Counting geodesic arcs in a fixed conjugacy class on negatively curved surfaces with boundary

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Abstract

We show how to derive an asymptotic estimates for the number of closed arcs γ on a surface V of (variable) negative curvature with non-empty geodesic boundary which lie in a given non-trivial conjugacy class.

1 Introduction

There are many asymptotic counting results for geodesic arcs on negatively curved surfaces, dating back to the work of Huber and Margulis [6] for closed surfaces. Motivated by interesting recent work of Parkkonen-Paulin [7] and Kenison-Sharp [4], one can ask whether there are analogous results when we count only geodesic arcs in a fixed conjugacy class. More precisely, let V be a compact surface with (variable) negative curvature. Let $\xi \in V$ be a fixed reference point then the fundamental group $\pi_1(V, \xi)$ consists of homotopy classes c of closed curves on V beginning and ending at ξ. By additionally assuming that V has non-empty geodesic boundary (i.e., $\partial V \neq \emptyset$) we can assume that $\pi_1(V)$ is a free group.

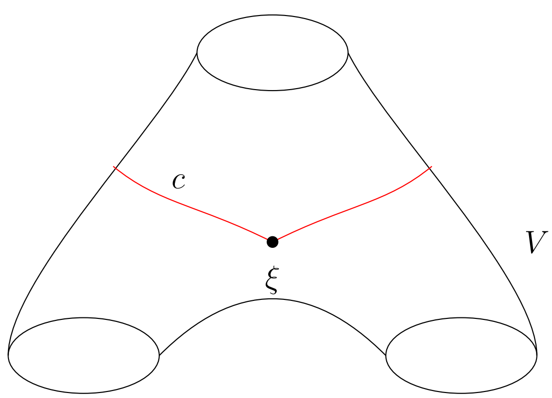

Figure 1: A geodesic arc c based at ξ on a surface V, represented by a "pair of pants".

Let us now fix some notation.

Notation 1.1. *Given $c \in \pi_1(V, \xi)$ we denote by γ_c the unique geodesic arc $\gamma_c : [0, L] \to V$ in the homotopy class c of length $L = L_c$ starting and finishing at ξ (i.e., $\gamma_c(0) = \gamma_c(L) = \xi$).*

We denote by $h = h(V) > 0$ the topological entropy of the geodesic flow $\phi_t : SV \to SV$ associated to V. In particular, h is the growth rate of the number of such geodesic arcs when ordered by length. Moreover, there is the stronger asymptotic estimate in the following classical result.

Theorem 1.2 (after Margulis)**.** *For each $\xi \in V$ there exists $C = C(V, \xi) > 0$ such that*

$$\#\{c \in \pi_1(V, \xi) \,:\, L_c \leq T\} \sim Ce^{hT} \text{ as } T \to +\infty$$

(i.e., $\lim_{T\to+\infty} \#\{c \in \pi_1(V, \xi) \,:\, L_c \leq T\}e^{-hT} = C$).

This result was originally proved when the surface is without boundary [6], but is also valid for surfaces with geodesic boundary too (see [9]).

Example 1.3 (Pair of pants)**.** *We can consider the concrete case of a pair of pants V. This is a compact surface of (variable) negative curvature with three geodesic boundary components, as illustrated in Figure 1. In this case the fundamental group is a free group on two generators $\alpha, \beta \in \pi_1(V, \xi)$, say (i.e., $\pi_1(V, \xi) = \langle \alpha, \beta \rangle$).*

We want to consider the restricted counting problem where we impose the additional restriction that we only count those elements $c \in \pi_1(V, \xi)$ which are conjugate to a fixed element $a \in \pi_1(V, \xi) - \{e\}$, say, i.e., $bab^{-1} \in \pi_1(V, \xi)$ for some $b \in \pi_1(V, \xi)$.

The main result of this note is the following.

Theorem 1.4 (Counting geodesic arcs in a fixed conjugacy class)**.** *Given $a \in \pi_1(M) - \{e\}$ and $\xi \in V$ there exists $C_a = C(V, \xi, a) > 0$ such that*

$$\#\{b \in \pi_1(V, \xi) \,:\, L_{bab^{-1}} \leq T\} \sim C_a e^{hT/2} \text{ as } T \to +\infty$$

(i.e.,$\lim_{T\to+\infty} \#\{b \in \pi_1(V, \xi) \,:\, L_{bab^{-1}} \leq T\}e^{-hT/2} = C_a$).

