcal{A}^{r_k}\right\}.$$

A set $X_F\subset[0,1]$ is defined by continued fraction expansions of its elements with extra Markov conditions:

$$\begin{aligned}X_F:=\Big\{[0;a_1,a_2,\dots]\mid a_n\in\{1,2\},\ &\text{such that for all } j\geqslant 1\\ &a_ja_{j+1}\dots a_{j+r_1}\neq d_1^{(1)}d_2^{(1)}\dots d_{r_1}^{(1)}\\ &a_ja_{j+1}\dots a_{j+r_2}\neq d_1^{(2)}d_2^{(2)}\dots d_{r_2}^{(2)}\\ &\vdots\ \ \vdots\qquad\vdots\qquad\quad\vdots\\ &a_ja_{j+1}\dots a_{j+r_k}\neq d_1^{(k)}d_2^{(k)}\dots d_{r_k}^{(k)}\Big\}\subsetneq E_2.\end{aligned}$$

We next want to introduce a Markov iterated function scheme of uniformly contracting maps whose limit set is X_F. We take two maps $T_1,T_2:[0,1]\to[0,1]$ defined by $T_1(x)=\frac{1}{1+x}$ and $T_2(x)=\frac{1}{2+x}$ and consider all possible compositions of length $n:=\max_{1\le i\le k}r_i-1$, i.e. we consider a collection of maps

$$\mathcal{T}_n=\{T_{\underline{a}_n}:=T_{a_1}\circ\dots\circ T_{a_n}\mid\underline{a}_n=a_1\dots a_n\in\{1,2\}^n\}.$$

The Markov condition can be written as a $2^n\times 2^n$–matrix $M=M(\underline{a}_n^j,\underline{b}_n^k)$ where

$$M(\underline{a}_n^j,\underline{b}_n^k)=\begin{cases}0, & \text{if concatenation } \underline{a}_n^j\underline{b}_n^k \text{ contains } d_1^{(i)}\dots d_{r_i}^{(i)} \text{ as a subword for some } 1\le i\le k;\\ 1, & \text{otherwise.}\end{cases}$$

It is a simple observation that the limit set of $(\mathcal{T}_n, M)$ is equal to X_F.

4.2. **The transfer operator.** In order to compute the Hausdorff dimension of the limit set of $(\mathcal{T}_n, M)$ we follow a general approach which dates back to Bowen and Ruelle [17]. More precisely, we use the connection between Hausdorff dimension of the limit set and the spectral radius of a transfer operator.

The transfer operator associated to a Markov iterated function scheme is a linear operator acting on the space of Hölder-continuous functions $C^\alpha(\{1,\dots,2^n\}\times[0,1])$, where $\{1,\dots,2^n\}\times[0,1]$ represents a disjoint union of 2^n copies of $[0,1]$ ([15], Section 2.4). It is defined by

$$\mathcal{L}_t\colon (f_{\underline{a}_n^1},\dots,f_{\underline{a}_n^{2^n}}) \mapsto (F^t_{\underline{b}_n^1},\dots,F^t_{\underline{b}_n^{2^n}}),$$

where

$$F^t_{b_n^k}(x) = \sum_{j=1}^{2^n} M(\underline{a}_n^j,\underline{b}_n^k)\cdot f_{\underline{a}_n^j}(T_{\underline{a}_n^j}(x))\cdot |T'_{\underline{a}_n^j}(x)|^t. \tag{4.1}$$

Our method is based on the following result (originally due to Ruelle, generalizing a more specific result of Bowen for limit sets of Fuchsian-Schottky groups):

Proposition 4.2 (after [17])**.** *Assume that the maximal positive eigenvalue of $\mathcal{L}_t$ is equal to 1. Then $\dim_H X_F = t$.*

In order to obtain lower and upper bounds on the maximal eigenvalue of $\mathcal{L}_t$ we use min-max inequalities as described in ([15], Section 3.1) which has the dual advantages of being easy to implement and also leading to rigorous results. More precisely, our numerical estimates are based on the following.

Lemma 4.3 ([15])**.** *Assume that there exist two positive functions*

$$\overline{f} = (f_{\underline{a}_n^1},\dots,f_{\underline{a}_n^{2^n}}),\ \overline{g} = (g_{\underline{a}_n^1},\dots,g_{\underline{a}_n^{2^n}}) \in C^\alpha(\{1,\dots,2^n\}\times[0,1])$$

such that for $\mathcal{L}_{t_0}\overline{f} = (F_{\underline{b}_n^1},\dots,F_{\underline{b}_n^{2^n}})$, $\mathcal{L}_{t_1}\overline{g} = (G_{\underline{b}_n^1},\dots,G_{\underline{b}_n^{2^n}})$ we have

$$\min_j \inf_x \frac{F_{\underline{b}_n^j}(x)}{f_{\underline{a}_n^j}(x)} > 1 \qquad \textit{and} \qquad \max_j \sup_x \frac{G_{\underline{b}_n^j}(x)}{g_{\underline{a}_n^j}(x)} < 1 \tag{4.2}$$

Then $t_0 \le \dim_H X_F \le t_1$.

