Effective extension of the pressure function

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1 Introduction

Let us assume that we have matrices $A_1, \cdots, A_k \in GL(d, \mathbb{R})$ ($d \geqslant 2$). Let $\mathbb{R}P^{d-1}$ be the $(d-1)$-dimensional real projective space then we can naturally associate the projective actions $\widehat{A}_i : \mathbb{R}P^{d-1} \to \mathbb{R}P^{d-1}$ on the $(d-1)$-dimensional real projective space $\mathbb{R}P^{d-1}$ (for $i = 1, \cdots, k$ and $k \geqslant 2$).

Let $C^\alpha(\mathbb{R}P^{d-1})$ (for $0 < \alpha \leqslant 1$) be the Banach space of α-Holder continuous functions $f : \mathbb{R}P^{d-1} \to \mathbb{R}$ with respect to the norm $\|f\| := \max\{\|f\|_\alpha, \|f\|_\infty\}$ where

$$\|f\|_\alpha = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)^\alpha} \text{ and } \|f\|_\infty = \sup_{x \in \mathbb{R}P^{d-1}} |f(x)|.$$

We are interested in the following operators (which feature in the work of Le Page [3], Guivarc'h-Le Page [2], etc.).

Definition 1.1 (Transfer operator) *For each $t \in \mathbb{R}$ and a probability vector $(p_1, \cdots, p_k)$ we define a linear operator $\mathcal{L}_t : C^\alpha(\mathbb{R}P^{d-1}) \to C^\alpha(\mathbb{R}P^{d-1})$ by*

$$L_t f(x) = \sum_{i=1}^k p_i |\det(D\widehat{A}_i)|^t f(\widehat{A}_i x)$$

where $x \in \mathbb{R}P^{d-1}$ and $f \in C^\alpha(\mathbb{R}P^{d-1})$.

The operator is well defined for any $0 < \alpha \leqslant 1$, but to have useful spctral properties we may require a relatively small choice of α.

We need the following technical hypothesis (when $t = 0$):

Hypothesis 1.2 (DFLY:Doeblin-Fortet, Lasota-Yorke) *There exists $0 < \alpha \leqslant 1$, $0 < \theta < 1$ and $C > 0$ such that*

$$\|L_0^n f\|_\alpha \leqslant C\|f\|_\infty + \theta^n \|f\|_\alpha \tag{1}$$

for all $n \geqslant 1$ and all $f \in C^\alpha(\mathbb{R}P^{d-1})$.

We can assume without loss of generality that $C \geqslant 1$ in (1), say.

It follows from the work of Le Page (and subsequently others) that the operator L_0 satisfies the DFLY condition if the family $\{A_1, \cdots, A_k\}$ is strongly transitive and proximal [3]. A consequence of this is that it follows that the operator L_0 has a simple maximal eigenvalue 1 (corresponding to the eigenspace of constant functions).

The aim of this note it to show the following.

Theorem 1.3 *Given matrices $A_1, \cdots, A_k$ satisfying Hypotheis 1.2 there exists an explict $\epsilon > 0$ such that the operator $\mathcal{L}_t$ has a simple eigenvalue $\lambda(t) \in \mathbb{R}$ of maximal modulus for $t \geqslant -\epsilon$.*

The result is known by work of Guivarc'h and Le Page for $t \geqslant 0$ [**?**]. It remains to find a value of ϵ for which the result holds for $-\epsilon \leqslant t < 0$.

Corollary 1.4 *For an explict $\epsilon > 0$ such that $\lambda(t) \in \mathbb{R}$ for $t \geqslant -\epsilon$.*

When $t = 0$ we have that $\lambda(0) = 1$.

2 Proof of Theorem 1.3

For our purposes it suffices to get bounds on the "spectral gap" of $\mathcal{L}_0$ (i.e., showing that the rest of the spectrum of $\mathcal{L}_0$ is contined in a disk centred at 0 of radius strictly smaller than $\lambda(t)$) and then using an Implicit Fuction Theorem.

