Regularity of homological entropy for winding cycles for Anosov and geodesic flows

April 13, 2026

1 Introduction

In 1957 Schwartzman [17] introduced the fundamental concept of an asymptotic winding cycle, which has proved to be a useful bridge between the ergodic theory of continuous flows and algebraic topology. In particular, they describe how the orbits of flows $\phi_t : M \to M$ on a compact manifold evolve in the homology. [1].

In this note we will consider the *homological entropy* for transitive Anosov flows given by the supremum of the entropies for ϕ-invariant measures which share a given Schwartzman asymptotic winding cycle. We will describe the dependence of the homological entropy as the Anosov flow changes.

1.1 The winding cycle and homological entropy

Let $\phi_t : M \to M$ be a C^1 flow on a compact manifold. For any ϕ-invariant probability measures μ one can define a linear map $\Phi_\mu : H^1(M, \mathbb{R}) \to \mathbb{R}$ on the real (de Rham) cohomology by

$$\Phi_\mu([\mu]) = \int \omega(X) d\mu$$

where ω is a closed one form on M and X is the vector field for the flow. By duality we can identify $\Phi_\mu \in H_1(M, \mathbb{R})$ as an element of homology.

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[1] A nice account appears in Schwartzman's scholarpedia article [18] (the only minor blemish being his misspelling of MP's name)

Henceforth we will assume that $\phi_t : M \to M$ is a C^{k+1} $(k \geq 0)$ transitive Anosov flow. The closed orbits are then dense in M and give rise to particularly simple examples.

Example 1.1 (closed orbits)**.** *For a ϕ-invariant probability measure μ^τ supported on a closed orbit τ with homology class $[\tau] \in H_1(M, \mathbb{Z})$ and least period $\lambda(\tau)$ we can write*

$$\Phi^\phi_{\mu^\tau} = \frac{[\tau]}{\lambda(\tau)} \in H_1(M, \mathbb{R}).$$

This naturally leads to the following definition.

Definition 1.2. *We denote by*

$$\mathcal{B}(\phi) = \{\Phi^\phi_\mu : \mu \in \mathcal{M}_\phi\} \subset H_1(M, \mathbb{R})$$

the image of the space $\mathcal{M}_\phi$ of ϕ-invariant probability measures.

Since $\mathcal{M}_\phi$ is well known to be convex and compact in the weak star topology, its image $\mathcal{B}(\phi)$ inherits these properties. [2]

1.2 Homological entropy

We are particularly interested in studying the following quantity, whose definition is in terms of the entropy of measures subject to homological restrictions.

Definition 1.3. *We define* the homological entropy $h(\phi, \cdot) : \mathrm{int}(\mathcal{B}(\phi)) \to \mathbb{R}$ *by*

$$h(\phi, \alpha) := \sup \left\{ h_\mu(\phi) \,:\, \mu \in \mathcal{M}_\phi \text{ with } \Phi^\phi_\mu = \alpha \right\} \qquad (1.1).$$

where $h_\mu(\phi)$ denotes the entropy of the (time one) flow ϕ with respect to the measure μ.

Recall that by the usual variational principle for the entropy of the flow that we have the trivial upper bound

$$h(\phi, \alpha) \leq h_{top}(\phi) := \sup \{h_\mu(\phi) \,:\, \mu \in \mathcal{M}_\phi\} \qquad (1.2)$$

(see [20] for more details on the variational principle).

It will be useful in the sequel to have a more convenient definition of the homological entropy. To this end, we note the following.

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[2] Massart studied the shape of the sets $\mathcal{M}_\phi$ for geodesic flows [13].

Lemma 1.4. *Let $\phi_t : M \to M$ be a C^1 Anosov flow. For $\alpha \in \mathrm{int}(\mathcal{B})(\phi)$ we have that*

$$h(\phi, \alpha) = \lim_{\delta \to 0} \lim_{T \to +\infty} \frac{1}{T} \log \#\{\tau : \ell(\tau) \leq T, \|[\tau] - \alpha \ell(\tau)\| \leq \delta \ell(\tau)\}.$$

This lemma follows from a well known local limit theorem due to Kifer (see §2.4 for more details of the proof). In fact, Babillot and Ledrappier proved a much stronger asymptotic formula under the mild additional assumption that the flow was non-cyclic ([1], Theorem 1.2). However, the simpler statement in Lemma 1.4 suffices for our purposes.

The value $h(\phi, \alpha)$ plays an important role in various counting problems associated to closed orbits for Anosov flows [10], [1]. More recently, it appears in work related to the abelian Livsic theorem [5].

1.3 The main results

For a given transitive Anosov flow $\phi_t : M \to M$, Babillot and Ledrappier described the dependence of the homological entropy $h(\phi, \alpha)$ on $\alpha \in \mathrm{int}(\mathcal{B}(\phi))$ [1]. We now want to understand the dependence of $h(\phi, \alpha)$ on the Anosov flow ϕ. In this context, we note that if $\alpha \in \mathrm{int}(\mathcal{B}(\phi))$ then for a sufficiently C^1 close flow ψ we also have that $\alpha \in \mathrm{int}(\mathcal{B}(\psi))$ (see §2.2 for more details). Our first result gives the dependence of $h(\phi, \alpha)$ on the flow ϕ.

Theorem 1.5. *Let $\phi_t : M \to M$ be a C^1 $(\beta > 0)$Anosov flow and $\alpha \in \mathrm{int}(\mathcal{B}(\phi))$. Then the dependence of $h(\phi, \alpha)$ on the flow ϕ is differentiable.*

This result is analogous to the classic work of Katok, Knieper and Weiss on differentiability of the topological entropy $h_{top}(\phi)$ as a function of a C^1 Anosov flow [7]. However, perhaps of particular interest is the formula for the derivative (which will appear in §3).

When $\phi_t : M \to M$ is a C^{k+1} Anosov flow with $k \geq 1$ then there is a correspondingly more regular dependence of $h(\phi, \alpha)$ on the flow. [3]

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[3] To interpret the regularity of $h(\phi, \alpha)$ as a function of ϕ, we associate to the C^{k+1} flow the corresponding C^k vector field X on M. The space of C^k vector fields is a Banach manifold leading to a natural interpretation of the C^k dependence of $h(\phi, \alpha)$ under C^{k+1} perturbations of C^{k+1} flows ϕ for $0 \leq k \leq +\infty$ and $k = \infty$. However, for easy of exposition we will consider the simpler formulation of a parameterized family $(\epsilon, \epsilon) \ni \lambda \mapsto \phi^{(\lambda)}$ as was done in [8].

Theorem 1.6. *For $k \geq 1$, let $\phi_t : M \to M$ be a C^{k+1} Anosov flow and $\alpha \in \operatorname{int}(\mathcal{B}(\phi))$.*

1. *For $k \geq 1$ the dependence of $h(\phi, \alpha)$ on the flow ϕ is C^{k-1}.*
2. *For $k = \infty$ the dependence of $h(\phi, \alpha)$ on the flow ϕ is C^{∞}.*

Clearly, Part 2 follows immediately from Part 1.
Finally, there is a result for real analytic flows.

