COUNTING GEODESIC LOOPS ON SURFACES OF GENUS AT LEAST 2 WITHOUT CONJUGATE POINTS

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Abstract. In this note we prove asymptotic estimates for closed geodesic loops on compact surfaces with no conjugate points. These generalize the classical counting results of Huber and Margulis and sector theorems for surfaces of strictly negative curvature. We will also prove more general sector theorems, generalizing results of Nicholls and Sharp for special case of surfaces of strictly negative curvature.

1. Introduction

For a closed surface M of negative curvature there are classical results which count the number of geodesic arcs starting and ending at a given reference point $p \in M$ and whose length at most t, say. For constant curvature surfaces these were proved by Huber in 1959, and for variable curvature surfaces these were proved by Margulis in 1969, In particular, they give simple asymptotic estimates for this counting function as $t \to +\infty$. In this brief note we will extend these results in Corollary 1.6 to the more general setting of surfaces without conjugate points.

There are refinements of the original counting results of Huber and Margulis whereby the geodesics are restricted to lie in a sector. These are due to shown for constant curvature surfaces by Nicholls in 1983, and for variable curvature surfaces by Sharp in 2001. We will describe generalizations of these results to surfaces without conjugate points in Corollaries 1.5 and 1.6. These will follow from a more general statement (Theorem 1.3) which appears below.

We begin with some general notation. Let (M, g) be a closed Riemannian manifold, SM the unit tangent bundle of M and let $\pi : SM \to M$ be the natural projection to the footpoint.

Let $t, \theta, \theta' > 0$ and $v_0, v_0' \in SM$ with $\pi v_0 = \pi v_0' = p$, say. We want to count geodesic loops $c : [0, \tau] \to M$ which:

(1) start and finish at p (i.e., $c(0) = c(\tau) = p$);
(2) have length τ less than t;
(3) leaves the fibre S_pM at an angle at most θ to v_0; and
(4) enters the fibre S_pM at an angle at most θ' to v_0'.

(see Figure 1).

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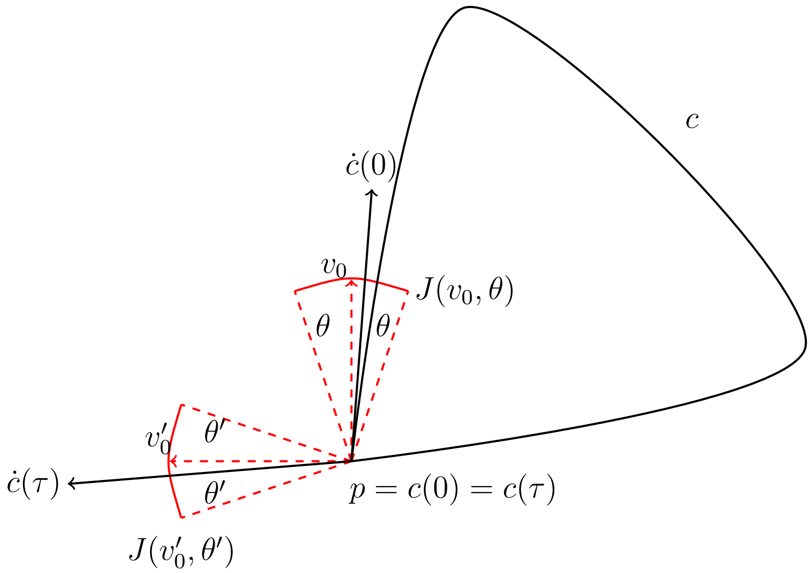

FIGURE 1. A geodesic loop c which starts within an angle θ of v_0 and ends within an angle θ' of v_0'

Definition 1.1. *Given an angle $0 < \theta \leq \pi$ and a unit tangent vector $v_0 \in SM$, we define the following arc in the fibre $S_{\pi v_0}M$:*

$$J(v_0, \theta) := \{w \in S_pM : \measuredangle_p(v_0, w) \leq \theta\},$$

i.e., the unit tangent vectors w in the same fibre as v_0 at an angle at most θ.

This allows us to introduce convenient notation for the collection of geodesic arcs satisfying properties (1)–(4).

Definition 1.2. *We let $\mathcal{C}(t, J(v_0, \theta), J(v_0', \theta'))$ denote the set of geodesic loops $c : [0, \tau] \to M$ based at $c(0) = c(\tau) = p \in M$ of length $\tau \leq t$ and satisfying $c'(0) \in J(v_0, \theta)$ and $c'(\tau) \in J(v_0', \theta')$.*

We will now consider the problem of counting geodesic the number

$$\#\mathcal{C}(t, J(v, \theta), J(v', \theta'))$$

of such geodesic arcs.

We will work in the general setting of closed surfaces M of genus at least 2 that have no conjugate points, i.e., for any two points $p, q \in M$ there is no geodesic from p to q along which there is a non-trivial Jacobi field vanishing at p and q. [1] By the Cartan-Hadamard theorem, an equivalent formulation is that there is a unique geodesic arc joining

[1]Recall that a Jacobi vector field is a vector field $J(t)$ on the geodesic $c(t)$ satisfying $\frac{D^2}{dt^2}J(t) = R(J(t)), \dot{\gamma}(t), \dot{\gamma}(t) = 0$, where D denotes the covariant derivative with respect to the Levi-Civita connection, R the Riemann curvature tensor $\dot{\gamma}(t) = d\gamma(t)/dt$ the tangent vector field.

distinct points in the universal cover $\widetilde{M}$. Examples include the special case that M has non-positive curvature. We refer to [1] for another well known example.