In the case of graphs, this problem has been studied by a number of authors including Kenison-Sharp [4], Parkkonen-Paulin [7], Broise-Alamichel-Parkkonen-Paulin [1], Douma [2] and Guillopé [3]. In proving corresponding results for geodesics on surfaces the additional difficulty is introducing the geometric lengths of the geodesic arcs.

2 Some preliminary lemmas

We begin with some preliminary results on free groups and the lengths of geodesic arcs. Let us assume for (notational) simplicity that V is a pair of pants. In particular, this means that $\pi_1(V)$ is a free group on two generators α, β, say (i.e., $\pi_1(V) = \langle \alpha, \beta \rangle$). We can then fix an element $a = a_1 \cdots a_m \in \pi_1(V)$ and conjugate elements $b^{-1}ab$, where $b = b_1 \cdots b_n$ (presented in reduced form in terms of generators $a_i, b_i \in \{\alpha^{\mp 1}, \beta^{\mp 1}\}$).

It is convenient to denote the word lengths by $|a| = m$ and $|b| = n$.

It suffices to consider only those conjugate elements for which bab^{-1} is already in reduced form, since otherwise the same conjugacy class would already be accounted for by an element of shorter (word) length. Thus without loss of generality we may assume that $b_1 \neq a_m$ and $b_n \neq a_1^{-1}$.

Notation 2.1. *If $l < n$ then we will denote $b_1^l = b_1 \cdots b_l$ and $b_{n-l+1}^n = b_{n-l+1} \cdots b_n$.*

The first lemma counts geodesic arcs subject to restrictions on the beginning and end of the presentations of the corresponding group elements.

Lemma 2.2. *Let $u, v \in \pi_1(V)$ with $|u| = |v| = l$ and satisfying $u_1 \neq a_m$ and $v_l \neq a_1^{-1}$. There exists a constant C_{uv} such that*

$$\#\{b \in \Gamma \,:\, b_1^l = u, b_{n-l+1}^n = v, L_b \leq T\} \sim C_{uv} e^{hT} \text{ as } T \to +\infty \tag{1}$$

The next lemma is more geometric in flavour and compares the lengths of (piecewise) geodesics arcs.

Lemma 2.3. *There exist:*

1. *values $\tau_l(u, v) > 0$, associated to pairs of words with $|u| = |v| = l$ and satisfying $u_1 \neq a_m$ and $v_l \neq a_1^{-1}$, for each $l \geq 1$; and*
2. *$K > 0$ and $0 < \rho < 1$,*

so that for any $b \in \Gamma$ with $|b| \geq 2l$ and satisfying $b_1^l = u$ and $b_{n-l+1}^n = v$ we have that

$$|L_{bab^{-1}} - L_a - 2L_b + \tau_l(u, v))| \leq K\rho^l$$

Finally, it easy to see that the constants in Lemma 2.2 satisfy $C_{uv} \to 0$ as $l \to +\infty$. However, we have the following additional estimate.

Lemma 2.4. *The following limit exists*

$$\lim_{l \to +\infty} \sum_{\substack{|u|=|v|=l \\ u_1 \neq a_m \\ v_l \neq a_1^{-1}}} C_{uv} e^{-\tau_l(u,v)} =: C > 0.$$

We will postpone the proof of these three lemmas to Section 5. In the meantime, we will use them in the next section to derive the Theorem.

3 Proof of Theorem 1.4 (assuming lemmas 2.2, 2.3 and 2.4)

We can prove the Theorem using Lemmas 2.2, 2.3 and 2.4 and an approximation argument. Let us denote by

$$N_a(T) := \#\{b \in \pi_1(V, \xi) \,:\, L_{bab^{-1}} \leq T\}$$

the counting function in Theorem 1.4. Given $u, v \in \pi_1(V)$ with $|u| = |v| = l$ satisfying $u_1 \neq a_m$ and $v_l \neq a_1^{-1}$ we can write

$$N^{u,v}(T) := \#\{b \in \Gamma \,:\, b_1^l = u, b_{n-l+1}^n = v, L_b \leq T\}.$$

