DIMENSION OF THE FEIGENBAUM ATTRACTOR

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ABSTRACT. We describe a simple method for estimating the dimension of the Feigenbaum attractor for unimodal intervals maps. Although the method will lead to a rigorous bound, we have not attempted to fully validate the value we have computated.

1. Introduction

The original experimental discoveries and conjectures of Feigenbaum [12] and Coullet-Tresser [9] in the mid-1970s have been an important catalyst in the development of the general theory of renormalization in dynamical systems. These important empirical results described the period doubling bifurcations of families of unimodal maps and presented a framework for understanding the underlying mechanism. The majority of these influential conjectures were subsequently proved in 1982 by Lanford, with later advances by Epstein, Lyubich, McMullen, Sullivan and others.

Theorem 1.1 (Feigenbaum Conjectures : Lanford's Theorem)**.** *There exists a C^ω unimodal map $g : [-1,1] \to [-1,1]$ such that:*

(1) $g(0) = 1$;

(2) g is unimodal (i.e., a functions which is first monotone increasing for $x < 0$ and then monotone decreasing for $x > 0$ after passing the critical point at 0 with $g'(0) = 0$ and $g''(0) < 0$); and

(3) g is symmetric (i.e., $g(x) = g(-x)$),

such that g satsifies the Cvitanovic-Feigenbaum Functional equation *$\mathcal{R}(g) = g$ where*

$$\mathcal{R}g(x) = \alpha g \circ g\left(\frac{x}{\alpha}\right) \text{ and } \alpha = -1/g(1). \tag{1.1}$$

In Lanford's original proof he constructed a power series solution

$$g(x) = 1 + \sum_{n=1}^{\infty} a_{2n} x^{2n}. \tag{1.2}$$

The solution was based on the use of fixed point theorems applied to a suitable family of holomorphic functions on an explicit domain. The proof is computer assisted and many of the values for a_m can be computed to high precision (e.g., for $m \leq 170$, say, the a_m are computed to in excess of 150 decimal places in [17]).

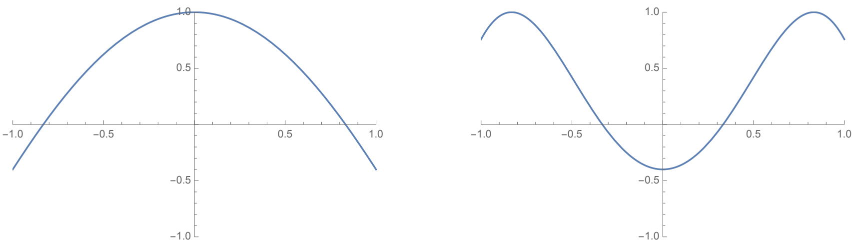

FIGURE 1. (i) A plot of the unimodal g which is a fixed point for the Feigenbaum-Cvitanovic functional equation (1); (ii) A plot of $g \circ g$, the middle portion of which looks like an inverted and rescaled plot of g.

1.1. **Numerical invariants.** Associated to the solution $\mathcal{R}(g) = g$ are a number of interesting numerical values (dubbed "Feigenvalues"). The best known such Feigenvalue is probably

$$\delta = 4.66920160910299067185320382046620161\ldots$$

which played an important role in Feigenbaum's original empirical discoveries for the logistic map $F_m u(x) = 1 - \mu x^2$ converge. More precisely, if $\mu_1 < \mu_2 < \mu_3 < \cdots$ are the parameter values for period doubling then

$$\frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n} \to \delta \text{ as } n \to +\infty.$$

A second Feigenvalue is

$$\alpha = -\frac{1}{g(1)} = -2.502907875095892822283902873218215 7\ldots$$

which can be characterized by the (signed) distances $(d_n)_{n=1}^{\infty}$ of the 2^n-attracting orbit from the origin at those parameter values μ for which 0 is superattracting. More precisely,

$$\frac{d_n}{d_{n+1}} \to \delta \text{ as } n \to +\infty.$$

The constants δ and α are universal in that the same constant arises for any similar family of unimodal maps. Furthermore, these values can be extracted from the function g and thus a detailed knowledge of the coefficients (a_m) in (21) allows them to be computed to high precision.

Finally, we want to consider a third numerical value associated to g. We first need the following definition.