Finally, using the following notation

$$S^2M := \{(v, v') \in SM \times SM : \pi v = \pi v'\}$$

we can formulate our main result.

Theorem 1.3. *Let M be a closed connected surface of genus at least 2 without conjugate points. Then there exists $\theta_0 > 0$, $h > 0$ and a measurable positive function $a : S^2M \times (0, \theta_0)^2 \to \mathbb{R}_{>0}$ such that*

$$\#\mathcal{C}(t, J(v, \theta), J(v', \theta')) \sim a(v, v', \theta, \theta')e^{ht}, \text{ as } t \to +\infty \tag{1}$$

i.e., $\lim_{t\to+\infty} \frac{\#\mathcal{C}(t,J(v,\theta),J(v',\theta'))}{a(v,v',\theta,\theta')e^{ht}} = 1$.
Moreover if the geodesic flow is expansive [2] *then the function $a(\cdot, \cdot, \cdot, \cdot)$ is continuous.*

In the statement of the theorem the value h is the topological entropy of the geodesic flow on the unit tangent bundle SM.

Remark 1.4. *In the special case that M has constant curvature then $a(\cdot)$ is a constant function, and when M has variable negative curvature it is known that $a(\cdot)$ is a continuous function (not least because it is expansive).*

Theorem 1.3 has corollaries which extend several classical results from the context of negative curvature. In particular, this leads to generalizations of classical counting and sector theorems. For example, when we set $\theta' = \pi$ then this gives the following.

Corollary 1.5 (Sector Theorem)**.** *Given $0 < \theta \le \pi$ there exists $a = a(p, \theta) > 0$ such that the number of geodesic arcs which: start at $p \in M$ and finish at $q \in M$; leave S_pM at an angle at most θ to v_0; and have length at most t, is asymptotic to ae^{ht} as $t \to +\infty$.*

This generalizes results from [8], [9], [10].

Furthermore, when $\theta = \theta' = \pi$ then this further reduces to the original counting result:

Corollary 1.6 (Arc counting)**.** *There exists $a = a(p) > 0$ such that the number of geodesic arcs which start at $p \in M$, finish at $q \in M$ and have length at most t is asymptotic to ae^{ht} as $t \to +\infty$.*

This generalizes results from [5], [6], [7].

Finally, we can describe equidistribution result of a slightly different flavour. Let $\widehat{M}$ be a finite cover for M. We can associate to any

[2]A flow $\phi_t : SM \to SM$ is *expansive* if for all $\delta > 0$ there exists $\epsilon > 0$ such that if $d(\phi_t(x), \phi_{s(t)}(y)) < \delta$ for all $t \in \mathbb{R}$ for $x, y \in SM$ and a continuous map $s : \mathbb{R} \to \mathbb{R}$ then $y = \phi_t(x)$ where $|t| < \epsilon$.

geodesic arc c on M which starts and ends at $p \in M$ (and has length L_c) a lift $\widehat{c}$ to $\widehat{M}$. The following corollary estimates the proportion of geodesic arcs such that $\widehat{c}$ on $\widehat{M}$ with $\widehat{c}(0) = \widehat{c}(L_c)$

Corollary 1.7 (Equidistribution in finite covers)**.** *The proportion of geodesic arcs c which start and end at $p \in M$, have lifts $\widehat{c}$ which start and end at the same point in $\widehat{M}$, and have length at most t is asymptotic to*

$$\frac{\mathrm{Area}(M)}{\mathrm{Area}(\widehat{M})} a e^{ht} \text{ as } t \to +\infty$$

This corollary can be used to prove a corollary related to the first Homology group $H_1(M, \mathbb{Z})$. Each closed loop c based at p gives rise naturally to an element $\langle c \rangle \in H_1(M, \mathbb{Z})$. Let us consider a finite index subgroup $G < H_1(M, \mathbb{Z})$ then for a geodesic arc c we can associate the coset $\langle c \rangle G \in H_1(M, \mathbb{Z})G$.

Corollary 1.8 (Homological Equidistribution)**.** *For a fixed coset $\alpha \in H_1(M, \mathbb{Z})/G$. The proportion of geodesic arcs c which start and finish at $p \in M$, satisfy $\langle c \rangle \Gamma = \alpha$ and have length at most t is asymptotic to*

$$\#(H_1(M, \mathbb{Z})/\Gamma) a e^{ht} \text{ as } t \to +\infty$$

Remark 1.9. *The theorem and each of the corollaries has a natural equivalent formulation in terms of the action $\Gamma \times X \to X$ of the covering group $\Gamma = \pi_1(M)$ on the universal cover X. For example, Corollary 1.6 gives an asymptotic estimate for $\#\{g \in \Gamma : d_X(\overline{p}, g\overline{p}) \leq t\}$ where $\overline{p} \in X$ and d_X is the lifted Riemannian metric to X.*

2. Closed arcs and isometries

The structure of the proof of Theorem 1.3 follows the lines of Margulis' original proof. However, it requires modifications using a number of recent techniques from [2], [3]. A key ingredient is the construction of the measure of maximal entropy for the geodesic flow $\phi_t : SM \to SM$.