Step 1 (The quotient space). We can consider the quotient space $B = C^\alpha(\mathbb{R}P^{d-1})/\mathbb{C}$ where the induced norm is $\|f\| = \|f\|_\alpha + \text{var}(f)$ where

$$\text{var}(f) = \sup_x f(x) - \inf_x f(x).$$

We would like to effectively bound the spectral radius of the quotient operator $\mathcal{L}_0 : B \to B$, since

$$\text{spectrum}\,(\mathcal{L}_0 : B \to B) = \text{spectrum}\left(\mathcal{L}_0 : C^\alpha(\mathbb{R}P^{d-1}) \to C^\alpha(\mathbb{R}P^{d-1})\right) - \{1\}.$$

We observe that on B the DFLY condition reduces to

$$\|\mathcal{L}_0^n f\|_\alpha \leqslant C\text{var}(f) + \theta^n \|f\|_\alpha. \tag{2}$$

Step 2 (A simplifying assumption). Given $f \in B$ with $\|f\| = 1$ then we can assume henceforth that $\text{var}(f) \geqslant \frac{1-\theta}{2C}$ since otherwise $\text{var}(f) \leqslant \frac{1-\theta}{2} < 1$ (since we are assuming $C \geqslant 1$) and the DFLY inequality with $n = 1$ gives

$$\|\mathcal{L}_0 f\|_\alpha \leqslant C\text{var}(f) + \theta\|f\|_\alpha \leqslant \frac{1-\theta}{2} + \theta = \frac{1+\theta}{2} < 1.$$

immediately leading to a uniform bound on the norm of the operator on the quatient operator, and this of the spectral gap of the original operator.

Step 3 (A bound on the $\|\cdot\|_\alpha$-semi-norm in terms of var$(\cdot)$). By replacing f by $\mathcal{L}_0^m f$ (with a value of m yet to be specified) in (2) we have that

$$\begin{aligned} \|\mathcal{L}_0^{n+m} f\|_\alpha &\leqslant C\mathrm{var}(\mathcal{L}_0^m f) + \theta^n \|\mathcal{L}_0^m f\|_\alpha \\ &\leqslant \theta^{n+m}\|f\|_\alpha + \theta^n C \mathrm{var}(f) + C\mathrm{var}(\mathcal{L}_0^m f) \end{aligned} \tag{3}$$

and since we are assuming $\|f\|_\alpha, \mathrm{var}(f) \leqslant 1$ we can bound this last expression in (3) by

$$\underbrace{\theta^{n+m} + \theta^n C}_{\to 0 \text{ as } n \to +\infty} + C\mathrm{var}(\mathcal{L}_0^m f).$$

In particular, providing $n > |\frac{\log(C+1)}{\log\theta}|$ then $\theta^n(C+1) < 1/3$ and we can bound the first part by $\frac{1}{3}$.

Step 4 (Bounds on $\mathrm{var}(\mathcal{L}_0^m f)$**).** We claim that we can choose m such that for all f with

$$\mathrm{var}(f) \geqslant \frac{1-\theta}{2C}, \mathrm{var}(f) \leq 1 \text{ and } \|f\|_\alpha \leqslant 1 \tag{4}$$

we have that the second term in (3) is bounded by $C\mathrm{var}(\mathcal{L}_0^m f) \leqslant \frac{1}{2}$. Then combining these bounds we would have a bound for (3) given by $\|\mathcal{L}_0^{n+m} f\|_\alpha \leqslant \frac{5}{6}$.