Definition 1.2. *The* Feigenbaum attractor $X = X(g) \subset [-1, 1]$ *is the g-invariant Cantor set which is the closure of the orbit of the critical point, i.e.,* $X = \overline{\cup_{n=0}^{\infty} g^n(0)}$.

The restriction $g : X \to X$ is topologically conjugate to a diadic odometer (or adding machine).

A natural numerical invariant is the Hausdorff dimension $\dim(X) \in (0,1)$ of the attractor X which is a universal value in the following sense:

Theorem 1.3 (Coullet-Tresser geometric rigidity of Cantor sets [9], [22])**.** *Any quadratic map f whose critical point has an orbit with the same combinatorics as g has an attracting Cantor set $X(f)$ with Hausdorff dimension* $\dim X(f) = \dim(X(g))$

Unfortunately, the numerical value of the Hausdorff dimension $\dim(X)$ is notoriously difficult to rigourously estimate despite the map g being known to great accuracy. The purpose of this note is to describe a way to convert the highly accurate estimates for g into better estimates on $\dim(X)$.

1.2. **Estimates on** $\dim(X)$**.** The difficulty is in finding an efficient way to convert a detailed knowledge of the coefficients (a_m) for g into good estimates on $\dim(X)$. Estimates for the Hausdorff dimension $\dim(X)$ of the attractor were given: by Grassberger [14] (to 8 decimal places), Bensimon, Jensen and Kadanoff [2] (to 10 decimal places) and Kovacs. However, the best *non-rigorous* (albeit computationally stable) numerical result to date is due Christiansen et al [8] given to 27 decimal places:

$$\dim(X) = 0.538045143580549911671415 6\ldots \tag{1.3}$$

This estimate is based on the method of cycle expansions which depends on periodic points for g whose exponential growth makes it difficult to make significant improvements using this method. Moreover, it is difficult to get effective error bounds (which is inherent in [15]). Other estimates due to Grassberger [14] are described in ([20], pp.80-81).

There are difficulties in getting rigorous error bounds. However, using the simple on the derivatives of T_1' and T_2, there is a basic bound in ([10], p.141) of $0.5345 < \dim(X) < 0.5544$. More recently, Burbanks, Osbaldstein and Thurlby [7] refined this approach to give bounds

$$0.53705\ldots < \dim(X) < 0.53917\ldots$$

which they claim to be the best rigorous values.

In this note we will use a different method to estimate $\dim(X)$ which appears to have the advantage that it is quite successful in exploiting the very precise knowledge of the values (a_n) to automatically give more accurate rigorous estimates for $\dim(X)$ Our main numerical result is the following. [1]

[1] Subject to transcription errors for the values of a_m and accepting the accuracy of the Mathematica computations

Theorem 1.4. *We have the estimate*

$$\dim(X) = 0.5380451435805499116714155673749862927379649658778569609071 91 \cdots \pm 10^{-50} \tag{1.4}$$

accurate to the number of decimal places presented.

In particular, this confirms the non-rigorous estimate of Christiansen et al in (1.3). The value in (1.4) was computed using Mathematica, and so its veracity depends on the internal checks within that programme. More generally, the potential sources of errors in this calculation are described in the Appendix.

Notwithstanding the merits of the knowledge of the numerical estimate for $\dim(X)$ in Theorem 1.4, .the main purpose of this note is to describe a useful method for converting a knowledge of the coefficients (a_m) in (1.2) into bounds for $\dim(X)$. The method of proof of the bound in Theorem 1.4 is based on a general minmax approach which seems to be more effective in converting the very precise knowledge of expansion for g in [17] into estimates on $\dim(X)$.

2. Iterated function schemes and Dimension

We begin by recalling some useful background material.

2.1. **An iterated function scheme.** There is a well known alternative construction of $X = X(g)$ as the limit set of an iterated function scheme. More precisely, we want to use the following lemma (see [13] and [10], Theorem 8.2.1 for more details).

Lemma 2.1. *The Feigenbaum attractor is the limit set of the iterated function scheme consisting of two contractions $T_1, T_2 : [1/\alpha, 1] \to [1/\alpha, 1]$ given by*

$$T_1(x) = \frac{x}{\alpha} \text{ and } T_2(x) = g^{-1}\left(\frac{x}{\alpha}\right)$$

(i.e., X is the smallest non-empty closed set such that $X = T_1(X) \cup T_2(X)$).

The existence of a non-empty limit set associated to contractions is a consequence of a general result of Hutchinson ([11], Theorem 9.1). In Lemma 2.1 we use g^{-1} to denote the inverse of g restricted to $[g^2(1), 1]$.