2.1. **Some Notation.** Let X be the universal cover of M with the lifted metric. The covering group $\Gamma \cong \pi_1(M)$ satisfies that $M = X/\Gamma$.

Let SX denote the unit tangent bundle for X and let $\overline{\pi} : SX \to X$ denote the canonical projection of a unit tangent vector in SX to its footpoint in X. Let $\overline{p} \in X$ be a lift of $p \in M$ and let $\underline{B}(\overline{p}, R) \subset X$ denote a ball of radius $R > 0$ about $\overline{p}$. We can use this to give a convenient definition of topological entropy [4].

Definition 2.1. *The* topological entropy $h = h(\phi)$ *is given by*

$$h = \lim_{R \to +\infty} \frac{\log \mathrm{Vol}(\underline{B}(\overline{p}, R))}{R}.$$

Given $\overline{v} \in SX$, let $c = c_{\overline{v}} : \mathbb{R} \to X$ denote the unique geodesic such that $c_{\overline{v}}(0) = \overline{\pi}(\overline{v})$ and $c'_{\overline{v}}(0) = \overline{v}$.

Definition 2.2. *Let ∂X denote the ideal boundary of X consisting of equivalence classes $[c]$ of geodesics $c : \mathbb{R} \to X$ which stay a bounded distance apart.*

(See [2, Section 2] for a detailed description of the construction and properties of ∂X). In particular, every geodesic $c_{\overline{v}} : \mathbb{R} \to X$ defines two points $c(\pm\infty) \in \partial X$, which it is convenient to denote $\overline{v}^- := c(\infty)$ and $\overline{v}^+ := c(+\infty)$. The natural acton $\Gamma \times X \to X$ extends to an action of Γ on ∂X given by $g[c] = [gc]$, where $g \in \Gamma$.

Definition 2.3. *Given $\overline{p} \in X$, the* Busemann function $b_{\overline{p}}(\cdot, \cdot) : X \times \partial X \to \mathbb{R}$ *is defined by*

$$b_{\overline{p}}(\overline{q}, \xi) = \lim_{t \to +\infty} d(\overline{q}, c_v(t)) - t$$

for $\overline{v} \in S_{\overline{p}}X$ satisfying $\xi = c_{\overline{v}}(+\infty)$ [2, Definition 2.16].

We next recall the characterization of Patterson-Sullivan measures on the boundary ∂X constructed in [2, Proposition 5.1].

Definition 2.4. *The* Patterson-Sullivan measures *on ∂X are a family of measures $\{\mu_{\overline{p}} : \overline{p} \in X\}$ which transform under the action of Γ on ∂X by*

$$\frac{d\mu_{\overline{p}}\gamma}{d\mu_{\overline{p}}}(\xi) = e^{-hb_{\overline{p}}(\gamma\overline{p}, \xi)}$$

for $\gamma \in \Gamma$ and $\xi \in \partial X$

The Busemann function is also used in defining horocycles.

Definition 2.5. *The* stable horocycle *is defined by*

$$H_\xi(\overline{p}) = \{\overline{q} \in X : b_{\overline{p}}(\overline{q}, \xi) = 0\}$$

and the unstable horocycle *is defined by*

$$H^-_\xi(\overline{p}) = \{q \in X : b_{\overline{p}}(\overline{q}, -\xi) = 0\}$$

where $-\xi$ is the antipodal vector to ξ.

Finally, we define a class of tangent vectors which will serve us well in the proof.

Definition 2.6. *We denote by $\mathcal{E} \subset SX$ the set of* expansive vectors *consisting of those unit tangent vectors whose stable and unstable horocycles intersect at exactly one point.*

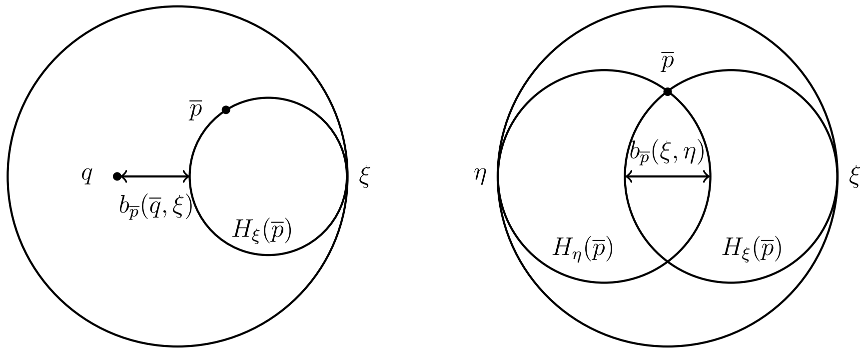

FIGURE 2. (i) The geometric interpretation of the Busemann function as the signed distance of $\overline{q}$ from $H_\xi(\overline{p})$ corresponds to $b_p(q,\xi)$; (ii) The distance between the horocycles $H_\xi(\overline{p})$ and $H_\eta(\overline{p})$ corresponds to $b_{\overline{p}}(\xi,\eta)$

2.2. **The measure of maximal entropy.** We begin with a correspondence which is useful in the construction of measures of maximal entropy.