We attempt to construct good choices of functions $f_{\underline{a}_n^j}$ and $g_{\underline{a}_n^j}$ for $1 \le j \le 2^n$ as positive polynomials of a relatively small degree using the collocation method. We fix a small natural m and define m Chebyshev nodes by

$$x_k := \frac{1}{2}\left(1+\cos\left(\frac{\pi(2k-1)}{2m}\right)\right) \in [0,1] \text{ for } k = 1,\dots,m.$$

The Lagrange interpolation polynomials are defined by $p_l(x) := \prod_{k=1}^m \frac{x-x_k}{x_l-x_k}$. These are the unique polynomials of minimal degree with the property that $p_l(x_k) = \delta_l^k$. We then consider the subspace of $C^\alpha(\{1,\dots,2^n\}\times[0,1])$ spanned by 2^n copies of the space $\langle p_k\rangle_{k=1}^m$:

$$\Pi(n,m) := \langle \{1,\dots,2^n\}\times\langle p_1,\dots,p_m\rangle\rangle \subset C^\alpha(\{1,\dots,2^n\}\times[0,1]).$$

Then the components of any $\bar{q} = (q_1,\dots,q_{2^n}) \in \Pi(n,m)$ are uniquely defined by their values at the Chebyshev nodes:

$$q_j(x) = \sum_{i=1}^m q_j(x_i)p_i(x) \in \mathbb{R}[x]; \quad j = 1,\dots,2^n. \tag{4.3}$$

In particular, the formula (4.3) defines a bijection $I\colon \mathbb{R}^{2^n m} \to \Pi(n,m)$. We introduce a projection operator $P\colon C^\alpha(\{1,\dots,2^n\}\times[0,1]) \to \mathbb{R}^{2^n m}$ given by

$$P(f_1,\dots,f_{2^n}) \mapsto (f_1(x_1),\dots,f_1(x_m),f_2(x_1),\dots,f_2(x_m),\dots,f_{2^n}(x_1),\dots,f_{2^n}(x_m)) \in \mathbb{R}^{2^n m}.$$

We may now consider a finite rank linear operator $B^t\colon \mathbb{R}^{2^n m} \to \mathbb{R}^{2^n m}$ defined by

$$B^t \bar{v} = P\mathcal{L}_t I \bar{v} \tag{4.4}$$

and construct the test functions $\bar{f}$ and $\bar{g}$ in (4.2) from the eigenvectors v_{t_0} and v_{t_1} corresponding to the leading eigenvalues of B^{t_0} and B^{t_1} respectively using the formulae $f = Iv_{t_0}$ and $g = Iv_{t_1}$.

Remark 4.4. This approach appears to be relatively straightforward to implement numerically compared to other methods. The bisection method can be used to get a refined estimate.

Nevertheless, practical implementation is challenging for large values of n. The first complication here is the computation of the matrix M that gives the Markov condition, since at first sight it requires analysing of 2^{2n} words of length $2n$ searching for forbidden substrings, and the resulting matrix of the size 2^{2n} would take about 2GB of computer memory[5] to store for a modest value $n = 17$ and for larger values $n > 19$ the resulting Markov matrix wouldn't fit into RAM memory of a personal computer.

Furthermore, the matrix B^t is even larger and requires much more space as it is not a binary matrix and its values need to be computed with higher accuracy. Typically we would like to work with 128 bits precision, so for modest values of $n = 17$ and $m = 6$ it would require 1512GB just to store.

A final complication is that the computation of the eigenvector of a huge matrix with high accuracy is also very time-consuming in practice. The best method here for us would be the power method, which has complexity of the matrix multiplication and depending on the realisation, but is no less than $O(n^{2.5})$.

In the remainder of the section we explain how to refine the basic algorithm to make it more practical.

4.3. **Simplifying the computation of the Markov matrix M.** The next statement gives the basis for our approach to making the computation tractable.

Proposition 4.5. *Assume that the columns j_1 and j_2 of the Markov matrix M are identical, i.e. for all $1 \le k \le 2^n$ we have that $M(\underline{a}_n^k, \underline{b}_n^{j_1}) \equiv M(\underline{a}_n^k, \underline{b}_n^{j_2})$. Then any eigenvector $\bar{f}$ of B^t lies in the subspace of $\Pi(n,m)$ for which $f_{\underline{a}_n^{j_1}} = f_{\underline{a}_n^{j_2}}$.*

We postpone the proof of this Proposition until Section 4.4. Fortuntately, it turns out that for the sets of forbidden words we need to deal with, the Markov matrix has a very small number of pairwise different columns (compared to its size).