2.2. **Dimension.** We want to consider the Hausdorff dimension of X which, in light of its characterization by Lemma 2.1, coincides with the Box dimension, whose simpler definition we briefly recall.

Definition 2.2. *Given $\epsilon > 0$ we let $N(X, \epsilon)$ be the smallest number of open intervals of length $\epsilon > 0$ needed to cover X. We can then write*

$$\dim(X) = \lim_{\epsilon \to 0} -\frac{\log N(X, \epsilon)}{\log \epsilon}.$$

We refer reader to [11] for more details.

Upper and lower bounds on the Hausdorff dimension of the attractor X may be obtained from properties of the iterated function scheme and with rigorous bounds on the renormalisation fixed-point f. Previous rigorous approaches (such as [10] and [7]) tend to be based on approximation of T_1 by (piecewise) affine maps. However, we will describe in the next section a different approach which allows a more effective use of the detailed knowledge of g to get more accurate estimates on $\dim(X)$.

3. Transfer operators

In contrast to previous approaches, we will use what we will refer to as a *minmax method* to estimate the dimension $\dim(X)$ by looking at suitable transfer operators acting on appropriate function spaces. The benefits of this approach will be shown with empirical estimates in the next section.

We can consider a family of bounded linear operators $\mathcal{L}_t$ (for $t \in \mathbb{R}$) on the Banach space $C^1([1/\alpha, 1])$ of continuously differential functions $h : [1/\alpha, 1] \to \mathbb{R}$ with the usual norm $\|h\| = \|h\|_\infty + \|h'\|_\infty$ where

$$\|h\|_\infty = \sup_{1/\alpha \le x \le 1} |h(x)|.$$

The definition of the operators is the following:

Definition 3.1. *The* transfer operators $\mathcal{L}_t : C^1([-1, 1]) \to C^1([-1, 1])$ *(for $t \in \mathbb{R}$) are bounded linear operators defined by*

$$\mathcal{L}_t h(x) = |T_1'(x)|^t h(T_1 x) + |T_2'(x)|^t h(T_2 x)$$

for $h \in C^1([1/\alpha, 1])$ and $x \in [1/\alpha, 1]$.

The spectra $\mathrm{sp}(\mathcal{L}_t) \subset \mathbb{C}$ (for $t \in \mathbb{R}$) of the transfer operator $\mathcal{L}_t$) consists of thse complex numbers z for which $(Iz - \mathcal{L}_t)^{-1} : C^1([1/\alpha, 1]) \to C^1([1/\alpha, 1])$ exists and is a bounded linear operator.

The spectra of the transfer operators have well understood properties which are described in the following lemma.

Lemma 3.2 (after Ruelle)**.** *For each $t \in \mathbb{R}$,*

1. *the operator $\mathcal{L}_t : C^1([-1, 1]) \to C^1([-1, 1])$ has a simple maximal positive eigenvalue $\lambda(t) > 0$, and*
2. *the rest of the spectrum is contained in a strictly smaller disk (i.e., $\exists \rho < \lambda(t)$ such that $\mathrm{sp}(\mathcal{L}_t) \setminus \{\lambda(t)\} \subset \{z \in \mathbb{C} : |z| \le \rho\}$).*

The relevance of these operators to the estimation of the Hausdorff dimension of X is the following simple lemma.

Lemma 3.3. *Let $t_0 < t_1$ be such that the maximal eigenvalues of the operators $\mathcal{L}_{t_0}$ and $\mathcal{L}_{t_1}$, respectively, satisfy*

$$\lambda(t_1) > 1 > \lambda(t_0).$$

Then we have that

$$t_0 < \dim(X) < t_1.$$

Proof. The result follows immediately from the following two facts. The maximal eigenvalue $\lambda(t)$ of the operator $\mathcal{L}_t$ satisfies:

(1) $t \mapsto \lambda(t)$ is a C^ω strictly monotone decreasing function. [2]; and
(2) $\lambda(\dim(X)) = 1$ (by the well known Bowen-Ruelle "pressure formula" [3], [21]).

□

In order to apply the bounds in Lemma 3.3 to obtain practical estimates on $\dim(X)$ we use the following useful criteria.

