Dimension of points with restricted even order partial quotients

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1 Introduction

Recently, there have been several instances where precise bounds on the Hausdorff dimension of a subset of the real numbers (defined in terms of restrictions on continued fraction expansions) has lead to striking progress on a variety of problems. For example, density one Zaremba-type estimates [2], [6]; Counter-examples to reciprocity conjectures and Apollonian packings [8]; and estimates for spanning trees in finite graphs [3].

Let us fix $c \in \mathbb{N}$ then we can consider the set

$$\mathbb{S}_c = \{[b_1, c, b_3, c, b_5, c, \cdots] \in [0,1] : b_n \in \mathbb{N} \text{ for } n \geq 1 \text{ odd}\}$$

consisting of numbers in the unit interval with continued fraction expansions with even digits equal to c. In the case that $c = 1$ a knowledge of the dimension [1] of this set plays an important role in the estimates in [3]. These sets were also studied in [4] in the setting of problems of when sums and products of such sets contain intervals and bounds on the dimension were presented (particularly when $c = 1$ and $c = 2$).

Our modest aim in this note is to recall the simple algorithm which gives accurate and efficient estimates on the dimension of such sets and to illustrate this with some of the examples $\dim(\mathbb{S}_c) \in (0,1)$ ($c \in \mathbb{N}$) which have attracted interest recently.

[1] For all of the sets we will cosnider their Hausdorff dimension and Box dimension agree and we will simply write $\dim(\cdot)$ for the dimension.

2 Results

We can consider the following concrete estimates.

2.1 The case $c = 1$.

The case $c = 1$ and the value of $\dim(\mathbb{S}_1)$ appears in the work of Chan, Kontorovich and Park [3]. It is important to their work that the dimension of the set is larger than

$$(\sqrt{40} - 4)/3 = 0.77485177344558622133...$$

In fact, they determine that the dimension is larger than 0.775 (cf. ([3], Theorem 1.19)). The following result gives a more precise value.

Theorem 2.1. *We can estimate*

$$\dim(\mathbb{S}_1) = 0.79885836696622033966...$$

accurate to the number of decimal places given.

The bounds on $\dim(\mathbb{S}_1$ allow one deduce that for sufficiently large A the sets

$$\mathbb{S}_{1,A} = \{[b_1, c, b_3, c, b_5, 1, \cdots] \in [0, 1] : b_n \in \mathbb{N}, b_n \leq A \text{ for } n \geq 1 \text{ odd}\}$$

where the $(b_i)_{i \in 2\mathbb{N}-1}$ terms are all bounded by A satisfies

$$\dim(\mathbb{S}_{1,A}) > (\sqrt{40} - 4)/3.$$

More precisely, in [3] they determine that for $A = 110$ one has the inequality $\dim(\mathbb{S}_{1,110}) > (\sqrt{40} - 4)/3$ (cf. Theorem 1.19 and Remark 1.20 in [3]). More precise estimates on the dimension some of these sets appear in the following result.

Theorem 2.2. *We can estimate the following*

$$\dim(\mathbb{S}_{1,108}) = 0.77474983773874196635...$$
$$\dim(\mathbb{S}_{1,109}) = 0.77490273977177039970.,.$$
$$\dim(\mathbb{S}_{1,110}) = 0.77505321465806719858...$$

accurate to the number of decimal places given.

In particular, it appears from these estimates that

$$\dim(\mathbb{S}_{1,109}) > (\sqrt{40} - 4)/3 > \dim(\mathbb{S}_{1,108}).$$

2.2 The case $c \geq 2$

Estimates on the dimension of the sets $\mathbb{S}_1$ and $\mathbb{S}_2$ also appear in the work of Hančl and Turek. In particular, they showed bounds

$$0.732 \leq \dim(\mathbb{S}_1) \leq 0.819$$

([4], Corollary 6.7) and ([4], Theorem 1.3) and

$$0.5 \leq \dim(\mathbb{S}_2) < 0.676$$

([4] Theorem 6.8) and ([4] Corollary 5.3).

We have more specific estimates.