To establish the claim we can fix

$$\delta^\alpha < \frac{1-\theta}{8C} (\leqslant \frac{1}{4}\mathrm{var}(\mathcal{L}_0^m f)) \tag{5}$$

and then choose $m \geqslant n$ sufficiently large that for any $x \in \mathbb{R}P^{d-1}$ we have that the set $X = \{\widehat{A}_{i_1}\widehat{A}_{i_2}\cdots\widehat{A}_{i_m}x\}$ is δ-dense. Given any f as above we can choose two points $x_{\min}, x_{\max} \in \mathbb{R}^{d-1}$ such that

$$f(x_{\max}) = f_{\max} := \max_\xi f(\xi) \text{ and } f(x_{\min}) = f_{\min} := \min_\eta f(\eta).$$

Moreover, we can choose points $x'_{\max}, x'_{\min} \in X$ with $d(x'_{\max}, x_{\max}), d(x'_{\min}, x_{\min}) < \delta$. If $p = \min_i p_i > 0$ then we can then bound

$$\begin{aligned} (\mathcal{L}_0^m f)_{\max} &\leqslant (1-p^m) f_{\max} + p^m(f_{\min} + \delta^\alpha) \\ (\mathcal{L}_0^m f)_{\min} &\geqslant (1-p^m) f_{\min} + p^m(f_{\max} - \delta^\alpha) \end{aligned}$$

and taking the difference gives

$$\begin{aligned}
\operatorname{var}(\mathcal{L}_0^m f) &= (\mathcal{L}_0^m f)_{\max} - (\mathcal{L}_0^m f)_{\min} \\
&\leqslant ((1-p^m)f_{\max} + p^m(f_{\min}+\delta^\alpha)) - ((1-p^m)f_{\min} + p^m(f_{\max}-\delta^\alpha)) \\
&= (1-p^m)\operatorname{var}(f) - p^m(\operatorname{var}(f) - 2\delta^\alpha) \\
&= (1-2p^m)\operatorname{var}(f) + 2\delta^\alpha p^m \\
&\leqslant \left((1-2p^m) + \frac{p^m}{4}\right)\operatorname{var}(f) \\
&\leqslant \left(1 - \frac{7}{4}p^m\right)\operatorname{var}(f).
\end{aligned} \tag{6}$$

In particular, using (3) and (6) we can finally deduce that the norm of $\mathcal{L}_0^m : B \to B$ is less than

$$\rho = \max\left\{\left(\frac{1+\theta}{2}\right)^m, \left(1-\frac{7}{4}p^m\right)\right\} < 1. \tag{7}$$

Step 5 (Effective perturbations). We can use the traingle inequality and (7) to deduce that the resolvant for $\mathcal{L}_0 : B \to B$ satisfies

$$\begin{aligned}
\|(I-\mathcal{L}_0)^{-1}\| &\leqslant \|\sum_{n=0}^{\infty} \mathcal{L}_0^n\| \\
&\leqslant (1 + \|L_0\| + \cdots + \|L_0\|^{m-1}) \sum_{n=0}^{\infty} \|\mathcal{L}_0^{mn}\| \\
&\leqslant \frac{(1 + \|L_0\| + \cdots + \|L_0\|^{m-1})}{1-\rho}.
\end{aligned}$$

Step 6 (Implicit Function Theorem). We can now combine the bound above with the following Implicit Function theorem based result.

Lemma 2.1 (Kloeckner) *If* $\|\mathcal{L}_t - \mathcal{L}_0\| \leqslant \frac{1}{6\|(I-L_0)^{-1}\|}$ *then* $\mathcal{L}_t : C^\alpha(\mathbb{R}P^{d-1}) \to C^\alpha(\mathbb{R}P^{d-1})$ *has a simple maximal eigenvalue.*

This completes the proof of the theorem.

References

[1] B. Kloeckner, Effective perturbation theory for simple isolated eigenvalues of linear operators. J. Operator Theory 81 (2019), no. 1, 175–194

[2] Y. Guivarc'h and E. Le Page, Spectral gap properties for linear random walks and Pareto's asymptotics for affine stochastic recursions, Ann. Inst. Henri Poincaré Probab. Stat.52(2016), no.2, 503–574.

[3] E. Le Page, Theoremes limites pour les produits de matrices aleatoires. In: Heyer, H. (eds) Probability Measures on Groups. Lecture Notes in Mathematics, vol 928. Springer, Berlin, 1982.