Theorem 1.7. *Let ϕ be a real analytic Anosov flow and $\alpha \in \operatorname{int}(\mathcal{B}(\phi))$ then the dependence of $h(\phi, \alpha)$ on the flow ϕ is C^{ω}.*

The proof of Theorem 1.5 is based on presenting an explicit formula for the derivative, which follows the lines of the argument in [7] for the differentiability of the topological entropy $h_{top}(\phi)$ for the flow $\phi_t : M \to M$. In contrast, the proof of Theorem 1.6 follows the ideas in [8] in making use of structural stability of the Anosov flow. The drop in smoothness in Part 1 is then implicit in this method of proof. The proof of Theorem 1.7 uses yet another approach based on zeta functions.

2 Anosov flows and winding cycles

We begin with some standard definitions and results. We first give the definition and describe some basic properties of the Anosov flow.

2.1 Anosov flows

Let $\phi_t : M \to M$ be a C^1 flow on a compact manifold.

Definition 2.1. *We say that $\phi_t : M \to M$ is a* transitive Anosov flow *if*

1. *There exists a continuous splitting $TM = E^0 \oplus E^s \oplus E^u$ into $D\phi_t$-invariant subbundles such that*
   - (a) *E^0 is a one dimensional bundle tangent to the flow direction;*
   - (b) *There exists $C > 0$ and $\lambda > 0$ such that $\|D\phi_t | E^s\| \leq Ce^{-\lambda t}$ and $\|D\phi_{-t} | E^u\| \leq Ce^{-\lambda t}$ for $t \geq 0$.*
2. *The flow ϕ_t is transitive (i.e., there exists a dense orbit).*

It is well known that the space of C^{k+1} Anosov flows (for $k \geq 1$) is open in the C^1 norm. Moreover, Anosov flows are structural stable in the following sense (cf. [7]).

Proposition 2.2 (Structural stability). *Given a C^1 Anosov flow $\phi^0 : M \to M$ any nearby C^1 (Anosov) flow $\phi : M \to M$ is conjugate to ϕ^0 up to a reprarameterization, i.e., there exists $\beta > 0$ and*

1. *a Hölder continuous change of velocity function $R = R^\phi \in C^\beta(M, \mathbb{R}^+)$, and*
2. *a Hölder continuous homeomorphism $\Theta = \Theta^\phi \in C^\beta(M, M)$ and Hölder continuous reparameterizations $\overline{t} : \mathbb{R} \times M \to \mathbb{R}$ and $\overline{s} : \mathbb{R} \times M \to \mathbb{R}$*

such that for $x \in M$ we have that $\Theta \circ \phi^0_{\overline{t}(s,x)}(x) = \phi_s \circ \Theta(x)$ with

$$\overline{t}(s, x) = \int_0^s R(\phi^0_u x)du \text{ (for } s \geq 0)$$

and $\Theta \circ \phi^0_t(x) = \phi_{\overline{s}(t,x)} \circ \Theta(x)$ with

$$\overline{s}(t, x) = \int_0^t (1/R(\phi^0_u x))du \text{ (for } t \geq 0).$$

In addition, the value $\beta > 0$ in Proposition 2.2 can be taken to be the same in a small neighbourhood of ϕ^0.

Definition 2.3. *The space $C^\beta(M)$ is a Banach space with respect to the natural norm*

$$\|F\| := \sup_{x \neq y} \frac{|F(x) - F(y)|}{d(x, y)^\alpha} + \sup_x |F(x)|, \text{ for } F \in C^\beta(M)$$

and $C^\beta(M, M)$ is a Banach manifold modelled on these Banach spaces (cf. [7]).

The following result is particularly useful in the proof of Theorem 1.6.

Lemma 2.4 (de la Llave, Marco and Moriyan [11]). *Let ϕ be a C^{k+1} Anosov flow ($k \geq 0$).*

1. *For $k = 0$, the function $R = R^\phi \in C^\beta(M)$ and the homeomorphism $\Theta = \Theta^\phi \in C^\beta(M, M)$ have a continuous dependence on the flow ϕ.*

2. *For $k \geq 1$, the function $R = R^\phi \in C^\beta(M)$ and the homeomorphism $\Theta^\phi \in C^\beta(M, M)$ have a C^{k-1} dependence on the flow ϕ.*

In particular, note that in Part 2 we lose two degrees of differentiability.

Remark 2.5. *For convenience, we have assumed that M is a C^∞ manifold. More generally, if we assume M is a C^r manifold then we require $r \geq k+1$ in order for the Banach manfiold $C^\beta(M, M)$ and space $C^\beta(M)$ to have C^{r-2} charts.*

2.2 The asymptotic winding cycle

Let $H^1(M, \mathbb{R})$ denote the first real cohomology group of the compact manifold M. Using de Rham cohomology (and Hodge theory) we can choose a basis of (harmonic) 1-forms $\omega_1, \cdots, \omega_b$ on M, where $b = \dim H^1(M, \mathbb{R})$ denotes the first Betti number. We can then identify $\mathbb{R}^b$ with $H^1(M, \mathbb{R})$ using the identification

$$\xi = (\xi_1, \cdots, \xi_b) \mapsto \omega_\xi := \sum_{i=1}^{b} \xi_i \omega_i.$$

Let the C^{k+1} Anosov flow $\phi_t : M \to M$ have an associated C^k vector field $X = X(\phi)$. Consider a ϕ-invariant probability measure μ. Given an element $[\omega] \in H^1(M, \mathbb{R})$ in the de Rham cohomology we can associate to the closed form ω a smooth function $\omega(X) : M \to \mathbb{R}$. We can now formally define the winding cycle briefly mentioned in the introduction.

Definition 2.6. *Given any ϕ-invariant probability measure μ we can associate its* Schwartzman asymptotic winding cycle $\Phi^\phi_\mu : H^1(M, \mathbb{R}) \to \mathbb{R}$ *by the linear functional*

$$\Phi^\phi_\mu([\omega]) = \int \omega(X) d\mu.$$

By Poincaré-de Rham duality Φ^ϕ_μ can be identified with an element of $H_1(M, \mathbb{R})$.

Remark 2.7. *There is a more intuitive definition of the asymptotic cycle associated to a flow in the case of a ergodic measure μ, which we briefly recall. For a typical point $x \in M$ (with respect to μ) and large $T > 0$ we can consider the (long) orbit segment curve $\phi_{[0,T]}(x) = \{\phi_t(x) : 0 \leq t \leq T\} \subset M$. We then construct a closed curve $\gamma_{x,T}$ by concatenating the orbit segment with a (much shorter) curve from x to $\phi_T(x)$ of length at most diam(M),* *say. Finally, let $[\gamma_{x,T}] \in H_1(M,\mathbb{Z})$ denote the associated element in the first homology group and then we can define*

$$\Phi_\mu^\phi = \lim_{n\to+\infty} \frac{1}{T}[\gamma_{x,T}] \in H_1(M,\mathbb{R}).$$

The limit exists for almost all (μ) points by the Birkhoff ergodic theorem and the limit is independent of the closing curve.