Fix $\delta > 0$. Writing $\epsilon_l := K\rho^l$ and using Lemma 2.3 and then Lemma 2.2 we have that

$$N_a(T) \leq \sum_{\substack{|u|=|v|=l \\ u_1 \neq a_m \\ v_l \neq a_1^{-1}}} N^{u,v}\left(\frac{T - L_a + \tau_l(u,v) + \epsilon_l}{2}\right)$$

$$\leq \left((1+\delta)e^{-L_a/2}e^{\epsilon_l/2} \sum_{\substack{|u|=|v|=l \\ u_1 \neq a_m \\ v_l \neq a_1^{-1}}} C_{uv}e^{-\tau_l(u,v)} \right) e^{hT/2},$$

for sufficiently large T. Let $\eta > 0$. We can then assume that l was chosen sufficiently large that

$$\sum_{\substack{|u|=|v|=l \\ u_1 \neq a_m \\ v_l \neq a_1^{-1}}} C_{uv} \leq (1+\eta)C$$

by Lemma 2.4. In particular, for T sufficiently large we have

$$N_a(T) \leq \left((1+\eta)(1+\delta)e^{-L_a/2}e^{\epsilon_l/2}C\right) e^{T/2}.$$

Similarly, we can get a lower bound

$$N_a(T) \geq \left((1-\eta)(1-\delta)e^{-L_a/2}e^{-\epsilon_l/2}C\right) e^{T/2}.$$

Comparing these two inequalities, and since $\eta, \delta > 0$ can be chosen arbitrarily small provided l and T are sufficiently large, then Theorem 1.4 follows with $C_a = e^{-L_a/2}C$.

4 Coding and transfer operators

Before proving the three lemmas from Section 2 in Section 5, we need to recall some preliminary results on the use of symbolic dynamics and lengths of geodesic arcs. We will use a dynamical approach to the counting problem. The particular method we will employ uses symbolic dynamics.

For definiteness we will continue to concentrate on the case of a pair of pants. We first introduce a matrix

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

where we label the rows and columns by $\mathcal{S} = \{0, a, b, a^{-1}, b^{-1}\}$.

Definition 4.1. *We can associate a compact metric space*

$$\Sigma_B = \{x = (x_n)_{n=1}^{\infty} \in \mathcal{S} \,:\, B(x_n, x_{n+1}) = 1, n \in \mathbb{N}\}.$$

A shift map $\sigma : \Sigma_B \to \Sigma_B$ defined by $(\sigma x)_n = x_{n+1}$, for $n \in \mathbb{N}$.

Observe that Σ_B contains the fixed point $\dot{0} = (0, 0, 0, \cdots)$.

We can define a natural metric on Σ_B by

$$d(x, y) = \sum_{n=1}^{\infty} \frac{e(x_n, y_n)}{2^n}$$

where $x = (x_n)_{n=1}^{\infty}$, $y = (y_n)_{n=1}^{\infty}$ and

$$e(i, j) = \begin{cases} 1 & \text{if } x_n \neq y_n \\ 0 & \text{if } x_n = y_n. \end{cases}$$

We can now relate the lengths of geodesic arcs to certain sequences in Σ_B.

Lemma 4.2 (after Lalley)**.** *There exists a Hölder continuous function $r : \Sigma_B \to \mathbb{R}$ such that for any sequence*

$$x = (x_1, x_2, x_3, \cdots, x_n, 0, 0, \dots) \in \Sigma_B$$

finishing with infinitely many 0s the associated group element

$$g = x_1 x_2 \cdots x_n \in \Gamma$$

corresponds to a geodesic arc of length $L_g = r^n(x) := \sum_{k=0}^{n-1} r(\sigma^k x)$.

This result was proved by Lalley in the case of constant negative curvature (see [5], p. 41) and generalized to the present context of variable negative curvature in [9]. For completeness, we briefly outline the construction. Let $\widetilde{V}$ be the Universal Cover for V with the lifted metric $\widetilde{d}$. Elements $g \in \pi_1(V)$ have a natural action on $\widetilde{V}$ as covering transformations. Let $\widetilde{\xi} \in \widetilde{V}$ be a lift of $\xi \in V$. The function $r : \Sigma_B \to \mathbb{R}$ is defined on the dense set of points $x = (x_1, x_2, \cdots, x_n, 0, 0, \dots)$ by

$$r(x) = \begin{cases} \widetilde{d}(x_1 x_2 \cdots x_n \widetilde{\xi}, \widetilde{\xi}) - \widetilde{d}(x_2 \cdots x_n \widetilde{\xi}, \widetilde{\xi}) & \text{if } n \geq 2 \\ \widetilde{d}(x_1 \widetilde{\xi}, \widetilde{\xi}) & \text{if } x = (x_1, 0, 0, \cdots) \\ 0 & \text{if } x = (0, 0, 0, \cdots) \end{cases}$$

and then extends (as a Hölder continuous function) to Σ_B.