Example 4.6. In particular, for the specific sets which we study in this paper, we have the following.

(1) In the case of the set B_1 which appears in Section §2.2.12 the Markov matrix has 41186 columns, of which only 138 are pairwise distinct.
(2) In the case of the set B_2 which appers in Section §2.2.11 the Markov matrix has 79034 columns of which only 184 are pairwise distinct.

[5] An optimistic estimate is that one needs at least 2^{2n} bits and for $n = 17$ we get 2^{34} bits, which is 2^{31} bytes, exactly 2GB.

(3) In the case of the set X which is used in Section §2.3.4 to obtain a lower bound on the transition value t_1, the Markov matrix has 3940388 columns of which only 429 are pairwise distinct.
(4) In the case of the set Y which is used in Section §2.3.5 to obtain the upper bound on the transition value t_1, the Markov matrix has 3940438 columns of which only 434 are pairwise distinct.
(5) In the case of the set Ω defined in [11] the Markov matrix has 45059 columns, of which only 114 are pairwise distinct.

Proposition 4.11 below gives an upper bound on the number of pairwise different columns in the transition matrix in terms of forbidden words.

Therefore instead of computing (and storing) the entire Markov matrix M it is sufficient to identify and to compute only unique columns, and to keep a record of the indices of columns which are identical. This is a significant saving in memory already, but there is room for even more.

Remark 4.7. We note that if the rows i_1 and i_2 and the columns j_1 and j_2 of the Markov matrix agree, i.e. $M(\underline{a}_n^{i_1},\underline{b}_n^k) \equiv M(\underline{a}_n^{i_2},\underline{b}_n^k)$ and $M(\underline{a}_n^k,\underline{b}_n^{j_1}) \equiv M(\underline{a}_n^k,\underline{b}_n^{j_2})$ for all $1 \le k \le 2^n$, then $M(\underline{a}_n^{i_1},\underline{b}_n^{j_1}) = M(\underline{a}_n^{i_1},\underline{b}_n^{j_2}) = M(\underline{a}_n^{i_2},\underline{b}_n^{j_2}) = M(\underline{a}_n^{i_2},\underline{b}_n^{j_1})$.

This simple observation allows us to reduce significantly the memory needed to store the Markov matrix M. In particular, in our considerations the Markov matrix M can be replaced by a smaller *reduced* Markov matrix $\widehat{M}$ as follows.

Step 1. Identify the words $\underline{a}_n^j$, $j = 1,\dots,K$ such that the rows $M(\underline{a}_n^j,\cdot)$ are pairwise different and define the map R which associates a row j with a row $R(j)$ from the set of unique rows.
Step 2. Identify the words $\underline{b}_n^j$, $j = 1,\dots,K$ such that the columns $M(\cdot,\underline{b}_n^j)$ are pairwise different and define the map C which associates a column j with a column $C(j)$ from the set of unique columns.
Step 3. Compute the reduced Markov matrix $\widehat{M} = \widehat{M}(\underline{a}_n^j,\underline{b}_n^k)$ of the size K^2.

It is clear that the huge Markov matrix M can be easily recovered from the reduced matrix $\widehat{M}$ using the correspondence maps R and C, since $M\left(\underline{a}_n^j,\underline{b}_n^k\right) = \widehat{M}\big(\underline{a}_n^{R(j)},\underline{b}_n^{C(k)}\big)$, $1 \le k \le 2^n$. Therefore, the main step in computing the Markov matrix M is the computation of the sets of words which give unique columns and rows of M.

4.3.1. *An upper bound for the number of unique rows and columns.*

Definition 4.8. We call the word $w' = w_n \dots w_1$ the semordnilap or reverse of the word $w = w_1 \dots w_n$.

The structure of the set of forbidden words we consider implies that the reduced Markov matrix is a square matrix.

Remark 4.9. If the set of forbidden words includes every word together with its reverse, then the number of pairwise different columns in the Markov matrix is equal to the number of pairwise different rows.

We need the following notation for the sequel.

Definition 4.10. For any $1 \le k \le n-1$ we call a subword $w_k \dots w_1$ *a suffix* of the word $w_n \dots w_1$ and a subword $w_n \dots w_{k+1}$ *a prefix* of the word $w_n \dots w_1$.

We now want to give an absolute upper bound on the number of unique columns or, equivalently, rows, of the reduced Markov matrix $\widehat{M}$ in terms of forbidden words.