Lemma 3.4. *Let $\mathcal{L}_t$ (for $t \in \mathbb{R}$) be the operators defined in Definition 3.1.*

(1) Assume that for $t_1 \in \mathbb{R}$ there exists a strictly positive C^1 function $h_1 : [1/\alpha, 1] \to \mathbb{R}^+$ such that

$$\sup_{\frac{1}{\alpha} \le x \le 1} \frac{(\mathcal{L}_{t_1} h_1)(x)}{h_1(x)} < 1 \tag{3.1}$$

then $\lambda(t_1) \le 1$.

(2) Assume that for $t_0 \in \mathbb{R}$ there exists a strictly positive C^1 function $h_0 : [1/\alpha, 1] \to \mathbb{R}$ such that

$$\inf_{\frac{1}{\alpha} \le x \le 1} \frac{(\mathcal{L}_{t_0} h_0)(x)}{h_0(x)} > 1 \tag{3.2}$$

then $\lambda(t_0) \ge 1$.

Proof. The proof is very short, so we include it for the reader's convenience. For part (1) we observe that applying $\mathcal{L}_{t_1}$ repeatedly gives that for any $x \in [1/\alpha, 1]$:

$$\cdots \le \mathcal{L}_{t_1}^n h_1(x) \le \cdots \le \mathcal{L}_{t_1}^2 h_1(x) \le \mathcal{L}_{t_1} g(x) \le h_1(x) \tag{3.3}$$

by virtue of the positivity of the operator and assumption (3.1). However, the spectral properties described in Lemma 3.2 imply that

$$\lim_{n \to +\infty} \|\mathcal{L}_{t_1}^n h_1\|_\infty^{1/n} = \lambda(t_1).$$

Moreover, since $h_1 > 0$ we trivially have $\lim_{n \to +\infty} \|h_1\|_\infty^{1/n} = 1$. The conclusion of part (1) comes from combining these observations with the inequalities in (3.3).

[2]The analyticity comes from $\lambda(t)$ being an isolated eigenvalue and analytic perturbation theoery. The strict monotonicity comes from an explicit form for the first derivative being negative)

For part (2) we similarly observe that by applying repeated $\mathcal{L}_{t_0}$ then the positivity of the operator and assumption (5) we have that for all $x \in [1/\alpha, 1]$

$$h_0(x) \leq \mathcal{L}_{t_0} h_0(x) \leq \mathcal{L}_{t_0}^2 h_0(x) \leq \cdots \leq \mathcal{L}_{t_0}^n h_0(x) \leq \cdots . \tag{3.4}$$

As in part (1), by Lemma 3.2 we see that

$$\lim_{n \to +\infty} \|\mathcal{L}_{t_0}^n h_0\|_\infty^{1/n} = \lambda(t_0)$$

and, again, trivially $\lim_{n \to +\infty} \|h_0\|_\infty^{1/n} = 1$ and so the conclusion comes from (3.4). □

Remark 3.5. *In Lemma 3.4 it would suffice to consider positive test functions $h_0, h_1 : [\alpha, 1] \to \mathbb{R}^+$ which are only continuous. However, in our application in the next section the test functions we construct will actually be polynomial, so there is no real loss of generating in assuming in the lemma that these functions are C^1.*

4. Implementation

We need to convert the theoretical bounds in the previous section into an effective method to numerically estimate $\dim(X)$. In light of the general Lemmas 3.3 and 3.4 we have useful criteria to check whether $\dim(X)$ lies in a given interval $[t_0, t_1]$.

We begin with the following reformulation.

Proposition 4.1 (Criteria for bounds on $\dim(X)$)**.** *Let $t_0 < t_1$. A sufficient condition that*

$$t_0 < \dim(X) < t_1$$

is the existence of positive continuous functions $h_0, h_1 : [1/\alpha, 1] \to \mathbb{R}^+$ such that

$$\inf_{1/\alpha \leq x \leq 1} \frac{\mathcal{L}_{t_0} h_0(x)}{h_0(x)} > 1 \text{ and } \sup_{1/\alpha \leq x \leq 1} \frac{\mathcal{L}_{t_1} h_1(x)}{h_1(x)} < 1. \tag{4.1}$$

To apply Proposition 4.1 we need to find appropriate functions h_0, h_1 and values t_0, t_1. We address these issues in the next subsections.

4.1. Interpolation.

Assume we have candidate values for $t_0 < t_1$. To choose the functions h_0, h_1 we proceed using a simple collocation method.