This function can now be used to define a useful linear operator. Let $0 < \alpha < 1$ be the Hölder exponent of r and let $C^\alpha(\Sigma_B, \mathbb{C})$ be the Banach space of complex valued α-Hölder continuous functions.

Definition 4.3. *There exists $0 < \alpha < 1$ such that for each $s \in \mathbb{C}$, we can associate a transfer operator $\mathcal{L}_s : C^\alpha(\Sigma_B, \mathbb{C}) \to C^\alpha(\Sigma_B, \mathbb{C})$ defined by*

$$\mathcal{L}_s w(x) = \sum_{\sigma y = x} e^{-sr(y)} w(y), \quad w \in C^\alpha(\Sigma_B, \mathbb{C}).$$

We have the following properties for these operators.

Lemma 4.4. *Let $s = \sigma + it$.*

1. *The operator* $\mathcal{L}_\sigma$ *has a simple maximal positive eigenvalue* $e^{P(-\sigma r)}$ *with an eigenfunction* ψ_σ *satisfying* $\psi_\sigma(\dot{0})$ *and an eigenmeasure* μ_σ*. The rest of the spectrum is contained in a disc of strictly smaller radius than* $e^{P(-\sigma r)}$*.*

2. *For* $t \neq 0$*, the spectrum of the operator* $\mathcal{L}_s$ *is contained in a disc of strictly smaller radius than* $e^{P(-\sigma r)}$*.*

Proof. The proof is essentially contained in [9]. The key observation is that spectral properties of the transfer operator in the statement coincide with those of the transfer operator $\widetilde{\mathcal{L}}_s : C^\alpha(\Sigma_A) \to C^\alpha(\Sigma_A)$ associated to the larger transitive component

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

of B, the associated space $\Sigma_A \subset \Sigma_B$ and subshift $\sigma : \Sigma_A \to \Sigma_A$, the restriction $\widetilde{r} : \Sigma_A \to \mathbb{R}$ of function $r : \Sigma_B \to \mathbb{R}$ and defined in the same way as $\mathcal{L}_s$.

For the first part of the Lemma the corresponding results on the spectra for Σ_A correspond to the well known Ruelle Operator Theorem for $\mathcal{L}_\sigma$ (see [8]). The property that $\psi_\sigma(\dot{0}) > 0$ follows by first showing $\psi_\sigma \geq 0$ and considering the dense set $\cup_{n=1}^\infty \sigma^{-n}(\dot{0})$.

For the second part of the lemma it suffices to show that the restriction $\widetilde{r} : \Sigma_A \to \mathbb{R}$ doesn't satisfy $\{\widetilde{r}^n(x) : \sigma^n x = x\} \subset a\mathbb{Z}$ for some $a > 0$. However, these values correspond to lengths of closed geodesics and a simply geometric argument shows this cannot hold. □

The value $P(-\sigma r)$ appearing in Lemma 4.4 is called the *pressure* and the characterization in terms of the maximal eigenvalue of the positive transfer operator is but one of many equivalent definitions. Furthermore, we have the following standard result (see [8] for details).

Lemma 4.5. *The function* $\mathbb{R} \ni t \mapsto P(-tr)$ *is a real analytic function such that:*

1. $P(-hr) = 0$*;*

2. $P(-\sigma r) < 0$ *for* $\sigma > h$*; and*

3. $\frac{d}{dt}P(-tr)|_{t=h} \neq 0$*.*

5 Proof of Lemmas 2.2, 2.3 and 2.4.

It remains is to complete the proofs of the lemmas, all of which are fairly straightforward variants on standard arguments.