Proposition 4.11. *Assume that there are k forbidden words which have S different suffixes in total. Then the number of pairwise distinct columns in the Markov matrix is no more than $S+1$.*

Remark 4.12. Note that the total number of suffixes S is bounded by the sum of the lengths of all forbidden words: $S \leq \sum_{j=1}^{k} r_k$.

The first step is to identify all the words which contain a forbidden word as a subword, since all of them give the zero row (or column) in the transition matrix. After removing them from our consideration, we obtain the set of allowed words.

$$A := \left\{ \underline{w}_n = w_1 \dots w_n \in \mathcal{A}^n,\ w_j \dots w_{j+r_i} \notin F, \text{ for all } 1 \leq j \leq n - r_i,\ 1 \leq i \leq k \right\}.$$

Once the set A is computed, it is of course possible to study all concatenations of allowed words and to identify those which give the unique rows and columns to the transition matrix. However, this would require $O\left(\left(\sum_{i=1}^{k} r_i\right) \cdot (\#A)^2\right)$ operations, which is prohibitively time-consuming, since $\#A$ is typically very large and in the examples we consider we have $\#A \approx 2^{n-1}$. In the next subsection we give a faster algorithm, which requires only $O\left(\#A\left(\sum_{i=1}^{k} r_i\right)\right) + O\left(\left(\sum_{i=1}^{k} r_i\right)^4\right)$ operations.

To prove Proposition 4.11 we need the following Lemmas. We denote by P_F the set of prefixes of forbidden words and we denote by S_F the set of suffixes of forbidden words.

Lemma 4.13. *Let us consider two words $\underline{a}_n^1, \underline{a}_n^2 \in A$. If for every prefix $p \in P_F$ either both $\underline{a}_n^1$ and $\underline{a}_n^2$ terminate with p, or neither $\underline{a}_n^1$ nor $\underline{a}_n^2$ terminate with p then the columns corresponding to $\underline{a}_n^1$ and $\underline{a}_n^2$ in the transition matrix are equal.*

Lemma 4.14. *For any word $\underline{a}_n^1 \in A$, a subset of prefixes $\{p \in P_F\}$ of forbidden words such that $\underline{a}_n^1$ terminates with p depends on the longest such prefix only.*

4.3.2. *Computation of the reduced Markov matrix $\widehat{M}$.* This can be realised by a number of technical steps. Let us denote by $|w|$ the length of the word w.

(1) Compute the sets $P_F = \{\overline{w} \text{ is a prefix of } w \mid w \in F\}$ and $S_F = \{\widehat{w} \text{ is a suffix of } w \mid w \in F\}$.
(2) For every word $w \in A$ we compute two sets $SF_w = \{\overline{w} \in S_F | \overline{w} \text{ is a prefix of } w\}$ and $PF_w = \{\widehat{w} \in P_F | \widehat{w} \text{ is a suffix of } w\}$.
(3) We say that two words $w_1, w_2 \in A$ are "suffix–equivalent" if $SF_{w_1} = SF_{w_2}$ and we say that $w_1, w_2 \in A$ are "prefix–equivalent" if $PF_{w_1} = PF_{w_2}$. Thus we split the set of allowed words in equivalence classes by suffixes $A/_{\sim S}$ and prefixes $A/_{\sim P}$. It turns out that there is relatively small number of equivalence classes compared to the number of allowed words.

    In the next steps we explain that in order to decide whether two words are compatible it is sufficient to work with their equivalence classes.
(4) The following encoding is handy to study compatibility of words based on equivalence classes. First, we fix enumeration of the set of forbidden words $F = \{d_1, \dots, d_k\}$. To every suffix $\widehat{d} \in S_F$ of a forbidden word we associate a set of pairs $\{(j, |\widehat{d}|) \mid \widehat{d} \text{ is a suffix of } d_j, d_j \in F\}$. To every prefix $\overline{d} \in P_F$ we associate a set pairs $\{(j, |d_j| - |\overline{d}|) \mid \overline{d} \text{ is a prefix of } d_j, d_j \in F\}$.

Note that concatenation of a prefix and a suffix is a forbidden word, if their encodings are the same.

(5) For any allowed word $w \in A$ we apply the encoding described above to the equivalence classes $A/_{\sim S}$ and $A/_{\sim P}$.
(6) It is clear that the concatenation of the words w_1 and w_2 doesn't have a forbidden subword, if and only if the corresponding equivalence classes AP_{w_1} and AS_{w_2} do not have any common pairs after encoding. Therefore, instead of computing the Markov matrix for the set of allowed words it is sufficient to compute the compatibility matrix for the equivalence classes.
(7) We identify unique rows and columns in the compatibility matrix for the equivalency classes and choose representatives from each class to obtain words which give unique rows and columns in the reduced matrix $\widehat{M}$.