Definition 2.7. *The* Hopf map $H\colon SX \to \partial^2 X \times \mathbb{R}$ *is defined by*

$$H(v) := (\overline{v}^-, \overline{v}^+, s(v)) \quad \textit{where} \quad s(\overline{v}) := b_p(\pi(\overline{v}, (\overline{v}^-). \tag{2}$$

In particular, following [2, Lemma 5.5] this family of measures defines a Γ-invariant measure $\overline{\mu}$ on $\partial X \times \partial X \setminus \mathrm{diag}$ (where $\mathrm{diag} \subset \partial X \times \partial X$ are the diagonal elements) characterized by

$$d\overline{\mu}(\xi,\eta) = e^{h\beta_{\overline{p}}(\xi,\eta)} d\mu_{\overline{p}}(\xi) d\mu_{\overline{p}}(\eta), \text{ for } \xi, \eta \in \partial X, \tag{3}$$

where $\beta_{\overline{p}}(\xi,\eta)$ is the distance in X between the horospheres $H_{\overline{p}}(\xi)$ and $H_{\overline{p}}(\eta)$, see Figure 2 (ii) (or [3, Figure 1]).

Definition 2.8. *The Hopf transform carries $d\overline{\mu} \times dt$ to a measure $d\overline{m} := H_*(d\overline{\mu} \times dt)$ on SX.*

There is a natural projection from SX to SM (taking v to $v\Gamma$). The following result was proved in [2, Theorem 1.1].

Lemma 2.9. *The measure $\overline{m}$ on SX projects (after normalization) to the measure $\underline{m}$ maximal entropy for the geodesic flow on SM (i.e., $\pi_* \overline{m} = \underline{m}$ and $h(\underline{m}) = h$). Moreover,*

(1) $\underline{m}$ is unique, strongly mixing [3] and fully supported; and
(2) $\underline{m}(\mathcal{E}) = 1$ (cf. [2, Equation (2.10)]*).*

We now turn to the final ingredients in the proof.

[3] $\underline{m}$ is even shown to be Bernoulli

2.3. **Flow boxes.** For the remainder of this section, we fix a choice of $(v_0, v_0') \in S^2M \cap \mathcal{E}^2$. We can then associate to the sets $J(v_0, \theta), J(v_0', \theta') \subset SM$ in Definition 1.2 a choice of lifts $\overline{J}(v_0, \theta), \overline{J}(v_0', \theta') \subset SX$.

To proceed we want to consider the natural images of these sets in ∂X:

Definition 2.10. *We can associate to $J(v_0, \theta)$ and $J(v_0', \theta')$ their "future" and "past" subsets of ∂X defined, respectively, by*

$$\mathbf{F} = \mathbf{F}_\theta := \{\bar{w}^+ : \bar{w} \in \overline{J}(v_0, \theta)\} \text{ and } \mathbf{P} = \mathbf{P}_\theta := \{\bar{w}^- : \bar{w} \in \overline{J}(v_0, \theta)\}$$

$$\mathbf{F}' = \mathbf{F}_{\theta'} := \{\bar{w}^+ : \bar{w} \in \overline{J}(v_0', \theta')\} \text{ and } \mathbf{P}' = \mathbf{P}_{\theta'} := \{\bar{w}^- : \bar{w} \in \overline{J}(v_0', \theta')\}.$$

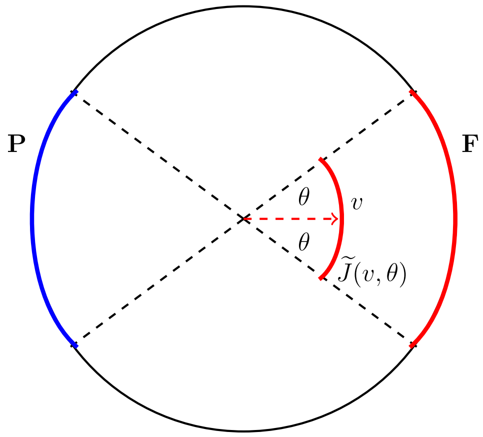

FIGURE 3. The sets $\mathbf{P}$ and $\mathbf{F}$ associated to $\widetilde{J}(v, \theta)$

The sets $\mathbf{F}, \mathbf{P}, \mathbf{F}', \mathbf{P}' \subset \partial X$ will be used to construct flow boxes for the geodesic flow. Assume first that $\epsilon > 0$ is small (with respect to the injectivity radius of M) and then choose $\theta_1 > 0$ such that for all $\theta < \theta_1$ we have

$$\operatorname{diam}\left(\pi H^{-1}(\mathbf{P} \times \mathbf{F} \times \{0\})\right) < \frac{\epsilon}{2}$$

(see [3, Lemma 3.9]). For $\alpha \leq \frac{3}{2}\epsilon$ and $\theta \in (0, \theta_1)$ we define two different flow boxes [4] B_θ^α and $B_{\theta'}^{\alpha'}$ (of different "lengths" α and ϵ^2, respectively) in SX by:

$$\begin{aligned} \overline{B}_\theta^\alpha &:= H^{-1}(\mathbf{P} \times \mathbf{F} \times [0, \alpha]) \text{ and} \\ \overline{B}_{\theta'}^{\epsilon^2} &:= H^{-1}(\mathbf{P}' \times \mathbf{F}' \times [0, \epsilon^2]). \end{aligned} \tag{4}$$

(cf. [3, (3.11) and (3.12)]).