Since we want to consider the behaviour of winding cycles as the Anosov flow ϕ changes the following elementary lemma on the dependence of $\mathcal{B}(\phi)$ on the flow is useful in giving context.

Lemma 2.8. *The sets $\mathcal{B}(\phi) \subset H_1(M,\mathbb{R})$ have a continuous dependence on ϕ with respect to the C^1 metric on the flow and the Hausdorff metric on sets.*

Proof. To see this we need only observe that the measures supported on closed ϕ-orbits τ are well known to be weak star dense in $\mathcal{M}_\phi$ and thus the associated values $\Phi_{\mu^\tau}^\phi$ are dense in $\mathcal{B}(\phi)$. However, as one easily sees from Example 1.1 the image $\Phi_{\mu^\tau}^\phi = [\tau]/\lambda(\tau)$ changes continuously with the flow ϕ since the homology class remains fixed but the lengths vary continuously in a uniform sense given by structural stability of the flow (see Proposition 2.2). □

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2.3 Legendre pairs

Given a C^1 Anosov flow $\phi_t : M \to M$ let $X = X(\phi)$ be the associated vector field on M. Our aim is to study the dependence of $h(\phi,\alpha)$ on the flow $\phi_t : M \to M$, but it is more convenient to proceed indirectly by instead introducing a quantity which has an explicit dependence on a second quantity $\xi \in \mathbb{R}^b$, which is closely related to $\alpha \in \mathrm{int}(\mathcal{B}(\phi))$.

Definition 2.9. *Given a C^1 Anosov flow $\phi_t : M \to M$ with vector field X, we define* the cohomological pressure $P(\phi,\cdot) : \mathbb{R}^b \to \mathbb{R}$ *by*

$$P(\phi,\xi) = \sup\left\{ h_\mu(\phi) + \int \omega_\xi(X)(x)d\mu(x) \, : \, \mu = \phi\text{-invariant probability}\right\}$$

where $\xi \in \mathbb{R}^b \cong H^1(M,\mathbb{R})$.

If $\phi_t : M \to M$ is a $C^{1+\alpha}$ Anosov flow there will be a unique ϕ-invariant probability measure μ^ξ realising this supremum. [4]

The cohomological pressure is merely a restriction of the usual pressure function $P(\phi, \cdot) : C^\beta(M) \to \mathbb{R}$ to the special class of functions of the form $\omega_\xi(X)(x)$. In particular, when the Anosov flow $\phi_t : M \to M$ is $C^{1+\beta}$ such functions are Hölder continuous the uniqueness of μ^ξ is well-known [20].

Since the pressure function $P(\phi, \cdot) : C^\beta(M) \to \mathbb{R}$ is real analytic, for a given Anosov flow $\phi_t : M \to M$ the cohomological pressure function $\xi \mapsto P(\phi, \xi)$ is also known to be real analytic.

Example 2.10. *In the special case that $\xi = 0$ then by the usual variational principle for entropy we have that $P(\phi, 0) = h_{top}(\phi)$, i.e. the topological entropy for the Anosov flow [20].*

For a fixed $C^{1+\beta}$ Anosov flow ϕ, the pressure $P(\phi, \xi)$ and the entropy $h(\phi, \alpha)$ are related by the following simple and elegant observation in [1].[5]

Lemma 2.11 (Babillot-Ledrappier)**.** *For a given $C^{1+\beta}$ Anosov flow $\phi_t : M \to M$ we have the following.*

1. *The map $\mathbb{R}^b \ni \xi \mapsto \alpha_\xi^\phi := \nabla P(\phi, \xi) \in \mathrm{int}(\mathcal{B}(\phi)) \subset \mathbb{R}^b$ is a smooth diffeomorphism.*
2. *Both $P(\phi, \cdot) : \mathbb{R}^b \to \mathbb{R}$ and $-h(\phi, \cdot) : \mathrm{int}(\mathcal{B}(\phi)) \to \mathbb{R}$. are C^∞ convex functions and are related as Legendre conjugates, [6] i.e.,*

$$P(\phi, \xi) = \sup_{\alpha \in \mathcal{B}} \{h(\phi, \alpha) + \langle \alpha, \xi \rangle\} \text{ and } -h(\phi, \alpha) = \sup_{\xi \in \mathbb{R}^d} \{\langle \alpha, \xi \rangle - P(\phi, \xi)\}.$$

Of course, since by Part 1 of Lemma 2.11 we have that

$$\mathbb{R}^b \ni \xi \to \alpha_\xi^\phi \in \mathrm{int}(\mathcal{B}(\phi))$$

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[4] This makes $\omega_\xi(X)$ Hölder continuous which is an element in the proof of uniqueness.

[5] The argument in [1] appears to depend on the smoothness of the pressure which, in turn, seems to require that the flow be $C^{1+\beta}$

[6] There is a simple geometric interpretation. If $\alpha = (\alpha_1, \ldots, \alpha_b) \in \mathbb{R}^b$ then one can consider the family of hyperplanes in $\mathbb{R}^{b+1}$ of the form $L_t := \{(y_1, \ldots, y_b, y_{b+1}) : \sum_{i=1}^b y_i \alpha_i - t y_{b+1} = 0\}$. Then $-h(\phi, \alpha) = t$ is the value such that L_t intersects the graph $\{(\xi, P(\phi, \xi)) : \xi \in \mathbb{R}^b\}$ in a single point. This helps how $h(\phi, \cdot)$ inherits its C^ω dependence from $P(\phi, \cdot)$.

is a smooth diffeomeorphism we can deduce the same for its inverse, which we denote by $\mathrm{int}(\mathcal{B}(\phi)) \ni \alpha \mapsto \xi_\alpha^\phi \in \mathbb{R}^b$.

We will now describe the dependence of these quantities on the Anosov flow $\phi_t : M \to M$. We first recall a result on the regularity of the cohomological pressure which follows from a result due to Contreras. [7]

Lemma 2.12 (after Contreras). *For $k \geq 2$, the map $P(\phi, \xi)$ has a C^{k-1} dependence on the C^{k+1} Anosov flow $\phi_t : M \to M$, i.e., if $\phi_t^\lambda : M \to M$, for $\lambda \in (-\epsilon, \epsilon)$, is a C^{k+1} family of Anosov flows then $(-\epsilon, \epsilon) \ni \lambda \mapsto P(\phi^{(\lambda)}, \xi)$ is C^{k-1}.*

Proof. By a result of Contreras ([3], Theorem C (a)) the pressure function $P(\phi, \cdot) : C^1(M) \to \mathbb{R}$ has a C^{k-1} dependence on ϕ, in addition to its usual analytic dependence on functions. Moreover, the associated vector field $X(\phi)$ has a C^{k+1} dependence on ϕ (albeit in a Banach manifold with one degree less of differentiability because of considering the vector field rather than the flow). But by Definition 2.9 we know that $P(\phi, \xi) = P(\phi, \omega_\xi(X(\phi))$ and the result follows. $\square$