The main advantage of this approach is that in order to compute the equivalency classes $A/_{\sim S}$ and $A/_{\sim P}$ it is sufficient to parse the huge set of allowed words only once. The number of operations on subsequent steps is $O\left(\left(\sum_{i=1}^k r_i\right)^4\right)$.

4.4. **Computation of the test functions.** In order to construct the test functions to use in Lemma 4.3 we need to compute the eigenvector of the matrix B^t defined by (4.4). By straightforward computation we can obtain the explicit form of B^t. Indeed for any $v \in \mathbb{R}^{2^n m}$ we have

$$Iv = (q_1(x), \dots, q_{2^n}(x)), \quad q_j(x) = \sum_{l=1}^m v_{(j-1)m+l} \cdot p_l(x), \text{ for } 1 \le l \le 2^n. \tag{4.5}$$

Therefore using (4.1) we get $\mathcal{L}_t I v = (Q_1, \dots, Q_{2^n})$ where for all $1 \le k \le 2^n$ we have

$$\begin{aligned} Q_k(x) &= \sum_{j=1}^{2^n} M(\underline{a}_n^j, \underline{b}_n^k) \cdot q_j(T_{\underline{a}_n^j}(x)) \cdot |T'_{\underline{a}_n^j}(x)|^t \\ &= \sum_{j=1}^{2^n} M(\underline{a}_n^j, \underline{b}_n^k) \cdot |T'_{\underline{a}_n^j}(x)|^t \cdot \left(\sum_{l=1}^m v_{(j-1)m+l} \cdot p_l(T_{\underline{a}_n^j}(x))\right). \end{aligned}$$

Hence the components of $u^t = (u_1^t, \dots, u_{2^n}^t) = P\mathcal{L}_t I v$ are given by

$$u_{(k-1)m+i} = Q_k(x_i) = \sum_{j=1}^{2^n} \sum_{l=1}^m M(\underline{a}_n^j, \underline{b}_n^k) \cdot |T'_{\underline{a}_n^j}(x_i)|^t \cdot p_l(T_{\underline{a}_n^j}(x_i)) \cdot v_{(j-1)m+l}.$$

Introducing 2^n small $m \times m$ matrices

$$B^{j,t}(i,l) := |T'_{\underline{a}_n^j}(x_i)|^t \cdot p_l(T_{\underline{a}_n^j}(x_i)) \tag{4.6}$$

we get

$$B^t = \begin{pmatrix} M(\underline{a}_n^1, \underline{b}_n^1) \cdot B^{1,t} & M(\underline{a}_n^2, \underline{b}_n^1) \cdot B^{2,t} & \dots & M(\underline{a}_n^{2^n}, \underline{b}_n^1) \cdot B^{2^n,t} \\ M(\underline{a}_n^1, \underline{b}_n^2) \cdot B^{1,t} & M(\underline{a}_n^2, \underline{b}_n^2) \cdot B^{2,t} & \dots & M(\underline{a}_n^{2^n}, \underline{b}_n^2) \cdot B^{2^n,t} \\ \vdots & \vdots & \ddots & \vdots \\ M(\underline{a}_n^1, \underline{b}_n^{2^n}) \cdot B^{1,t} & M(\underline{a}_n^2, \underline{b}_n^{2^n}) \cdot B^{2,t} & \dots & M(\underline{a}_n^{2^n}, \underline{b}_n^{2^n}) \cdot B^{2^n,t} \end{pmatrix}. \tag{4.7}$$

We are now ready to prove Proposition 4.5.

Proof. (of Proposition 4.5). Since by assumption $M(\underline{a}_n^k, \underline{b}_n^{j_1}) = M(\underline{a}_n^k, \underline{b}_n^{j_2})$ for all $k = 1, \dots, 2^n$, using representation (4.7) of the matrix B^t we conclude that

$$B^t\left((j_1-1)m+l,k)\right) = B^t\left((j_2-1)m+l,k\right)$$

for all $1 \le l \le m$ and $1 \le k \le 2^n$. Therefore for any $v \in \mathbb{R}^{2^n m}$ and $u = B^t v$ we have

$$u_{(j_1-1)m+l} = \sum_{k=1}^{2^n} B^t((j_1-1)m+l,k)\cdot v_k = \sum_{k=1}^{2^n} B^t((j_2-1)m+l,k)\cdot v_k = u_{(j_2-1)m+l}.$$

The result follows from (4.5) applied to u. ☐

We proceed to computing the leading eigenvector of B^t. By Proposition 4.5 it belongs to the subspace of dimension $\mathbb{R}^{Km}$ where K is the number of pairwise different columns of the Markov matrix M. We have already mentioned that it is not possible to work with the matrix B^t itself. The next Lemma allows us to replace the matrix B^t with a smaller reduced matrix $\widehat{B}^t$ in our considerations.