Proof of Lemma 2.2. This follows by analogy with the argument in [9]. For each $l \geq 1$, we can formally define the complex function

$$\eta_{uv}(s) = \sum_{n=2l}^{\infty} \sum_{\substack{b:|b|=n \\ b_1^l = u \\ b_{n-l+1}^n = v}} e^{-sL_b}, \quad s \in \mathbb{C}.$$

We associate to u the cylinder set in Σ_B defined by

$$[u]_{\Sigma_B} = \{x = (x_n)_{n=1}^{\infty} \in \Sigma_B : x_i = u_i \text{ for } 1 \le i \le l\}$$

and its characteristic function

$$\chi_{[u]_{\Sigma_B}}(x) = \begin{cases} 1 & \text{if } x \in [u] \\ 0 & \text{otherwise.} \end{cases}$$

Using Lemma 4.2 and the definition of the transfer operator we can write that

$$\sum_{\substack{b:|b|=n \\ b_1^l=u \\ b_{n-l+1}^n=v}} e^{-sL_b} = \sum_{\substack{b:|b|=n \\ b_1^l=u \\ b_{n-l+1}^n=v}} e^{-sr^n(b\dot{0})} = e^{-sr^l(v\dot{0})} \mathcal{L}_s^{n-l} \chi_{[u]}(\dot{0}),$$

since $r^n(b\dot{0}) = r^{n-l}(b\dot{0}) + r^l(v\dot{0})$, where $b\dot{0}$ denotes the concaternation of the finite word b and the infinite sequence $\dot{0}$. By Lemma 4.4 and the first part of Lemma 4.5 we can deduce that $\eta_{uv}(s)$ converges for $Re(s) = \sigma > h$ (since then $e^{P(-\sigma r)} < 0$, as we remarked above). Moreover, by using both parts of Lemma 4.4 one can easily show (following [9]) that $\eta_{uv}(s)$ has a meromorphic extension

$$\eta_{uv}(s) = \frac{C_{uv}}{s-h} + R(s),$$

where $R(s)$ is analytic in a neighbourhood of $Re(s) \ge h$ and, using the first and last part of Lemma 4.5, we can write

$$C_{uv} = -\frac{e^{-sr^l(v\dot{0})}\mu([u])\psi_h(\dot{0})}{\frac{d}{dt}P(-tr)|_{t=h}} > 0.$$

where $\mu := \mu_h$.

If we write

$$\eta(s) = \int_0^1 e^{-sT} dN^{uv}(T)$$

as a Riemann-Stieltjes integral then we can apply the Ikehara-Wiener tauberian theorem to deduce the asymptotic result (cf. [9]). □

The dependence of C_{uv} on v can be made more explicit using the symmetry in the counting function $N^{uv}(T)$. First, note that if we denote

$$[u]_{\Sigma_A} = \{x = (x_n)_{n=1}^{\infty} \in \Sigma_A : x_i = u_i \text{ for } 1 \le i \le l\}$$

then it is easy to see that $\mu([u]_{\Sigma_A}) = \mu([u]_{\Sigma_B})$. Next consider the right shift map on

$$\Sigma_A^- = \left\{x = (x_n)_{n=-\infty}^{-1} : A(x_n, x_{n+1}) = 1 \text{ for } n \le -2\right\}$$

then we can write

$$[v]_{\Sigma_A^-} = \left\{x = (x_n)_{n=-\infty}^{-1} \in \Sigma_A^- : x_i = v_{n+1+i} \text{ for } -l \le i \le -1\right\}$$

then there exists a constant $K > 0$ and a probability measure μ_- on Σ_A^- such that

$$C_{uv} = K\mu_h([u]_{\Sigma_A})\mu_-([v]_{\Sigma_A^-}). \tag{2}$$

Proof of Lemma 2.3. The geodesic arcs γ_a, $\gamma_{b^{-1}}$ and γ_b on V, each starting and finishing at ξ, can each be lifted to geodesic arcs $\widetilde{\gamma}_a$, $\widetilde{\gamma}_{b^{-1}}$ and $\widetilde{\gamma}_b$ on the Universal Cover $\widetilde{V}$. Moreover, fixing a lift $\widetilde{\xi}$ of ξ we can choose the lifts $\widetilde{\gamma}_a$, $\widetilde{\gamma}_{b^{-1}}$, $\widetilde{\gamma}_b$ on V such that

$$\begin{aligned}\widetilde{\gamma}_b(0) &= \widetilde{\xi},\\ \widetilde{\gamma}_b(L_b) &= \widetilde{\gamma}_a(0) \text{ and}\\ \widetilde{\gamma}_a(L_a) &= \widetilde{\gamma}_{b^{-1}}(0).\end{aligned}$$