Lemma 2.13. *Assume that $\phi_t^\lambda : M \to M$, for $\lambda \in (-\epsilon, \epsilon)$, is a C^{k+1} family of Anosov flows.*

1. *The diffeomorphism $\xi \mapsto \alpha_\xi^{\phi^\lambda}$ has a C^{k-1} dependence on $\lambda \in (-\epsilon, \epsilon)$.*
2. *Similarly, the diffeomorphism $\alpha \mapsto \xi_\alpha^{\phi^\lambda} \in \mathbb{R}^b$ has a C^{k-1} dependence on $\lambda \in (-\epsilon, \epsilon)$.*

Proof. The C^{k-1} dependence of the pressure function, and thus ∇P, follow from the result of Contreras ([3], Theorem C (a)). In particular, $\alpha_\xi^{\phi^\lambda} = \nabla P(\phi^\lambda, \xi)$ has a C^{k-1} dependence on λ. Since $\xi_\alpha^{\phi^\lambda}$ represents the inverse of $\alpha_\xi^{\phi^\lambda}$ it necessarily inherits the C^{k-1} dependence on λ. $\square$

Finally, we have the following alternative characterization of the invariant measure μ^ξ from Definition 2.9 in terms of equation (1.1).

Lemma 2.14. *Let $\phi_t : M \to M$ be a $C^{1+\beta}$ transitive Anosov flow. Let $\alpha \in \mathrm{int}(\mathcal{B}(\phi))$.*

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[7] The proof uses symbolic dynamics to model the flow and then reduces to the implicit characterization of $P(\phi, \xi)$ in terms of the analytic pressure function on Hölder continuous functions, which in turn have a C^{k-1} dependence by virtue of Lemma 2.4

1. *There is a unique ϕ-invariant probability realising the maximum in (1.1) which we denote by $\mu_\alpha = \mu_\alpha^\phi$ (i.e., $h(\phi, \alpha) = h_{\mu_\alpha}(\phi)$ and $\Phi_{\mu_\alpha}^\phi = \alpha$).*

2. *For the associated value $\xi = \xi_\alpha^\phi \in \mathbb{R}^b$ given in part 1 of Lemma 2.11 we have that $\mu_\alpha = \mu^\xi$, i.e., the unique (equilibrium) measure for the function $\omega_\xi(X) \in C^\infty(M)$ with*

$$\begin{aligned} P(\phi, \xi) :=& \sup \left\{ h(\phi, \mu) + \int \omega_\xi(X) d\mu \, : \, \mu = \ \phi\text{-invariant probability} \right\} \\ &= h(\phi, \mu^\xi) + \int \omega_\xi(X) d\mu^\xi. \end{aligned} \tag{2.1}$$

Proof. This is described in ([1], pp.19-20). The uniqueness of μ^ξ can be understood at the symbolic level for C^1 flows. The flow $\phi_t : M \to M$ is modelled by the suspension flow $\psi_t : \Sigma_A^r \to \Sigma_A^r$ associated to a Hölder continuous function $r : \Sigma_A \to \mathbb{R}^+$ over a topologically mixing subshift of finite type $\sigma : \Sigma_A \to \Sigma_A$. The measure μ^ξ necessarily corresponds to the measure $m^\xi := d\nu^\xi \times dt / \int r d\nu^\xi$ where ν^ξ is the unique equilibrium state for the Hölder continuous function

$$\int_0^{r(\cdot)} \omega_\xi(X)(\sigma_t^r \cdot) dt \circ \pi - P(\phi, \xi) r(\cdot) : \Sigma \to \mathbb{R}$$

where $\pi : \Sigma^r \to M$ is the semi-conjugacy with $\pi \circ \psi_t = \phi \circ \pi$, i.e., $m^\xi = \mu^\xi \circ \pi^{-1}$. □

The following corollary will be useful later.

Corollary 2.15. *If $(\alpha_n)_{n=1}^\infty$ are contained in* $\operatorname{int}(\mathcal{B}(\phi))$ *and $\alpha_n \to \alpha \in$* $\operatorname{int}(\mathcal{B}(\phi))$ *then $h(\phi, \alpha_n) \to h(\phi, \alpha)$*

Proof. By Part 1 of Lemma 2.11 (and the comments after the lemma) we can associate to α a value $\xi_\alpha^\phi \in \mathbb{R}^b$ and to $(\alpha_n)_{n=1}^\infty$ to a sequence $(\xi_{\alpha_n}^\phi)_{n=1}^\infty$ in $\mathbb{R}^b$ such that $\xi_{\alpha_n}^\phi \to \xi_\alpha^\phi$ as $n \to +\infty$. By the Fenchel-Young (in)equality we can write

$$P(\phi, \xi_{\alpha_n}^\phi) - h(\phi, \alpha_n^\phi) = \langle \xi_{\alpha_n}^\phi, \alpha_n \rangle \text{ and } P(\phi, \xi_\alpha^\phi) - h(\phi, \alpha^\phi) \ = \langle \xi_\alpha^\phi, \alpha \rangle \tag{2.2}$$

By the continuity of pressure we can see $\lim_{n\to+\infty} P(\phi, \xi_{\alpha_n}^\phi) = P(\phi, \xi_\alpha^\phi)$ and since $\lim_{n\to+\infty} \langle \xi_{\alpha_n}^\phi, \alpha_n \rangle = \langle \xi_\alpha^\phi, \alpha \rangle$ the result follows from (2.2). □

2.4 Proof of Lemma 1.4

We now describe how Lemma 1.4 follows from a result of Kifer.

Proof. Let $U = \{\mu \in \mathcal{M}(\phi) : \|\Phi_\mu - \alpha\| < \delta\}$ and $K = \{\mu \in \mathcal{M}(\phi) : \|\Phi_\mu - \alpha\| \leq \delta\}$ then $U \subset K$ and we can apply a large deviation result of Kifer ([9], Theorem 2.1) to deduce that

$$\limsup_{T\to+\infty} \frac{1}{T} \log \#\{\tau : \ell(\tau) \leq T, \mu_\tau \in K\} \leq \sup_{m\in K} h(\phi, m).$$

and

$$\liminf_{T\to+\infty} \frac{1}{T} \log \#\{\tau : \ell(\tau) \leq T, \mu_\tau \in U\} \geq \sup_{m\in U} h(\phi, m).$$

Thus

$$\begin{aligned}\sup_{\|\beta-\alpha\|\leq\delta} h(\phi, \mu_\beta) &\geq \limsup_{T\to+\infty} \frac{1}{T} \log \#\{\tau : \ell(\tau) \leq T, \|[\tau] - \alpha\ell(\tau)\| \leq \delta\ell(\tau)\} \\ &\geq \liminf_{T\to+\infty} \frac{1}{T} \log \#\{\tau : \ell(\tau) \leq T, \|[\tau] - \alpha\ell(\tau)\| < \delta\ell(\tau)\} \\ &\geq \sup_{\|\beta-\alpha\|<\delta} h(\phi, \mu_\beta).\end{aligned}$$

We can also write the upper and lower bounds

$$\sup_{m\in K} h(m) = \sup_{\|\beta-\alpha\|\leq\delta} h(\mu_\beta) \text{ and } \sup_{m\in U} h(m) = \sup_{\|\beta-\alpha\|<\delta} h(\mu_\beta)$$

and observe that both of these tend to $h(\mu_\alpha) = h(\phi, \alpha)$ as $\delta \to 0$ by Corollary 2.15. ☐

3 Proof of Theorem 1.5

We can now give the formula for the derivative of $h(\phi, \alpha)$ as a function of the Anosov flow. It is convenient for the exposition to continue to formulate this in terms of a parameterized family of $C^{1+\beta}$ Anosov flows $\{\phi^\lambda\}_{\lambda\in(-\epsilon,\epsilon)}$, the general case being similar.