Definition 4.15. Assume that the Markov matrix M has K pairwise different columns. Let $j_1, \dots, j_K$ be the indices of the unique columns of M and let $i_1 \dots, i_K$ be the indices of the unique rows of M. Let R be the correspondence map as constructed in Step 1. We define the reduced matrix $\widehat{B}^t$ by

$$\widehat{B}^t((k-1)m+i,(l-1)m+j) = \sum_{\substack{s:R(s)=l\\ 1\le s\le 2^n}} M(i_k,s)B^{s,t}(i,j), \quad 1 \le l \le K. \tag{4.8}$$

Remark 4.16. Note that in order to compute the matrix $\widehat{B}^t$ there is no need to *store* the matrix B^t. It is sufficient to add elements of B^t to the corresponding elements of $\widehat{B}^t$ as we compute them.

Lemma 4.17. *Let $\hat{v}$ be the eigenvector of $\widehat{B}^t$. Then the eigenvector of B^t can be computed using the formula $v_{(j-1)m+l} := \hat{v}_{(R(j)-1)m+l}$ for $1 \le l \le m$ and $1 \le j \le 2^n$.*

Proof. Let $P\colon \mathbb{R}^{2^n m} \to \mathbb{R}^{Km}$ be the orthogonal projector onto the subspace defined by the system of equations $v_{i_k} = v_s$ where $R(s) = i_k$ for all $1 \le s \le 2^n$ and $1 \le k \le K$. Then $\widehat{B}^t = PB^tP^*$, where P^* stands for the transposed matrix P. ☐

Therefore, in order to recover the eigenvector of B^t and to compute the test functions, it is sufficient to compute the eigenvector of a much smaller reduced matrix $\widehat{B}^t$, defined above. The latter can be realised using simple iterations method.

4.5. **Verification of the min-max inequalities.** Finally, to verify the conditions of Lemma 4.3 numerically, we follow the same method as proposed in [15].

First, we compute the coefficients of the polynomials $p_1, \dots, p_K$ from the eigenvector of $\widehat{B}^t$.

The transfer operator $\mathcal{L}_t$ given by (4.1) can be written using the reduced Markov matrix $\widehat{M}$ and the correspondence map R. More precisely, let $j_1, \dots, j_K$ be the indices of the unique columns in matrix M. Let $p_1, \dots, p_K$ be the polynomials constructed from the eigenvector of the $\widehat{B}^t$. Then the transfer operator takes the form $\mathcal{L}_t\colon (p_1, \dots, p_K) \mapsto (Q_1^t, \dots, Q_K^t)$ where

$$Q_k^t(x) = \sum_{i=1}^{2^n} \widehat{M}(\underline{a}_n^{R(i)}, \underline{b}_n^{j_k}) \cdot p_{R(i)}(T_{\underline{a}_n^i}(x)) \cdot |T'_{\underline{a}_n^i}(x)|^t. \tag{4.9}$$

In order to obtain upper and a lower bounds on $\frac{Q_K^t}{p_K}$ we take a partition of the interval $[0,1]$ into 256 equal intervals. Then we evaluate $\frac{Q_k}{p_k}$ at the centre and to compute $\sup\left|\frac{d}{dx}\frac{Q_k(x)}{p_k(x)}\right|$ on each interval. The latter is realised using arbitrary precision ball arithmetic [6].

4.6. **Computational aspects.** Here we give some numerical data. The decimal numbers which we give in this section are truncated, not rounded.

4.6.1. *Set* B_1. We apply the method described above to estimate the dimension of the set B_1. In this case, the set of forbidden words constitutes of 27 words of length from 5 to 17.

$$F = \{21212,\ 21112121,\ 12111212,\ 211121222,\ 111112121,\ 1111212111, \\ 21112122112,\ 222111212211,\ 12211112121122,\ 112211121221111, \\ 222111121211221,\ 2112211121221112,\ 2111122121112212,\ 22221111212112222, \\ \text{and their reverses}\ \}$$

Therefore we will consider the words of length 16 and we begin by computing the set of allowed words A_F, which do not contain a forbidden word as a subword. The computation leaves us with 41186 allowed words (down from $2^{16} = 65536$). We also compute the coefficients of the Möbius maps corresponding to allowed compositions $T_{j_1} \circ \ldots \circ T_{j_{16}}$.

Then we employ the algorithm described in §4.3.2 to identify the words which give unique columns and rows to the Markov matrix together with correspondence maps R and C. It turns out that there are 138 such words. Finally, we calculate the Markov matrix itself. The computations we have done so far take less than a minute.