Definition 4.2. *Fix a natural number $m \geq 2$.*

(1) Consider the Chebychev points

$$x_m = \left(\frac{1 + 1/|\alpha|}{2}\right) \cos\left(\frac{n\pi}{2m}\right) + \left(\frac{1}{\alpha} + 1\right) \text{ for } 0 \leq n \leq m$$

scaled to lie in the interval $[1/\alpha, 1]$.

(2) *Consider the Lagrange polynomials* $\ell_m : [1/\alpha, 1] \to \mathbb{R}$ *defined by*

$$\ell_k(x) = \frac{\prod_{n\neq k}(x - x_m)}{\prod_{n\neq k}(x_k - x_m)} \text{ for } 0 \le n \le m$$

scaled to be defined on the interval $[1/\alpha, 1]$.

In particular, we have the useful property

$$\ell_k(x_n) = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}$$

for $0 \le k, n \le m$. This allows us to proceed as follows.

(i) Assume we are given an interval $[t_0, t_1]$ within which we want to verify that the value $\dim(X)$ can be found.

(ii) For each of the choices $i = 1, 2$ we can associate the $(m + 1) \times (m + 1)$-matrices

$$M_i = \left((\mathcal{L}_{t_i}\ell_k)(x_n)\right)_{k,n=0}^m .$$

(iii) Providing m is sufficiently large, there is a maximal positive eigenvalue $\lambda_i > 0$ and a corresponding right eigenvector $\underline{v}^i = (v_0^i, \cdots, v_m^i)$ has strictly positive entries, i.e., $\lambda_i \underline{v}^i = \underline{v}^i M_i$ and $v_i > 0$ for $0 \le i \le m$.

(iv) Associate the two polynomial functions

$$h_i(x) = \sum_{n=0}^m v_n^i \ell_n(x) \text{ for } x \in [1/\alpha, 1] \quad (\text{for } i = 0, 1).$$

(v) Finally, if

$$\inf_{1/\alpha \le x \le 1} \frac{(\mathcal{L}_{t_0}h_0)(x)}{h_0(x)} > 1 \text{ and } \sup_{1/\alpha \le x \le 1} \frac{(\mathcal{L}_{t_1}h_1)(x)}{h_1(x)} < 1 \tag{4.2}$$

then we can deduce that $t_0 < \dim(X) < t_1$.

Remark 4.3 (Motivation for the construction of h_0 and h_1). *By way of giving some motivation for this construction let us assume that we knew* h_i *is close to the (positive) eigenvector for the maximal eigenvalue* $\lambda(t_i)$ *for* $\mathcal{L}_{t_i}$, *and* h_1, *for* $i = 1, 2$. *Then the left hand sides of the inequalities in (4.2) will be close to* $\lambda(t_0) > 1$ *and* $\lambda(t_1) < 1$, *respectively. However, the functions* h_i *are constructed using eigenvectors* $\underline{v}^i$ *for* $i = 1, 2$. *Thus since* M_i *can be viewed as an approximation to* $\mathcal{L}_{t_i}$ *(in a suitable sense) for* $i = 1, 2$, *perturbation theory supports the idea of the closeness of* h_i *to the eigenvector for* $\mathcal{L}_{t_i}$.

Remark 4.4 (Effectiveness of the algorithm). *If* $t_0 < \dim X < t_1$ *but (4.2) isn't satisfied, then by increasing* m *it can be achieved (as claimed above in (iii)). More precisely, providing* m *is sufficiently large and* $t_0 < \dim(X) < 1$ *(or equivalently* $\lambda(t_0) > 0 > \lambda(t_1)$*) we have that*

(a) for $h_i(x) > 0$ *(*$i = 0, 1$*); and*

(b) the appropriate inequality in (4.2) holds.

This follows by analytic perturbation theory, but for the present concrete setting if suffices to check this conclusion empirically.

4.2. **Bisection method.** In order to obtain candidate values of $t_0 < t_1$ which give upper and lower bounds on $\dim X$ we proceed by a simple bisection method. This generates a sequence of improving bounds $t_0^{(k)} < \dim X < t_1^{(k)}$, for $k \geq 0$ where $|t_0^{(k)} - t_1^{(k)}| \to 0$ as $k \to +\infty$.

We can follow the iterative steps described below.