Let $\underline{B}_\theta^\alpha = \pi(B_\theta^\alpha)$ and $\underline{B}_{\theta'}^{\epsilon^2} = \pi(B_{\theta'}^{\epsilon^2})$ be their projections onto SM.

[4]for the geodesic flow $\phi_t : SX \to SX$ on SX

Remark 2.11. *Since the function $\rho' \to m(\underline{B}^{\epsilon^2}_{\rho'})$ is nondecreasing, and thus has countably many discontinuties (by Lebesgue's Theorem), we can suppose without loss of generality that $\theta' \in (0, \theta_1)$ is a continuity point, and so, in particular,*

$$\lim_{\rho' \to \theta'} m(\underline{B}^{\epsilon^2}_{\rho'}) = m(\underline{B}^{\epsilon^2}_{\theta'}). \tag{5}$$

In order to give a dynamical approach to the counting problem the following two definitions will prove useful. Let $\phi^t : SX \to SX$ denote the geodesic flow on SX.

Definition 2.12. *For $t > 0$ we can define two subsets of Γ by:*

$$\Gamma_{\theta,\theta'}(t) := \{\gamma \in \Gamma : \overline{B}^{\epsilon^2}_{\theta'} \cap \phi^{-t}\gamma_* \overline{B}^{\alpha}_{\theta} \neq \emptyset\} \tag{6}$$

$$\Gamma^*_{\theta,\theta'}(t) := \{\gamma \in \Gamma_{\theta,\theta'}(t) : \gamma\mathbf{F} \subset \mathbf{F}' \text{ and } \gamma^{-1}\mathbf{P} \subset \mathbf{P}'\}. \tag{7}$$

where the sets have an implicit dependence on $\epsilon, \alpha, v_0, v_0'$. (cf. [3, (4.4) and (4.14)].*)*

By definition we have $\Gamma^*_{\theta,\theta'}(t) \subset \Gamma_{\theta,\theta'}(t)$ and although we may not expect the reverse inclusion to be true, we have the following slightly more modest result.

Lemma 2.13. *For every $\rho' \in (0, \theta')$ and $\rho \in (0, \theta)$, there exists $t_0 > 0$ such that*

$$\Gamma_{\rho,\rho'}(t) \subset \Gamma^*_{\theta,\theta'}(t) \quad \text{for all} \quad t \geq t_0.$$

We postpone the proof of Lemma 2.13 until Appendix A.

The next lemma shows there is an inclusion of the set defined in Definition 1.2 into $\Gamma(t)$.

Lemma 2.14. *We have an injection*

$$\mathcal{C}(t, J(v_0, \theta), J(v_0', \theta')) \hookrightarrow \Gamma(t)$$

which associates to a geodesic c the associated homotopy class $[c] \in \pi_1(M) \cong \Gamma$.

We postpone the proof of Lemma 2.14 until Appendix A.

Although we may not expect the reverse inclusion in Lemma 2.14 to be true, we at least have the following partial result.

Lemma 2.15. *For every $\rho' \in (0, \theta')$, there exists $t_0 > 0$ such that there is an inclusion*

$$\Gamma_{\theta,\rho'}(t) \hookrightarrow \mathcal{C}(t \pm 2\epsilon, J(v_0, \theta), J(v_0', \theta')) \quad \forall t > t_0.$$

Again we postpone the proof of Lemma 2.15 until Appendix A.

3. Proof of the counting results

In this section we will use results from the previous section to prove the following proposition, which easily implies Theorem 1.3.

Proposition 3.1. *We have an asymptotic expression for the cardinality of* $\Gamma(t)$ *of the form:*

$$\#\Gamma(t) \sim e^{ht}\overline{m}(B)\frac{\mu_{\overline{p}}(\mathbf{F}')}{\mu_{\overline{p}}(\mathbf{F})} \text{ as } t \to +\infty. \tag{8}$$

Remark 3.2. *The constant on the righthand side of (8) depends on* p*, but not then on the choice of* $\bar{p} \in \pi^{-1}(p)$*.*

We begin with a little more notation. Let

$$S_\theta = H^{-1}\left(\mathbf{P} \times \mathbf{F} \times [0, \epsilon^2]\right) \subset SX \tag{9}$$

be another flow box and let

$$\Gamma^*(t, \alpha) := \{\gamma \in \Gamma^* : S_\theta \cap \gamma_* \phi^{-t} B_\theta^\alpha \neq \emptyset\}.$$

The proof of Proposition 3.1 now depends on the following two technical lemmas.