Afterwards, we choose $m = 8$, $t_0 = 0.5$ and compute the reduced matrix $\widehat{B}^t$ using the formulae (4.6) and (4.8) and find its eigenvector using the power method. We work 128 bit for precision and the eigenvector is computed with an error of 10^{-26}. We recover 138 polynomials from the eigenvector applying the formula (4.3). Then we take a uniform partition of the interval $[0,1]$ into 256 intervals and estimate the ratios $\frac{F_j}{f_j}$ on each of the intervals using ball-precision arithmetic. To obtain accurate bounds on the numerator of the derivative $F_j' f_j - f_j' F_j$, we compute the first 4 of its derivatives. We omit the eigenvector here, but we note that for $t_0 = 0.5$ the leading eigenvalue of $\widehat{B}^{t_0}$ is $1.0004258\ldots > 1$ and the ratios satisfy

$$1.000425 < \frac{F_j^{t_0}}{f_j} < 1.000426 \text{ for } j = 1, \ldots, 138.$$

We then test another two values to get a more accurate estimate. For $t_1 = 0.50001$ we have that the leading eigenvalue of $\widehat{B}^{t_1}$ is $1.0002239\ldots > 1$ and the ratios can be bounded as

$$1.000223 < \frac{F_j^{t_1}}{f_j} < 1.000225 \text{ for } j = 1, \ldots, 138.$$

For $t_2 = 0.50005$ we get that the leading eigenvalue of $\widehat{B}^{t_2}$ is $0.99941699\ldots < 1$ and the ratios can be bounded as

$$0.999416 < \frac{F_j^{t_2}}{f_j} < 0.999418 \text{ for } j = 1, \ldots, 138.$$

It takes about 20 minutes to complete the estimates for a single value of t using 8 threads running in parallel. [6]

4.6.2. *Set* B_2. In this case we have 33 forbidden words of length from 5 to 18. Thus we consider the words of length 17 and after removing those which contain a forbidden word as a subword, obtain 79034 allowed words. Among those we identify 184 words which give unique columns to the Markov matrix M and (another) 184 words which give unique rows. Using the same parameters $m = 8$ and $t_0 = 0.5$ we compute the matrix $\widehat{B}^{t_0}$ and its eigenvector with an error of 10^{-40}. The leading eigenvalue is $0.9996\ldots$ and after another 40 minutes we have lower and upper bounds on the ratios

$$0.999606 < \frac{F_j^{t_0}}{f_j} < 0.999607 \text{ for } j = 1, \ldots, 184.$$

Therefore we deduce that $\dim_H B_2 < 0.5$. In order to obtain more refined estimates we consider another two values $t_1 = 0.499975$ and $t_2 = 0.499995$. It turns out that the largest eigenvalue or $\widehat{B}^{t_1}$ is $1.0001426\ldots > 1$ and we have the following bounds for the ratios

$$1.000141 < \frac{F_j^{t_1}}{f_j} < 1.000143 \text{ for } j = 1, \ldots, 184.$$

The largest eigenvalue of $\widehat{B}^{t_2}$ is $0.99971391\ldots < 1$ and the bounds for the ratios are

$$0.999713 < \frac{F_j^{t_2}}{f_j} < 0.999714 \text{ for } j = 1, \ldots, 184.$$

Therefore we conclude that $0.499975 < \dim_H B_2 < 0.499995$. It takes about 90 minutes to obtain estimates on $\frac{F_j^t}{f_j}$ for a single value of t. The time is evidently affected by the number of functions.

4.6.3. *Set* X. The set X is specified by exclusion of 46 words of length from 5 to 24. To compute its Hausdorff dimension we consider an iterated function scheme of compositions of length 23. After removing all compositions which correspond to forbidden words, we are left with 3940388 maps, which is slightly less than a half of 2^{23}. The algorithm also identifies 429 unique columns and rows in the Markov matrix M; some of them are repeated as many as 141030 times. These computations take about 5 minutes.

We then choose $m = 8$ and $t_0 = 0.5$ as initial dimension guess and work with precision of 190 bits. It takes about 2.5 hours to compute the eigenvector of the reduced matrix $\widehat{B}^{t_0}$ of dimension $429 \cdot 8$ with an error of 10^{-40} and to obtain coefficients of 429 polynomials of degree 7. The corresponding eigenvalue is $0.999973\ldots < 1$.

Most of the time is then taken by calculation of the images of these polynomials under the map $\mathcal{L}_{t_0}$ as it involves taking compositions with all 3940388 maps. We use a partition of the interval $[0, 1]$ into 1024 intervals. The computation takes around 15 days with 8 threads running in parallel. Finally, we obtain

$$0.9999732 < \frac{F_j^{t_0}}{f_j} < 0.9999738 \qquad \text{for } j = 1, \ldots, 429.$$

which allows us to conclude that $\dim_H X < t_0 = 0.5$.