Proposition 3.1. *Let ϕ^λ, for $\lambda \in (-\epsilon, \epsilon)$, be a C^{k+1} family of Anosov flows and let*

$$R^\lambda = 1 + \lambda R^{(1)} + o(\lambda) \in C^\alpha(M, \mathbb{R})$$

be the associated structural stability reparameterization. Then

$$\frac{dh(\phi^\lambda,\alpha)}{d\lambda}|_{\lambda=0} = h(\phi^0,\alpha)\int R^{(1)} d\mu_\alpha.$$

Remark 3.2. *This is similar to the original formula of Katok-Knieper-Weiss for the derivative of the usual topological entropy [7]*

$$\frac{dh_{top}(\phi^\lambda)}{d\lambda}|_{\lambda=0} = h_{top}(\phi^0)\int R^{(1)} d\mu_{\mathrm{top}}$$

(where the measure μ_α is replaced by the measure of maximal entropy μ_{top}) the proof of which we adapt.

Proof of Proposition 3.1. We present the proof in five steps.

Step 1. The value $h(\phi^\lambda,\alpha)$ was characterized in Lemma 1.4 in terms of the growth rate of closed ϕ^λ-orbits τ^λ of least period $\ell(\tau^\lambda) \leq T$.

In the case of the unperturbed flow $\phi = \phi^0$ we can consider the further restriction of the closed orbits $\tau = \tau^0$ as follows.

Lemma 3.3. *Given $\alpha \in \mathrm{int}(\mathcal{B})$ and $\delta, \eta > 0$,*

$$\frac{\#\{\tau : \left|\int R^\lambda d\mu_\tau - \int R^\lambda d\mu_\alpha\right| < \delta, \ell(\tau) \leq T \text{ and } \|[\tau] - \alpha\ell(\tau)\| \leq \delta\ell(\tau)\}}{\#\{\tau : \ell(\tau) \leq T, \|[\tau] - \alpha\ell(\tau)\| \leq \delta\ell(\tau)\}} > 1-\eta,$$

for T sufficiently large.

Proof. This can be deduced from a large deviation theorem of Kifer ([9], Theorem 2.1). Compare also with ([1], Theorem 1.3) where Babillot and Ledrappier deduce a stronger result from the same source. □

Step 2. Fix now any $0 < \eta < 1$ then comparing Lemmas 1.4 and 3.3 we have that the value $h(\phi,\alpha)$ also occurs as the growth rate of the closed orbits with these additional conditions, i.e.,

$$h(\phi,\alpha) = \lim_{T\to+\infty}\frac{1}{T}\log\#\left\{\tau : \left|\int R^\lambda d\mu_\tau - \int R^\lambda d\mu_\alpha\right| < \delta, \ell(\tau) \leq T \text{ and } \|[\tau] - \alpha\ell(\tau)\| \leq \delta\ell(\tau)\right\}.$$

Moreover, by structural stability we can naturally associate to each ϕ^0-closed orbit $\tau = \tau^0$ a corresponding ϕ^λ-closed orbit denoted τ^λ using the homeomorphism $\Theta^\lambda := \Theta^{\phi^\lambda} : M \to M$ from Proposition 2.2. We note that trivially the homology class $[\tau^\lambda] = [\tau] \in H^1(M, \mathbb{Z})$ remains unchanged. Furthermore, by the definition of R^λ, for $\lambda \in (-\epsilon, \epsilon)$, we can write

$$\frac{\ell(\tau^\lambda)}{\ell(\tau)} = \int R^\lambda d\mu_\tau \leq \int R^\lambda d\mu + \delta,$$

provided we restrict to those τ satisfying $\left|\int R^\lambda d\mu_\tau - \int R^\lambda d\mu_\alpha\right| < \delta$.

Step 3. We claim that for $\lambda \in (-\epsilon, \epsilon)$ we have the inequalities

$$\frac{h(\phi, \alpha)}{\left(\int R^\lambda d\mu_\alpha\right)^2} \leq h(\phi^\lambda, \alpha) \leq h(\phi, \alpha) \int \left(\frac{1}{R^\lambda \circ \Theta^\lambda}\right)^2 d\mu_\alpha^\lambda \tag{3.1}$$

To get the lower bound in (3.1) we can combine Lemma 1.4 and the results in Step 2 to write

$$\begin{aligned}
h(\phi, \alpha) &\leq \lim_{\delta \to 0} \lim_{T \to +\infty} \frac{1}{T} \log \#\{\tau : \ell(\tau) \leq T, \|[\tau] - \alpha \ell(\tau)\| \leq \delta \ell(\tau)\} \\
&= \lim_{\epsilon \to 0} \lim_{\delta \to 0} \lim_{T \to +\infty} \frac{1}{T} \log \#\{\tau : \ell(\tau) \leq T, \|[\tau] - \alpha \ell(\tau)\| \leq \delta \ell(\tau) \text{ and} \\
&\qquad\qquad \left|\int R^\lambda d\mu_\tau - \int R^\lambda d\mu_\alpha\right| < \epsilon\Big\} \\
&\leq \lim_{\epsilon \to 0} \lim_{\delta \to 0} \lim_{T \to +\infty} \frac{1}{T} \log \#\Big\{\tau : \ell(\tau) \leq T, \|[\tau] - \alpha \ell(\tau)\| \leq \delta \ell(\tau) \text{ and} \\
&\qquad\qquad \left(\int R^\lambda d\mu_\alpha - \epsilon\right) \ell(\tau) \leq \ell(\tau^\lambda) \leq \left(\int R^\lambda d\mu_\alpha + \epsilon\right) \ell(\tau)\Big\} \\
&\leq \lim_{\epsilon \to 0} \lim_{\delta \to 0} \lim_{T \to +\infty} \frac{1}{T} \log \# \left\{\tau^\lambda : \ell(\tau') \leq T \left(\int R^\lambda d\mu_\alpha + \epsilon\right) \text{ and} \right. \\
&\qquad\qquad \left. \left\|[\tau^\lambda] - \left(\alpha \int R^\lambda d\mu_\alpha\right)\right\| \leq \left(\epsilon \int R^\lambda d\mu_\alpha + \|\alpha\| \delta\right) \ell(\tau^\lambda)\right\} \\
&= \lim_{\epsilon \to 0} \left(\int R^\lambda d\mu_\alpha + \epsilon\right) \lim_{\delta \to 0} \lim_{T \to +\infty} \frac{1}{T} \log \#\{\tau : \ell(\tau^\lambda) \leq T \text{ and} \\
&\qquad\qquad \left\|[\tau^\lambda] - \left(\alpha \int R^\lambda d\mu_\alpha\right)\right\| \leq \left(\epsilon \int R^\lambda d\mu_\alpha + \|\alpha\| \delta\right) \ell(\tau^\lambda)\}.
\end{aligned}$$