Lemma 3.3. *For* $\gamma \in \Gamma^*(t, \alpha)$*, we have*

$$B_{\theta'}^{\epsilon^2} \cap \phi^{-(t+2\epsilon^{\frac{3}{2}})}\gamma_* B_\theta^{\alpha+4\epsilon^{\frac{3}{2}}} = H^{-1}(\mathbf{P}' \times \gamma\mathbf{F} \times [0, \epsilon^2]) =: S^\gamma.$$

The next lemma describes the $\overline{m}$-measure of the set S^γ.

Lemma 3.4. *For each* $\gamma \in \Gamma^*$*, we have*

$$\overline{m}(\overline{S}^\gamma) = \epsilon^2 e^{\pm 4h\epsilon} e^{-ht}\mu_p(\mathbf{P}')\mu_p(\mathbf{F}),$$

and similarly with $\overline{m}$ *and* $\overline{S}^\gamma$ *on* SX *replaced by the projections* m *and* $S^\gamma = \pi(\overline{S}^\gamma)$ *onto* SM*.*

We postpone the proofs of both of these lemmas until Appendix B.

Proof of Proposition 3.1. This follows the general lines of §5.2 in [3]. It follows from Lemmas 2.13 and 3.3 that given any $\alpha \in (0, \frac{3}{2}\epsilon]$ and $\rho' \in (0, \theta'), \rho \in (0, \theta)$, for all sufficiently large t we have

$$\underline{B}_{\rho'}^{\epsilon^2} \cap \phi^{-t}\underline{B}_\theta^\alpha \subset \bigcup_{\gamma \in \Gamma^*(t,\alpha)} \underline{S}^\gamma \subset \underline{B}_{\theta'}^{\epsilon^2} \cap \phi^{-(t+2\epsilon^2)}\underline{B}_\theta^{\alpha+4\epsilon^2}$$

by proving the corresponding result on SX and projecting to SM.

Using Lemma 3.4, for all $\gamma \in \Gamma^*(t)$, we have

$$\begin{aligned} e^{-4h\epsilon}\underline{m}(\underline{B}_{\rho'}^{\epsilon^2} \cap \phi^{-t}\underline{B}) &\leq \epsilon^2 \#\Gamma^*(t, \alpha) e^{-ht}\mu_p(\mathbf{P}')\mu_p(\mathbf{F}) \\ &\leq e^{4h\epsilon}\underline{m}(\underline{B}_{\theta'}^{\epsilon^2} \cap \phi^{-(t+2\epsilon^2)}\underline{B}_\theta^{\alpha+4\epsilon^2}). \end{aligned}$$

Sending $t \to \infty$, using mixing, and dividing through by $\underline{m}(\underline{B}_{\theta'}^{\epsilon^2})\underline{m}(\underline{B}_{\theta}^{\alpha}) = \overline{m}(B_{\theta'}^{\epsilon^2})\overline{m}(B_{\theta}^{\alpha})$, we get

$$e^{-4h\epsilon}\frac{\overline{m}(B_{\rho'}^{\epsilon^2})}{\overline{m}(B_{\theta'}^{\epsilon^2})} \lesssim \frac{\epsilon^2 \#\Gamma^*(t,\alpha)\mu_p(\mathbf{P}')\mu_p(\mathbf{F})}{e^{ht}\overline{m}(B_{\theta"}^{\epsilon^2})\overline{m}(B_{\theta}^{\alpha})} \lesssim e^{4h\epsilon}\frac{\overline{m}(B_{\theta}^{\alpha+4\epsilon^2})}{\overline{m}(B_{\theta}^{\alpha})}.$$

By (5), assuming that θ' is a point of continuity for $\rho' \mapsto m(B'_{\rho'})$, so we can send $\rho' \nearrow \theta'$ and obtain

$$e^{-5h\epsilon} \lesssim \frac{\#\Gamma^*(t,\alpha)}{e^{ht}\overline{m}(B)}\frac{\mu_p(\mathbf{F})}{\mu_p(\mathbf{F}')} \lesssim e^{5h\epsilon}(1+4\epsilon^2/\alpha). \tag{10}$$

Finally we need to replace $\#\Gamma^*(t,\alpha)$ by $\#\Gamma(t)$. (cf. Compare with [3, (5.4)].)

This ends the proof of Proposition 3.1. □

In order to allow for arbitrary θ and θ' in the main theoem we can break the arcs $J(\cdot,\cdot)$ into smaller pieces and apply the proposition.

Appendix A. Proofs of lemmas on isometries and closed arcs

This section is devoted to the proof of Lemmas 2.13, 2.14 and 2.15. The proof of Lemma 2.14 is relatively easy while Lemma 2.13 and 2.15 both uses a geometric featture of surfaces without conjugate point that we first recall here.