Theorem 2.3. *We can rigorously bound*

$$\begin{aligned}
\dim(\mathbb{S}_2) &= 0.67171130453414746839... \\
\dim(\mathbb{S}_3) &= 0.62151193454786357 40... \\
\dim(\mathbb{S}_4) &= 0.5953428935241985233... \\
\dim(\mathbb{S}_5) &= 0.57843385662193144418... \\
\dim(\mathbb{S}_6) &= 0.5667527341568113816... \\
\dim(\mathbb{S}_7) &= 0.55817917553465229170... \\
\dim(\mathbb{S}_8) &= 0.55160763566459691142...
\end{aligned}$$

accurate to the number of decimal places given

It would be straightforward to extend Theorem 2.3 to other values of $c \in \mathbb{N}$ (providing c is not too large).

Remark 2.4. *It is not difficult to show that* $\lim_{c \to +\infty} \dim(\mathbb{S}_c) = \frac{1}{2}$.

2.3 The case of random choices of c

We can also consider the larger sets

$$\begin{aligned}
\widetilde{\mathbb{S}}_c = \{[b_1, c_2, b_3, c_4, b_5, 1, \cdots] \in [0,1] : \; & b_n \in \mathbb{N} \text{ for } n \geq 1 \text{ odd, and} \\
& c_n \in \{1, \cdots, c\} \text{ for } n \text{ even}\}.
\end{aligned}$$

In the case that $c = 2$ there is a bound in ([4] Theorem 5.4) of

$$\dim(\widetilde{\mathbb{S}}_2) < 0.966.$$

The following result gives a more precise bound.

Theorem 2.5. *We can estimate*

$$\dim(\widetilde{\mathbb{S}}_2) = 0.8736198690238783954...$$

accurate to the number of decimal places given.

Remark 2.6. *Finally, in ([4], Theorem 5.7) there is a bound on the dimension of the set*

$$\mathcal{S} = \{[a_1, a_2, a_3, \cdots] \, : \, a_{2n-1} \in \mathbb{N}, a_{4n} = 1, a_{4n-2} = 2, n \in \mathbb{N}\}$$

of the form $\dim(\mathcal{S}) < 0.732$. *It is not immediately clear how to adapt our approach to this setting.*

3 The method

We briefly recall the set-up from [7]. We will concentrate on the cases of Theorem 2.1 and Theorem 2.2.

3.1 Transfer operators

The starting point is the following classical characterization in terms of the family of linear *transfer operators* $\mathcal{L}_t : C([0,1]) \to C([0,1])$ on the Banach space of continuous functions $C([0,1])$ (with the supreimum norm) defined by

$$\mathcal{L}_t h(x) = \sum_{b=1}^{\infty} \frac{1}{(1+cb+cx)^{2t}} h\left(\frac{b+x}{1+cb+cx}\right)$$

for $t \geq \frac{1}{2}$ (which guarantees convergence of the series). This definition is based on the contractions

$$T_b(x) = \frac{1}{c + \frac{1}{b+x}} = \frac{b+x}{1+cb+cx}$$

and the absolute value of its derivative:

$$|T_b'(x)| = \frac{1}{(1+cb+cx)^2}.$$

The connection between the transfer operator and the dimension $\dim(\mathbb{S}_c)$ comes from the following useful lemma.

Lemma 3.1 (after Bowen, Ruelle, Urbanski)**.** *The spectral radius $\rho(t)$ of $\mathcal{L}_t$ is continuous and monotone decreasing in t and there is a unique value $t = t_0$ for which $\rho(t_0) = 1$. Furthermore, $t_0 = \dim_H(\mathbb{S}_c)$.* [2]

Thus we can reduce the problem of estimating the dimension to that of estimating a value t_0 such that $\rho(t_0) = 1$. The following lemma gives a way to test if t_0 lies in an interval $[a, b]$.

Lemma 3.2. *Let $0 < a < b < 1$.*

(i) Assume that we can find a positive polynomial $f : [0, 1] \to \mathbb{R}^+$ such that

$$\mathcal{L}_a f(x) > f(x) \text{ for all } 0 \leq x \leq 1$$

then $\rho(a) > 1$ and therefore $t_0 = \dim_H(\mathbb{S}_c) \geq a$.

(ii) Assume that we can find a positive polynomial $g : [0, 1] \to \mathbb{R}^+$ such that

$$\mathcal{L}_b g(x) < g(x) \text{ for all } 0 \leq x \leq 1$$

then $\rho(b) < 1$ and therefore $t_0 = \dim_H(\mathbb{S}_c) \leq b$.

In particular, the final conclusion in each part comes from Lemma 3.1 The proof of this useful lemma is surprisingly simple and can be found in [7].