We can now deduce that

$$h(\phi,\alpha) \leq \left(\int R^\lambda d\mu_\alpha \right) h\left(\phi^\lambda, \alpha \int R^\lambda d\mu_\alpha\right).$$

Moreover, we have that

$$\begin{aligned} h\left(\phi^\lambda, \alpha \int R^\lambda d\mu_\alpha\right) &= \sup\left\{ h_{\phi^\lambda}(\mu) \,:\, \Phi_\mu^{\phi^\lambda} = \alpha \int R^\lambda d\mu_\alpha \right\} \\ &= \sup\left\{ h_{\phi^\lambda(\int R^\lambda d\mu_\alpha)}(\mu) \,:\, \Phi_\mu^{\phi^\lambda} = \alpha \right\} \\ &= \left(\int R^\lambda d\mu_\alpha \right) \sup\left\{ h_\phi(\mu) \,:\, \Phi_\mu^\lambda = \alpha \right\} \\ &= \left(\int R^\lambda d\mu_\alpha \right) h(\phi,\alpha). \end{aligned}$$

This gives the first inequality in (3.1). If we reverse the roles of ϕ and ϕ^λ (with μ_α replaced by μ_α^λ; and R^λ replaced by $1/R^\lambda \circ \Theta^\lambda$) then this gives the second inequality in (3.1).

Furthermore, using the expansion $R^\lambda = 1 + \lambda R^{(1)} + o(\lambda)$ as $\lambda \to 0$ we can deduce that

$$\begin{aligned} \frac{h(\phi,\alpha)}{\left(1 + \lambda \int R^{(1)} d\mu_\alpha^\lambda + o(\lambda)\right)} &\leq h(\phi^\lambda, \alpha) \\ &\leq h(\phi,\alpha) \int \left(\frac{1}{1 + \lambda R^{(1)} \circ \Theta + o(\lambda)} \right) d\mu_\alpha + o(\lambda). \end{aligned}$$

Step 4. A final ingredient we need is the following.

Lemma 3.4. *Given $\alpha \in \mathrm{int}(\mathcal{B}(\phi))$ the measures μ_α^λ converge to μ_α in the weak star topology as $\lambda \to 0$.*

Proof. Let μ_α^* be a weak star limit of the measures μ_α^λ as $\lambda \to 0$. Since μ_α^λ maximizes the inequality in (3.1) for the flow ϕ^λ, and:

1. the associated function $\omega_{\xi^\lambda}(X^{\phi^\lambda})$ converges to $\omega_{\xi^0}(X^{\phi^0})$; and
2. the corresponding vector ξ^λ converges to ξ as $\lambda \to 0$ (since $\xi^\lambda = (\nabla P(\phi.\cdot))^{-1}(\alpha)$ has an analytic dependence on λ inherited through the analyticity of the pressure)

then, as $\lambda \to 0$, we have that by upper semicontinuity of the entropy that

$$\begin{aligned} P(\phi,\xi) &\geq h_{\mu^*_\alpha}(\phi) + \int \omega_{\xi^{\phi^0}}(X^{\phi^0}) d\mu^*_\alpha \\ &\geq \limsup_{\lambda \to 0} \left(h_{\mu^\lambda_\alpha}(\phi^\lambda) + \int \omega_{\xi^\lambda}(X^{\phi^\lambda}) d\mu^\lambda_\alpha \right) = \lim_{\lambda \to 0} P(\phi^\lambda, \xi^\lambda) = P(\phi,\xi) \end{aligned}$$

and thus by uniqueness of the measure realizing the maximum we have that $\mu^*_\alpha = \mu_\alpha$. □

Step 5. Finally, from (3.1) and Lemma 3.4 we deduce that

$$\frac{\partial h(\phi^\lambda)}{\partial \lambda}|_{\lambda=0} = \lim_{\lambda \to 0} \frac{h(\phi^\lambda,\alpha) - h(\phi,\alpha)}{\lambda} = h(\phi,\alpha) \int R^{(1)} d\mu_\alpha$$

which completes the proof of the theorem.

□

4 Proof of Theorem 1.6

Since $P(\phi,\cdot)$ and $-h(\phi,\cdot)$ are Legendre conjugates by Lemma 2.11 and they are related by (2.2). In particular, we can write

$$h(\phi^\lambda,\alpha) = P(\phi^\lambda, \xi_\alpha^{\phi^\lambda}) - \langle \xi_\alpha^{\phi^\lambda}, \alpha \rangle \tag{4.1}$$

and note that by Lemma 2.12 the map $P(\phi,\xi)$ has a C^{k-1} dependence on the C^{k+1} Anosov flow ϕ. To complete the proof of Theorem 1.6 we need only the following.

Lemma 4.1. *The map* $(\alpha,\lambda) \mapsto \xi_\alpha^{\phi_\lambda} \in \mathbb{R}^b$ *is* C^{k-1}.

Proof. This follows from the dependence of $\xi \mapsto \alpha_\xi^{\phi^\lambda} = \nabla P(\phi^\lambda,\cdot)$ on ϕ^λ and the characterization of $\alpha \mapsto \xi_\alpha^{\phi^\lambda}$ as its inverse (with the same regularity of dependence on ϕ^λ). □

5 Proof of Theorem 1.7

The proof of Theorem 1.7 follows from the fact that for the C^ω Anosov flow $\phi_t : M \to M$ the cohomological pressure $P(\phi^\lambda, \xi_\alpha^\phi)$ has a real analytic dependence on $\xi \in \mathbb{R}^b$. This can be shown by using its characterization as the pole of the Ruelle zeta function. More precisely, we can associate to each closed orbit τ for the flow ϕ^λ its least period $\ell^\lambda(\tau)$. The zeta function is then formally defined by

$$\zeta_\xi^\lambda(s) = \prod_\tau \left(1 - \exp\left(\ell_\xi^\lambda(\tau) - s\ell^\lambda(\tau)\right)\right)^{-1}$$

where $\ell_\xi^\lambda(\tau) = \int_0^{\lambda(\tau)} \omega_\xi(X(\phi^\xi))(\phi_t^\lambda x)dt$ for any x_τ is a point on the orbit and $s \in \mathbb{C}$

Lemma 5.1. *The zeta function $\zeta_\xi^\lambda(s)$ is analytic for $Re(s) > P(\phi^\lambda, \xi_\alpha^\phi)$ and has a meromorphic extension to a neighbourhood of $P(\phi^\lambda, \xi_\alpha^\phi)$ where $s = P(\phi^\lambda, \xi_\alpha^\phi)$ is a simple pole.*