(1) We begin by choosing obvious upper and lower bounds $t_0^{(1)} < \dim X < t_1^{(1)}$ which are trivially valid. For example, we can let $t_0^{(1)} = 3/10$ and $t_1^{(1)} = 7/10$.
(2) We proceed to construct the sequences $t_0^{(k)}$ and $t_1^{(k)}$ $(k \geq 0)$ inductively. Given $t_0^{(k)} < \dim X < t_1^{(k)}$ (for $k \geq 1$) we can provisionally set
$$s = \frac{1}{2}(t_1^{(k)} - t_0^{(k)})$$
(3) We can associate to $\mathcal{L}_s$ a function h_s using the interpolation method described (ii), (iii) and (iv) above. Typically, we have one of the following two cases.
   (a) If we have
   $$\sup_{1/\alpha \leq x \leq 1} \frac{(\mathcal{L}_s h_s)(x)}{h_s(x)} < 1$$
   then set $t_1^{(k+1)} = s$ and $t_0^{(k+1)} = t_0^{(k)}$.
   (b) If we have
   $$\inf_{1/\alpha \leq x \leq 1} \frac{(\mathcal{L}_s h_s)(x)}{h_s(x)} > 1$$
   then set $t_0^{(k+1)} = s$ and $t_1^{(k+1)} = t_1^{(k)}$.

This produces a sequence of shrinking intervals $[t_0^{(k)}, t_1^{(k)}]$ $(k \geq 0)$ such that:

(1) $t_0^{(k)} \leq \dim(X) \leq t_1^{(k)}]$; and
(2) $|t_1^{(k)} - t_0^{(k)}| = 2^{-k+1}|t_1^{(1)} - t_0^{(1)}|$.

Remark 4.5. *Of course, if neither of the hypotheses in (a) and (b) holds then it will be necessary to increase the value of m, as mentioned in Remark 4.4.*

4.3. **Numerical estimates.** Our stating point is to use the approximation to g given in [17]. In particular, this gives estimates for the first 156 terms in the Chebychev polynomial expansion, each coefficient presented to in excess of 150 decimal places.

However, before addressing the stronger estimate in Theorem 1.4 we will first describe a more modest approximation where choices of h_0 and h_1 can be explicitly presented.

Example 4.6 (Estimate to 2 decimal places)**.** *If we only require a more modest rigorous bound* $0.53 \le \dim(X) \le 0.54$ *then we can apply the criteria for dimension bounds on* X *using simpler explicit polynomials*

(1) *For* $t_0 = 0.53$ *and*
$$h_0(x) = -2933 + 148x$$
we can estimate
$$\min \frac{\mathcal{L}_{t_0} h_0(x)}{h_0(x)} = 1.00023\ldots$$

(2) *For* $t_1 = 0.54$ *and*
$$\begin{aligned} h_1(x) = &-293406 + 15032x - 9718x^2 + 4200x^3 \\ &- 2391x^4 + 1336x^5 - 773x^6 + 432x^7 \end{aligned}$$
we can estimate
$$\max \frac{\mathcal{L}_{t_1} h_1(x)}{h_1(x)} = 0.999964\ldots$$

In particular, we can deduce from Proposition 4.1 that $0.53 \le \dim(X) \le 0.54$.

We can now turn to the estimate in Theorem 1.4.

Example 4.7 (Estimate to 50 decimal places)**.** *Working to* 100 *decimal places we can choose* $m = 300$ *and implement the approach described in the previous section. Starting from* $t_0^{(1)} = 3/10$ *and* $t_1^{(1)} = 7/10$ *we can apply the bisection method until we get suitable values. For*
$$\begin{aligned} t_0 =&0.538045143580549911671415567374986292737 \\ &964965877856960907191\cdots \end{aligned}$$
we can associate a suitable polynomials h_0 *and then estimate*
$$\begin{aligned} \min \frac{\mathcal{L}_{t_0} h_0(x)}{h_0(x)} =&1.0000000000000000000000000000000000000000000000 \\ &00000000000002043024411083008731521919696166\ldots \end{aligned}$$
For
$$\begin{aligned} t_1 =&0.538045143580549911671415567374986292737 \\ &964965877856960907193 \end{aligned}$$
we can associate a suitable polynomials h_0 *and then estimate*
$$\begin{aligned} \max \frac{\mathcal{L}_{t_0} h_0(x)}{h_0(x)} =&0.9999999999999999999999999999999999999999999999 \\ &999999999999995657268412268496192385300641278\ldots \end{aligned}$$

In particular, we can deduce from Proposition 4.1 that $t_0 \leq \dim(X) \leq t_1$. *This gives an estimate to approximately* 60 *places which we have more conservatively presented in Theorem 4.4.*

Remark 4.8. *The above calculations were carried out using Mathematica using its internal routines and error estimates. However, for more complete confidence in the estimate it would probably be best to use an open source programme.*

5. Final comments

Since the aim of this note is to explain one way in which precise estimates on g can be converted into good bounds on $\dim(X)$ it is useful to review the potential sources of loss of accuracy and the restrictions on this approach.