On the other hand, $bab^{-1} \in \pi_1(V,\xi)$ corresponds to a geodesic arc $\gamma_{bab^{-1}}$ and we can choose a lift $\widetilde{\gamma}_{bab^{-1}}$ with

$$\gamma_{bab^{-1}}(0) = \widetilde{\xi} \text{ and } \gamma_{bab^{-1}}(L_{bab^{-1}}) = \widetilde{\gamma}_{b^{-1}}(L_{b^{-1}}).$$

This is illustrated in Figure 2.

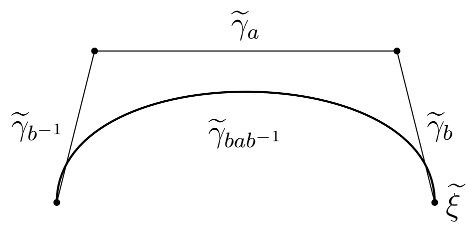

Figure 2: The lifts of the geodesics corresponding to a and b

The conclusion of the lemma follows from the properties of negative curvature. The side $\widetilde{\gamma}_a$ of the rectangle is fixed. Specifying the first and last l generators of b determines the angle between the sides corresponding to $\gamma_{b^{-1}}$ and γ_a, and the angle between the sides corresponding to γ_b and γ_a, up to an exponentially small error (in l). In particular, the relative difference $L_{bab^{-1}} - L_a - 2L_b$ in the lengths of the sides is determined up an exponential small error (in l). This allows us to choose a value $\tau_l(u,v)$ with the required property. □

Proof of Lemma 2.4. From (5.1) we know that we can write that

$$\sum_{\substack{|u|=|v|=l\\ u_1\neq a_m\\ v_l\neq a_1^{-1}}} C_{uv} e^{-\tau(u,v)} = K \sum_{\substack{|u|=|v|=l\\ u_1\neq a_m\\ v_l\neq a_1^{-1}}} \mu([u]_{\Sigma_A})\mu_-([v]_{\Sigma_A^-})e^{-\tau(u,v)}.$$

We can define

$$\Delta := \left\{\left((x_n)_{n=0}^{\infty}, (y_n)_{-\infty}^{-1}\right) \in \Sigma_A \times \Sigma_A^- : x_0 \neq a_m, y_{-1} \neq a_1^{-1}\right\}$$

and for each $l \geq 1$ we can define a locally constant function $\mathcal{T}_l : \Delta \to \mathbb{R}$ by

$$\mathcal{T}_l(x,y) = \sum_{|u|=|v|=l} e^{-\tau(u,v)} \chi_{[u]_{\Sigma_A}}(x)\chi_{[v]_{\Sigma_A^-}}(y).$$

Letting l tend to infinity this converges to a continuous function $\mathcal{T} : \Delta \to \mathbb{R}$ by Lemma 2.3 and

$$\lim_{l \to +\infty} \sum_{\substack{|u|=|v|=l \\ u_1 \neq a_m \\ v_l \neq a_1^{-1}}} C_{uv} e^{-\tau(u,v)} = \int_\Delta \mathcal{T}(x,y) d\mu(x) d\mu_-(y)$$

showing that the limit exists.

□

Remark 5.1. There are a number of questions that naturally arise.

1. The proof can probably be modified without too much difficulty to count geodesic arcs with certain additional restrictions. For example, those for which b is null in homology or where $\widetilde{\gamma}_b(L_b)$ lies in a sector. One can also show other equidistribution results for the geodesic arcs on V.

2. The same arguments would apply in higher dimensions, provided what we can still assume that the fundamental groups is a free group. For, example if V is the quotient of d-dimensional hyperbolic space by a Kleinian Schottky group.

3. It is natural to ask if Theorem 1.4 is still valid in the case that V is a closed surface with $\partial V = \emptyset$ (or where $\pi_1(V)$ is not a free group). However, there are significant additional difficulties in analysing the conjuhacy classes of a when $\pi_1(V)$ is not a free group.

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