3.2 Transfer operators applied to polynomials

In order to apply Lemma 2, given polynomials we need to get good estimates on $\mathcal{L}_a f$ and $\mathcal{L}_b g$. Since we are considering polynomials of the general form

$$a_0 + a_1 x + \cdots + a_N x^N,$$

[2] The heuristic is that the definition of Hausdorff dimension involves covers by intervals. Here these can be taken to be of the form $\{[a_1, a_2, \cdots, a_N + u] : 0 \leq u \leq 1\}$ and their diameter is estimated using $[a_1, a_2, \cdots, a_N + u]^2$. Then $\mathcal{L}_t^N 1(0)$, say, can be compared with a sum of diameters raised to the power t

for appropriately large N, this comes down to applying the transfer operator to monomials $P_r(x) := x^r$ for $r \in \mathbb{Z}_+$. In particular, we can explicitly write for $0 \leq x \leq 1$ and $r \geq 1$:

$$(\mathcal{L}_t P_r)(x) = \sum_{b=1}^{\infty} \left(\frac{b+x}{1+cb+cx} \right)^r \frac{1}{(1+cb+cx)^{2t}}.$$

Using the binomial expansion

$$\begin{aligned} \left(\frac{b+x}{1+cb+cx} \right)^r &= \frac{1}{c^r} \left(1 - \frac{1}{1+cb+cx} \right)^r \\ &= \frac{1}{c^r} \sum_{n=0}^{r} (-1)^n \binom{r}{n} \frac{1}{(1+cb+cx)^n} \end{aligned}$$

we can expand

$$\begin{aligned} (\mathcal{L}_t P_r)(x) &= \frac{1}{c^r} \sum_{n=0}^{r} (-1)^n \binom{r}{n} \left(\sum_{b=1}^{\infty} \frac{1}{(1+cb+cx)^{2t+n}} \right) \\ &= \sum_{n=0}^{r} \frac{(-1)^n}{c^{2t+n+r}} \binom{r}{n} \left(\sum_{b=1}^{\infty} \frac{1}{\left(\frac{1}{c}+b+x\right)^{2t+n}} \right). \end{aligned}$$

Because of the special form of the operator (and the choice of function P_r it is applied to) the image can be identified with

$$(\mathcal{L}_t P_r)(x) = \sum_{n=0}^{r} \frac{(-1)^n}{c^{2t+n+r}} \binom{r}{n} \zeta\left(1 + \frac{1}{c} + x, 2t+n\right)$$

where

$$\zeta(z,s) = \sum_{l=0}^{\infty} \frac{1}{(z+l)^s}$$

is the Hurwicz zeta function, which converges for $Re(s) > 1$ and has a meromorphic extension to $\mathbb{C}$ [1]. The advantage of this formulation is that there are methods [3] for rapidly evaluating the Hurwitz zeta function numerically with arbitrary precision $\zeta(z,s)$ which then give very efficient estimates to $(\mathcal{L}_t P_r)(x)$.

[3] For example, using the Euler-Maclaurin formula [5]

Example 3.3 ($c = 1$)**.** *In the particular case $c = 1$ the transfer operator becomes*

$$(\mathcal{L}_t P_r)(x) = \sum_{b=1}^{\infty} \left(\frac{b+x}{1+b+x} \right)^r \frac{1}{(1+b+x)^{2t}}$$

and we can then write

$$(\mathcal{L}_t P_r)(x) = \sum_{n=0}^{r} (-1)^n \binom{r}{n} \zeta(2+x, 2t+n)$$

3.3 Lagrange Polynomials and Chebychev points

It remains to find good choices of the polynomials f and g. This is achieved using collocation and interpolation (adapted from standard techniques in numerical analysis). To this end is convenient to represent arbitrary polynomials in terms of Lagrange polynomials. We can fix $m \geq 1$ sufficiently large (e.g., $m = 20$).

(i) We can associate the standard Chebychev points $(x_i)_{i=1}^m$ on the interval $[-1, 1]$ by

$$x_i = \cos(\pi(2i-1)/(2m))$$

and then translate them to points $(y_i = (x_i+1)/2)_{i=1}^m$ on the the interval $[0, 1]$.