5.1. **Approximation of the transfer operators.** The contractions $T_1, T_2 : [1/\alpha, 1] \to [1/\alpha, 1]$ used in the definition of the transfer operators $\mathcal{L}_t$ are defined in terms of g (and, consequently, the value α). If $g(z)$ has a uniform polynomial approximation by a polynomial $\tilde{g}(z)$ on the disk

$$[1/\alpha, 1] \subset D = \{z \in \mathbb{C} : |z| < \sqrt{8}\}$$

satisfying $\overline{T_1(D)}, \overline{T_2(D)} \subset D$ (see [16]) then the corresponding contractions T_i have comparable approximations $\widetilde{T}_i$ $(i = 1, 2)$ on $[1/\alpha, 1]$. The same conclusion applies to their derivatives by virtue of Cauchy's theorem. The operator $\widetilde{\mathcal{L}}_t$ defined in terms of $\widetilde{T}_i$ $(i = 1, 2)$ can be applied to the approximating polynomial $h : [\alpha, 1] \to \mathbb{R}$ (of degree m, say) and has a uniform bound

$$\begin{aligned} &\|\widetilde{\mathcal{L}}_t h - \mathcal{L}_t h\|_\infty \\ &\leq \sum_{i=1}^{2} \|(T_i')^t - (S_i')^t\|_\infty \|h \circ T_i\|_\infty + \|(T_i')^t\|_\infty \|h \circ T_i - h \circ S_i\|_\infty \end{aligned} \tag{5.1}$$

and by the mean value theorem:

$$\|h \circ T_i - h \circ S_i\|_\infty \leq \|h'\|_\infty \|T_i - S_i\|_\infty.$$

Let π be the projection onto polynomials of degree m then one can take h_0 to be the maximal eigenfunction of $\mathcal{L}\pi$ (and h to be the maximal eigenfunction for $\mathcal{L}_t$) then by [1] there exists $C > 0$ and $0 < \theta < 1$ such that

$$\|\mathcal{L}_t - \mathcal{L}_t \pi\|_\infty \leq C \|\mathcal{L}_t\| \theta^m$$

on D for $m \geq 1$. In particular, $\|h - h_0\| \to 0$ on D as $m \to +\infty$ and bound $\|h'\|_\infty$ on $[1/\alpha, 1]$ by Cauchy's theorem. This gives boundes on (5.1).

5.2. **Checking the minimax bounds.** In §4 we need to effectively estimate the supremum or infimum of functions $\frac{\mathcal{L}_t h}{h}(x)$ over $[1/\alpha, 1]$. To this end the following observation is helpful.

Lemma 5.1. *If $h : D \to \mathbb{R}$ is the collocation function associated to m then*

$$\sup_{1/\alpha \leq x \leq 1} \left| \left(\frac{\mathcal{L}_t h}{h} \right)' (x) \right| \to 0 \text{ as } m \to +\infty.$$

Proof. By the quotient rule

$$\left(\frac{\mathcal{L}_t h}{h} \right)' (z) = \frac{(\mathcal{L}_t h)'(z) h(z) - (\mathcal{L}_t h)(z) h'(z)}{h(z)^2} \text{ for } z \in D.$$

As above, by [1] we have that $\|\mathcal{L}_t - \mathcal{L}_t \pi\|_\infty \to 0$ on D as $m \to +\infty$ which gives that $\|h - h_0\| \to 0$. Thus $\sup_{\alpha \leq x \leq 1} |h'(x) - h_0'(x)| \to 0$ by Cauchy's theorem and similarly

$$\sup_{\alpha \leq x \leq 1} |(\mathcal{L}_t h)'(x) - (\mathcal{L}_t h_0)'(x)| \to 0.$$

Finally, since $\inf_{x \in I} h_m(x) \to \inf_{x \in I} h(x) > 0$ as $n \to +\infty$, combining these results gives the conclusion. □

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