Proof. This follows from [15], [14]. ☐

For each of the closed orbits τ the maps $(-\epsilon, \epsilon) \ni \lambda \mapsto \ell_\xi^\lambda(\tau), \ell^\lambda(\tau) \in \mathbb{R}$ are uniformly real analytic, i.e., there exists an neighbourhood $(-\epsilon, \epsilon) \subset U \subset \mathbb{C}$ such that these maps have analytic extensions to U. Moreover, this gives an bi-analytic map $U \ni \lambda \mapsto 1/\zeta_\xi^\lambda(s)$ for s in a small neighbourhood $P(\phi^0, \xi_\alpha^\phi)$. We can then choose sufficiently small $\delta > 0$ and write

$$P(\phi^\lambda, \xi_\alpha^\phi) = \frac{1}{2\pi i}\int_\Gamma \frac{\zeta_\xi^{\lambda\prime}(s)}{\zeta_\xi^\lambda(s)} ds$$

where $\Gamma = \{s \in \mathbb{C} : |s - P(\phi^0, \xi_\alpha^\phi)| = \delta\}$, choosing ϵ smaller, if necessary. In particular, the real analytic dependence of the zeta function translates into a real analytic dependence of $P(\phi^\lambda, \xi_\alpha^\phi)$ on λ. This gives analyticity of $h(\phi^\lambda, \alpha)$ by using (2.2).

6 Applications

6.1 Sharp minimizers

Sharp showed that given an Anosov flow $\phi_t : M \to M$ the cohomological pressure function $P(\phi, \cdot) : \mathbb{R}^b \to \mathbb{R}$ has a unique global minimum which occurs at ξ_ϕ, i.e.,

$P(\phi, \xi_\phi) \leq P(\phi, \xi)$ for all $\xi \in \mathbb{R}^b$ with equality precisely when $\xi = \xi_\phi$ [19]. The value ξ_ϕ is called the *Sharp minimizer*. The value ξ_ϕ plays a role in counting problems for closed orbits in a fixed homology class as we briefly recall. For Anosov flows where the measure of maximal entropy μ_{max} has a non-vanishing winding cycle (i.e., $\Phi^\phi_{\mu_{max}} \neq 0$) the counting function for all orbits must be modified to account for the distribution across different homology classes. The Anosov flow $\phi_t : M \to M$ is called *homologically full* if every homology class in $H_1(M, \mathbb{Z})$ contains a closed orbit. We recall the following elegant result.

Proposition 6.1 (Sharp [19]). *For a given $\alpha \in \mathrm{int}(\mathcal{B}(\phi)) \subset H_1(M, \mathbb{Z})$ there is an asymptotic for the counting function of the form*

$$\#\{\gamma : \ell(\gamma) \leq T \text{ and } [\gamma] \in \alpha\} \sim C e^{-\langle \xi_\phi, \alpha \rangle} \frac{e^{h^* T}}{T^{1+b/2}}, \text{ as } T \to +\infty$$

where $h^* = h(\phi, 0) = \sup\{h(\phi, \mu) : \mu \in \mathcal{M}_\phi, \Phi_\mu = 0\} \leq h_{top}(\phi)$.

Example 6.2. *In the particular case that ϕ is a reversible flow (such as a geodesic flow) we have that $\xi_\phi = 0$ and thus $h^* = h_{top}(\phi)$ and one recovers the asymptotic*

$$\#\{\gamma : \ell(\gamma) \leq T \text{ and } [\gamma] \in \alpha\} \sim C \frac{e^{h_{top}(\phi) T}}{T^{1+b/2}}, \text{ as } T \to +\infty.$$

Now that we have explained the role of ξ_ϕ we can state the following result on its dependence.

Proposition 6.3. *For C^{k+1} Anosov flows the Sharp minimizer ξ_ϕ has a C^{k-1} dependence on the flow.*

Proof. Let us consider a C^{k+1} family of Anosov flows $\phi_t^\lambda : M \to M$, for $\lambda \in (-\epsilon, \epsilon)$. We know that cohomological pressure $P(\phi^\lambda, \cdot) : \mathbb{R}^b \to \mathbb{R}$, which is itself real analytic, has a C^{k-1} dependence on the flow $\phi_t : M \to M$ and thus the map $P : (-\epsilon, \epsilon) \times \mathbb{R}^b \to \mathbb{R}$ is C^{k-1}.

Since $\xi_{\phi^\lambda} \in \mathbb{R}^k$ is a critical point for $P(\phi^\lambda, \cdot) : \mathbb{R}^b \to \mathbb{R}$ then for each $\lambda \in (-\epsilon, \epsilon)$, we have $F(\lambda, \xi_{\phi^\lambda}) = 0$ where $F : (-\epsilon, \epsilon) \times \mathbb{R}^b \to \mathbb{R}^b$ is defined by

$$F(\lambda, \xi) := \nabla_\xi P(\phi^\lambda, \xi).$$

We observe that if we write $\xi = (\xi_1, \ldots, \xi_b)$ and $F(\xi) = (F_1(\xi), \ldots, F_b(x))$ then

$$\det\left(\frac{\partial F_i}{\partial \xi_j}(0, \xi_{\phi^0})\right) \neq 0$$

because of the strict convexity of the pressure function. We can now apply the implicit function theorem to the function $F(\cdot,\cdot)$ to deduce that $(-\epsilon,\epsilon) \ni \lambda \mapsto \xi_{\phi^\lambda} \in \mathbb{R}^b$ is C^{k-1}. $\square$

6.2 Scarcity of Anosov flows with $h(\phi, 0) = h_{top}(\phi)$.

In some examples (including all geodesic flows) we have that $h^* = h(\phi, 0) = h_{top}$. However, within all transitive Anosov flows this property is fairly rare. The subset of C^{k+1} Anosov flows given by

$$\{\phi_t : M \to M \,:\, h_{top}(\phi) = h(\phi, 0)\}$$

is easily seen closed in the C^{k-1} topology (and nowhere dense). We can formulate this as follows.

Lemma 6.4. *The space of transitive Anosov flows ϕ for which $h^*(\phi) = h_{top}(\phi)$ is is the level set for a C^{k-1} map.*

Proof. We can consider the function $G(\phi) = h(\phi, 0) - h_{top}(\phi)$ on C^{k+1} Anosov flows. By Theorem 1.5 the function $h(\phi, 0)$ has a C^{k-1} dependence on ϕ and by [8] the function $h_{top}(\phi)$ has a C^{k-1} dependence on ϕ. Thus $G(\phi)$ has a C^{k-1} dependence on ϕ. $\square$

6.3 Dependence of associated measures.

We can now turn to the associated probability measures. Given any $\alpha \in \mathrm{int}(\mathcal{B})$ we know by Lemma 2.11 that there exists a ϕ-invariant probability measure μ_α^ϕ such that $\Phi_{\mu_\alpha^\phi} = \alpha$ and $h_{\mu_\alpha^\phi}(\phi) = h(\phi, \alpha)$

Example 6.5. *We have the equality $h_{top}(\phi) = h(\phi, \alpha)$ precisely when $\alpha = \Phi^\phi_{\mu_{max}}$, where μ_{max} is the unique measure of maximal entropy (i.e., $h_{top}(\phi) = h_{\mu_{max}}(\phi)$).*

The following theorem describes the dependence of the measure μ_α^ϕ on the flow ϕ when evaluated on sufficiently regular functions (e.g., for any $F \in C^1(M)$ the dependence of $\int F d\mu_\alpha^\phi \in \mathbb{R}$ on ϕ).