(ii) We can then associate the Lagrange polynomials $\ell^{(i)} : [0, 1] \to \mathbb{R}$ (for $1 \leq i \leq m$) by

$$\ell^{(i)}(x) = \frac{\prod_{j \neq i}(x - x_j)}{\prod_{j \neq i}(x_i - x_j)} =: \sum_{r=0}^{m} a_r^{(i)} x^r = \sum_{r=0}^{m} a_r^{(i)} P_r(x)$$

where $a_0^{(i)}, \cdots, a_m^{(i)}$ are the coefficients of the polynomial $\ell^{(i)}(x)$. By construction $\ell^{(i)}(x_j) = \delta_{ij}$ for $1 \leq i, j \leq m$ (where δ_{ij} is the usual Dirac Delta function.).

To prove the estimates in Theorem 2.1 We can then apply the transfer operator to write for $0 \leq x \leq 1$

$$\mathcal{L}_t \ell^{(i)}(x) = \sum_{r=0}^{m} a_r^{(i)} \left(\sum_{n=0}^{r} \frac{(-1)^n}{c^{2t+n+r}} \binom{r}{n} \left(\zeta(1 + \frac{1}{c} + x, 2t+n) \right) \right).$$

We can then get candidate polynomials for f and g by the following approach: Consider the square matrix $M_t = \ell^{(i)}(x_j)_{i,j=1}^m$ and let $v^{(t)} = (v_1^{(t)}, \ldots, v_m^{(t)})^T$ be the left eigenvector for the maximal eigenvalue (which is known to exist for m sufficiently large [7]). We can let

$$f(x) = \sum_{i=1}^{m} v_i^{(a)} \ell^{(i)} \text{ and } g(x) = \sum_{i=1}^{m} v_i^{(b)} \ell^{(i)}$$

for $0 \leq x \leq 1$.

Remark 3.4. *There are a number of practical considerations in the computation.*

1. *The choice of intervals $[a, b]$ can be chosen by a simple bisection method. Starting from $[a, b] = [0, 1]$, say, we can consider the subintervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ and estimates for $t = 0, \frac{1}{2}, 1$ to which of these intervals $\dim(\mathbb{S}_c)$ lies in (by Lemma 3.2). Proceeding iteratively, if we know $\dim_H(\mathbb{S}_c)$ lies in a sub-interval $[t_0, t_1]$ then we can consider sub-intervals $[t_0, (t_0 + t_1)/2]$ and $[(t_0 + t_1)/2, t_1]$ and use Lemma 3.2 to determine which of these smaller intervals lies in. We repeat this process until we have the desired level of accuracy in the estimate.*

2. *The value m determines the degree of the polynomials in Lemma 3.2. When at some step the bisection method becomes inconclusive this can be resolved by increasing m. (In our examples we chose $m = 20$).*

3. *The numerical errors that naturally occur when the algorithm is implemented can be controlled using ball-arithmetric.*

3.4 Modication to the proof of Theorems 2.3 and 2.5

The proofs of the remaining theorems uses the same methods applied to modified versions of the transfer operators.

3.4.1 Proof of Theorem 2.3

The proof in the case of Theorem 2.3 is simpler in as much as it doesn't involve the Hurwicz zeta function. We can now consider the associated transfer

operator $\widetilde{\mathcal{L}}_t : C([0,1]) \to C([0,1])$ given by

$$(\mathcal{L}_t h)(x) = \sum_{b=1}^{A} h\left(\frac{b+x}{1+cb+cx}\right) \frac{1}{(1+cb+cx)^{2t}} \text{ for } h \in C([0,1])$$

which involves a finite summation which can be effectively esimated (providing A is not too large).

3.4.2 Proof of Theorem 2.5

In the case of Theorem 2.5 the associated operator is $\widetilde{\mathcal{L}}_t : C([0,1]) \to C([0,1])$ defined by

$$\widetilde{\mathcal{L}}_t h(x) = \sum_{j=1}^{c} \sum_{b=1}^{\infty} \frac{1}{(1+jb+cx)^{2t}} h\left(\frac{b+x}{1+jb+cx}\right)$$

for $t \geq \frac{1}{2}$. We can then write

$$(\widetilde{\mathcal{L}}_t P_r)(x) = \sum_{j=1}^{c} \sum_{n=0}^{r} \frac{(-1)^n}{j^{2t+n+r}} \binom{r}{n} \left(\zeta(1+\frac{1}{j}+x, 2t+n)\right).$$

The proof then proceeds as before.

Remark 3.5. *We have typically presented our estimates to* 20 *decimal places. However, it would be easy to get results to greater accuracy by running the programme for longer.*

References

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