Definition A.1. *A simply connected Riemannian manifold X without conjugate points is a (uniform) visibility manifold if for every $\epsilon > 0$ there exists $L > 0$ such that whenever a geodesic $c : [a,b] \to X$ stays at a distance at least L from some point $p \in X$, then the angle sustained by c at p is less than ϵ, that is*

$$\measuredangle_p(c) = \sup_{a \le s,t \le b} \measuredangle_p((c(s), c(t)) < \epsilon.$$

Proof of Lemma 2.13. The proof uses [3, Lemma 4.9] with the choices $R = \mathbf{F}'_{\rho'}$, $Q = \mathbf{P}'_{\rho'}$, $V = int(\mathbf{F}'_{\theta'})$ and $U = int(\mathbf{P}'_{\theta'})$. □

Proof of Lemma 2.14. Let $\underline{c} \in \mathcal{C}(t, J(v_0,\theta), J(v'_0,\theta'))$ and c be the lift of $\underline{c}$ on X with $\underline{c}(0) = p$. There exists $\gamma \in \Gamma$ such that $c(t) = \gamma p = \gamma c(0)$. Let $\mathrm{pr}_* : SX \to SM$ be the map associated to $\pi : X \to M$ then by definition of $\mathcal{C}(t, J(v_0,\theta), J(v'_0,\theta'))$, for $w = c'(t)$, $\mathrm{pr}_* w \in B_{\theta'}^{\epsilon^2}$ and $\phi^{-t}w = c'(0) \in B_{\theta}^{\alpha}$ implies that $\bar{w} := \gamma_* w \in B_{\theta'}^{\epsilon^2}$ for some $\gamma \in \Gamma$. Therefore $\bar{w} \in B_{\theta'}^{\epsilon^2} \cap \phi^{-t}\gamma_* B_{\theta}^{\alpha}$. □

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[5]Compare with [2, (4.8)]

Proof of Lemma 2.15. Let $\gamma \in \Gamma_{\theta,\rho'}(t)$ and $w \in B^{\epsilon^2}_{\rho'} \cap \phi^{-t}\gamma_* B^{\alpha}_{\theta}$. By the triangle inequaity

$$d(p,\gamma p) \leq d(p,\pi w) + d(\pi w, \pi\phi^t w) + d(\pi\phi^t w, \gamma p).$$

By [3, Lemma 3.10], we have $d(p,\pi w) \leq \operatorname{diam}(B') \leq 2\epsilon$ and $d(\pi\phi^t w, \gamma p) \leq \operatorname{diam}(B) \leq 2\epsilon$. Substituting these into the above display inequality gives

$$d(p,\gamma p) \leq t + 4\epsilon.$$

We are left to prove that the geodesic $c := c_{p,\gamma_p}$ connecting p to γp satisfies $c'(0) \in J(v_0,\theta)$ and $c'(d(p,\gamma p)) \in J(v_0',\theta')$.

Let $v \in S_pX$ such that $v^+ = w^+ \in \mathbf{F}$, in particular, there exists $R > 0$ such that $d(c_v(t), c_w(t)) \leq R$ and therefore the geodesic connecting γp to $c_v(t)$ stays at distance at least $t - 2R$. Then using the uniform visabilty, there exists t_0 such that for all $t > t_0$ we have $\measuredangle_p(v, c'(0)) \leq \theta - \rho$ which implies that $c'(0) \in \mathbf{F}$. Therefore by the uniform visibility, we have $\measuredangle_p(c'_{p,\gamma p}(0), c'_v(0)) \leq \theta - \rho$, in particular $c'_{p,\gamma p}(0) \in J(v_0,\theta)$. Similarly we use the same visibility condition for the point γp and the geodesic joining p and $c_v(-t)$ where $v \in S_{\gamma p}X$ with $v^- = w^-$. Thus the geodesic $c_{p,\gamma p}$ belongs to $\mathcal{C}(t \pm 2\epsilon, J(v_0,\theta), J(v_0',\theta'))$. □

Appendix B. Counting

This section is devoted to the proof of Lemmas 3.3 and 3.4. The proof uses some geometric quantities that we will define first.

Definition B.1. *For $\xi \in \partial X$ and $\gamma \in \Gamma$, we let $b^{\gamma}_{\xi} := b_{\xi}(\gamma p, p)$*

Lemma B.2. *Given any $\gamma \in \Gamma^* = \{\gamma \in \Gamma : \gamma\mathbf{F} \subset \mathbf{F}$ and $\gamma^{-1}\mathbf{P} \subset \mathbf{P}\}$ and any $t \in \mathbb{R}$, we have*

$$B^{\epsilon^2}_{\theta'} \cap \phi^{-t}\gamma_* B^{\alpha}_{\theta} = \{w \in E^{-1}(\mathbf{P}' \times \gamma\mathbf{F}) : s(w) \in [0,\epsilon^2] \cap (b^{\gamma}_{w^-} - t + [0,\alpha])\}.$$

[6]

Proof of Lemma B.2. To prove that $B^{\epsilon^2}_{\theta'} \cap \phi^{-1}\gamma_* B^{\alpha}_{\theta} \subset E^{-1}(\mathbf{P}' \times \gamma\mathbf{F})$, we observe that if $E(w) \notin \mathbf{P}' \times \gamma\mathbf{F}$, then either $w^- \notin \mathbf{P}'$, so $w \notin B^{\epsilon^2}_{\theta'}$, or $w^+ \notin \gamma\mathbf{F}$, so $w \notin \phi^{-t}\gamma_* B^{\alpha}_{\theta}$.