[6] Computations were done using 4 Core 8 Threads Intel(R) Core(TM) i7-6700 CPU @ 3.40GHz

4.6.4. *Set* Y. The set Y is specified by exclusion of 48 words of length from 5 to 24. To compute its Hausdorff dimension we consider an iterated function scheme of compositions of length 23. After removing all compositions which correspond to forbidden words, we are left with 3940438 maps, which is slightly less than a half of 2^{23}. The algorithm also identifies 434 unique columns and rows in the Markov matrix M; some of them are repeated as many as 176015 times. These computations take about 5 minutes.

We then choose $m = 8$ and $t_0 = 0.5$ as initial dimension guess and work with precision of 190 bits. It takes about 2.5 hours to compute the eigenvector of the reduced matrix $\widehat{B}^{t_0}$ of dimension $434 \cdot 8$ with an error of 10^{-40} and to obtain coefficients of 434 polynomials of degree 7. The corresponding eigenvalue is $1.0000162\ldots > 1$.

Most of the time is then taken by calculation of the images of these polynomials under the map $\mathcal{L}_{t_0}$ as it involves taking compositions with all 3940438 maps. This time in attempt to make the computation faster we use a uniform partition of the interval $[0,1]$ into 256 intervals. The computation takes around 3 days with 8 threads running in parallel. Finally, we obtain

$$1.0000160 < \frac{F_j^{t_0}}{f_j} < 1.0000166 \qquad \text{for } j = 1, \ldots, 434.$$

which allows us to conclude that $\dim_H Y > t_0 = 0.5$.

4.6.5. *Set* Ω. The set Ω is specified by exclusion of 26 words of length from 5 to 15. To compute its Hausdorff dimension we consider an iterated function scheme of compositions of length 14. After removing all compositions which correspond to forbidden words, we are left with 45059 maps, which is ten times less than 3^{14}. The algorithm also identifies 114 unique columns and rows in the Markov matrix M; some of them are repeated as many as 3745 times, but some occur only once. These computations take less than a minute.

We then choose $m = 8$ and $t_0 = 0.5$ as initial dimension guess and work with precision of 190 bits. It less than a minute to compute the eigenvector of the reduced matrix $\widehat{B}^{t_0}$ of dimension $114 \cdot 8$ with an error of 10^{-40} and to obtain coefficients of 114 polynomials of degree 7. The corresponding eigenvalue is $1.956\ldots > 1$.

The subsequent estimates of the ratios $\frac{F_j^{t_0}}{f_j}$ using partition of the interval $[0,1]$ into 256 intervals take about 15 minutes and give

$$1.956990 < \frac{F_j^{t_0}}{f_j} < 1.9569915 \qquad \text{for } j = 1, \ldots, 114.$$

We therefore conclude that $\dim_H \Omega > 0.5$ and apply bisection method to get a better estimate. Taking the value $t_1 = 0.537152$ we get the leading eigenvalue $1.000031\ldots > 1$ and

$$1.0000315 < \frac{F_j^{t_1}}{f_j} < 1.0000320 \qquad \text{for } j = 1, \ldots, 114.$$

and for $t_2 = 0.537155$ we get the leading eigenvalue of $\widehat{B}^{t_2}$ to be $0.999977 < 1$ and

$$0.999977 < \frac{F_j^{t_2}}{f_j} < 0.999979 \qquad \text{for } j = 1, \ldots, 114.$$

We therefore conclude that $0.537152 < \dim_H \Omega < 0.537155$.

This information is summarized in Table 7 below.

Set	$\mathcal{A}$	$\#F$	n	$\#A_F$	K	t	λ_{max}	r_1	r_2	time
B_1	$\{1,2\}$	27	16	41186	138	0.5	1.000425	1.000424	1.000426	15mins
						0.50001	1.000223	1.000222	1.000225	
						0.50005	0.999416	0.499415	0.499418	
B_2	$\{1,2\}$	33	17	79034	184	0.5	0.999606	0.999600	0.999607	90 mins
						0.499975	1.000142	1.000141	1.000143	
						0.499995	0.999713	0.999712	0.999714	
X	$\{1,2\}$	46	23	3940388	429	0.5	0.999973	0.999972	0.999974	15 days
Y	$\{1,2\}$	48	23	3940438	434	0.5	1.000016	1.000015	1.000017	4 days
Ω	$\{1,2,3\}$	26	14	45059	114	0.5	1.956	1.955	1.957	12mins
						0.537152	1.000031	1.000030	1.000032	
						0.537155	0.999977	0.999976	0.999979	

TABLE 7. Numerical output of the algorithm for computing dimension of the sets B_1, B_2, X, Y, Ω. Time refers to the time needed to compute the lower and the upper bounds on the ratios $r_1 < \frac{F_j^t}{f_j} < r_2$ for a single value of t.

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