It remains to show that given $w \in E^{-1}(\mathbf{P}' \times \gamma\mathbf{F})$, we have

$$w \in B^{\epsilon^2}_{\theta'} \;\Leftrightarrow\; s(w) \in [0,\alpha], \text{ and} \tag{11}$$

$$w \in \phi^{-t}\gamma_* B^{\alpha}_{\theta} \;\Leftrightarrow\; s(w) \in b^{\gamma}_{w^-} - t + [0,\alpha]. \tag{12}$$

The first of these is immediate from the definition of B'. For the second, we observe that $s(v) = b_{v^-}(\pi v, p) = b_{\gamma v^-}(\gamma\pi v, \gamma p)$, and thus

$$\begin{aligned} \gamma_* B &= \{\gamma_* v : v \in E^{-1}(\mathbf{P} \times \mathbf{F}) \text{ and } b_{v^-}(\pi v, p) \in [0,\alpha]\} \\ &= \{w \in E^{-1}(\gamma\mathbf{P} \times \gamma\mathbf{F}) : b_{w^-}(\pi w, \gamma p) \in [0,\alpha]\} \end{aligned}$$

[6]Compare this to the proof of [3, Lemma 4.13]

By [3, Equation (3.1)] and [3, Equation (3.2)], we have

$$b_{w^-}(\pi w, \gamma p) = b_{w^-}(\pi w, p) + b_{w^-}(p, \gamma p) = s(w) - b^{\gamma}_{w^-};$$

moreover, since $s(\phi^t w) = s(w) + t$ by [3, Equation (3.8)], we see that $\phi^t w \in \gamma_* B$ if and only if $s(w) - b^{\gamma}_{w^-} + t \in [0, \alpha]$, which proves (12) and completes the proof of the lemma. □

Proof of Lemma 3.3. By Lemma B.2, the fact that $B^{\epsilon^2}_{\theta'} \cap \phi^{-t}\gamma_* B^{\alpha}_{\theta} \neq \emptyset$ implies existence of $\eta \in \mathbf{P}'$ such that

$$(b^{\gamma}_{\eta} - t + [0, \alpha]) \cap [0, \epsilon^2] \neq \emptyset$$

from which we deduce that

$$b^{\gamma}_{\eta} - t - \epsilon^{\frac{3}{2}} + [0, \alpha + 2\epsilon^{\frac{3}{2}}] \supset [0, \epsilon^2]$$

By [3, Lemma (4.11)], it follows that every $\xi \in \mathbf{P}'$ has

$$(b^{\gamma}_{\xi} - t - \epsilon^{\frac{3}{2}} + [0, \alpha + 2\epsilon^{\frac{3}{2}}]) \cap [0, \epsilon^2] \neq \emptyset$$

which in turn implies that

$$b^{\gamma}_{\xi} - t - 2\epsilon^{\frac{3}{2}} + [0, \alpha + 4\epsilon^{\frac{3}{2}}] \supset [0, \epsilon^2].$$

By Lemma B.2, this completes the proof. □

Proof of Lemma 3.4. By definition of $\underline{m}$, we have $\underline{m}(\underline{S}^{\gamma}) = \overline{m}(\overline{S}^{\gamma}) = \epsilon^2 \bar{\mu}(\mathbf{P} \times \gamma \mathbf{F})$. Then we need to prove that $\bar{\mu}(\mathbf{P} \times \gamma \mathbf{F}) = e^{\pm 4h\epsilon} e^{-ht} \mu_p(\mathbf{P}')\mu_p(\mathbf{F})$

Given $(\xi, \eta) \in \mathbf{P}' \times \gamma \mathbf{F}$, we can take q to lie on a geodesic connecting ξ and η, with $b_{\xi}(q, p) = 0$; then we have

$$|\beta_p(\xi, \eta)| := |b_{\xi}(q, p) + b_{\eta}(q, p)| \leq d(q, p) < \epsilon/2,$$

where the last inequality uses [3, Lemma 3.9]. Using this together with (3) gives

$$\bar{\mu}(\mathbf{P}' \times \gamma \mathbf{F}) = e^{\pm h\epsilon/2} \mu_p(\mathbf{P}')\mu_p(\gamma \mathbf{F}),$$

Using [2, Proposition 5.1 (a)] gives

$$\mu_p(\gamma \mathbf{F}) = \mu_{\gamma^{-1}p}(\mathbf{F}),$$

and [2, Proposition 5.1 (b)] gives

$$\frac{d\mu_{\gamma^{-1}p}}{d\mu_p}(\eta) = e^{-hb_{\eta}(\gamma^{-1}p, p)}.$$

When $\eta = c(-\infty)$, where $c := c_{p,\gamma^{-1}p}$. Using the visibility condition as in the proof of Lemma 2.15, for t large enough $\eta \in \mathbf{F}'_{\theta'+\iota}$ for some $\iota > 0$ very small. Using Lemma 2.15, $b_{\eta}(p, \gamma p) = t \pm 4\epsilon$. By [3, Lemma 4.11], for $\xi \in \mathbf{F}'$, $b_{\xi}(\gamma^{-1}p, p)$ varies by at most ϵ^2. We conclude that $\mu_p(\gamma \mathbf{F}) = e^{\pm 5\epsilon} e^{-ht} \mu_p(\mathbf{F})$, and and this proves the lemma. □

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