Theorem 6.6. *Let ϕ be a C^{k+1} Anosov flow and $\alpha \in \mathrm{int}(\mathcal{B}(\phi))$.*

1. *For $k \geq 2$ the dependence of measure μ_α^ϕ on the flow is C^{k-1}.*

2. For $k = \infty$ the dependence of measure μ_α^ϕ on the flow is C^∞.

Proof. We can consider the pressure function $P(\phi, \cdot) : C^1(M) \to \mathbb{R}$ evaluated at $\omega_\xi(X) + tF$. By [3] the function $P(\phi, \omega_\xi(X) + tF)$ is real analytic in t. Furthermore, is known that

$$\frac{\partial P(\phi, \omega_\xi(X) + tF)}{\partial t} = \int F d\mu_\alpha$$

cf. [14]. Finally, $P(\phi, \omega_\xi(X) + tF)$ has a C^{k-1} dependence on ϕ from which we can deduce the same for the derivative and the integral. ☐

7 Geodesic flows

A simple and amusing application is to Ricci flows on metrics on surfaces V of strictly negative curvature. However, first we need the formula for the first derivative of the entropy for the geodesic flows in terms of perturbations of the metric.

7.1 First derivative for geodesic flow

For a fixed metric of negative curvature g on a compact surface of negative curvature the associated geodesic flow ϕ^g is known to be Anosov, with vector field X^g. Given a C^1 path in the space of C^2 metrics of negative curvature g_λ ($\lambda \in (-\epsilon, \epsilon)$) we can denote the derivative at $\lambda = \lambda_0$ by $g_{\lambda_0}^{(1)}(v, v)$, i.e.,

$$g_\lambda(v, v) = g_0(v, v) + \lambda g_{\lambda_0}^{(1)}(v, v) + o(\lambda).$$

Given $\alpha \in \mathrm{int}(\mathcal{B}(\phi))$, let us denote $h(g_\lambda, \alpha) = h(\phi^{g_\lambda}, \alpha)$. In this context the formula for the derivative of the entropy takes the following form.

Lemma 7.1. *Given $\alpha \in \mathrm{int}(\mathcal{B}(\phi))$ we have that*

$$\frac{\partial h(g_\lambda, \alpha)}{\partial \lambda}|_{\lambda=0} = -\frac{1}{2} h(g_0, \alpha) \int g^{(1)}(v, v) d\mu_\alpha(v)$$

where as before μ_α is the unique measure satisfying

$$\begin{aligned} &h_{\mu_\alpha}(\phi^{g_0}) + \int \alpha(X(\phi)) d\mu_\alpha \\ &= \sup\{h_\mu(\phi^{g_0}) + \int \alpha(X(\phi)) d\mu : \mu = \ \phi^{g_0}\text{-invariant measure}\}. \end{aligned}$$

Proof. This is similar to the thermodynamic proof for $\alpha = 0$ in [8] adapted along the lines for Anosov flow in the previous section (where geometric estimates on norms replace structural stability in the dynamical approach) [8] Let g_0 and g be metrics of negative curvature then the associated geodesic flows $\phi_t^{g_0} : (SM_{g_0}) \to (SM_{g_0})$ and $\phi_t^{g_0} : (SM_g) \to (SM_g)$ are Anosov. In particular, Proposition 8.1 in [7] generalizes to

$$\frac{h(\phi^{g_0}, \alpha)}{\int_{(SM)_{g_0}} \|v\|_g d\mu_{g_0}^\alpha} \leq h(\phi^g, \alpha) \leq h(\phi^{g_0}, \alpha) \int_{(SM)_{g_0}} \|v\|_g d\mu_{g_0}^\alpha .$$

Let $\epsilon > 0$. Let c_{g_0} be a closed geodesic with respect to the metric g_0 of length $\ell_{g_0}(c_{g_0}) \leq T$ and for which

$$\left| \int_{(SM)_g} \|v\|_{g_0} d\mu_g - \frac{1}{\ell_g(c_g)} \int_0^{\ell_g(c_g)} \|\dot{c}_g(\phi_t^g v)\| \right| \leq \epsilon \text{ where } v \in c_g .$$

As observed in [7] for a family of metrics g_λ ($\lambda \in (-\epsilon, \epsilon)$) we can expand

$$\|v\|_{g_\lambda} = \|v\|_{g_0} + \lambda \left(\frac{\partial \|v\|_{g_\lambda}}{\partial \lambda} |_{\lambda=0} \right) + o(\lambda)$$

Combining these bounds gives

$$\begin{aligned} & h(\phi^{g_\lambda}, \alpha) \left(-\lambda \left(\int_{(SM)_{g_0}} \frac{\partial \|v\|_{g_\lambda}}{\partial \lambda} |_{\lambda=0} d\mu^0 \right) + o(\lambda) \right) \\ & \leq h(\phi^{g_\lambda}, \alpha) - h(\phi^{g_\lambda}, \alpha) \\ & \leq h(\phi^{g_\lambda}, \alpha) \left(-\lambda \left(\int_{(SM)_{g_\lambda}} \frac{\partial \|v\|_{g_\lambda}}{\partial \lambda} |_{\lambda=0} d\mu^\lambda \right) + o(\lambda) \right) \end{aligned}$$

Using that μ_α^λ converges to μ_α in the weak star topology as $\lambda \to 0$ and that

$$\frac{\partial \|v\|_{g_\lambda}}{\partial \lambda} = \frac{1}{2} \frac{\partial g_\lambda(v, v)}{\partial \lambda} .$$

the result follows.

□

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[8] The reason for the factor of $\frac{1}{2}$ is in the last line of the proof

7.2 Normalised Ricci flows

We can denote by $\kappa_g : M \to \mathbb{R}$ the Gauss curvature function for the metric g and denote the total curvature for the surface by

$$\overline{\kappa} = \int \kappa_g d\mathsf{vol}_g = 2\pi\chi(M)$$

where $\chi(M) < 0$ is the Euler characteristic of the surface M, by the Gauss-Bonnet theorem, and which, in particular, is independent of the metric.

Definition 7.2. *We can define the normalized Ricci flow $(g_t)_{t\geq 0}$ on the space of smooth (two dimensional) Riemannian metrics by the differential equation on metrics:*

$$\frac{\partial g_t(v,v)}{\partial t} = -2(\kappa_g(\pi(v)) - \overline{\kappa}) \text{ for } v \in (SM)_{g_t}$$

(where $\pi : (SM)_{g_t} \to M$ is the canonical projection).

It is known that starting from any surface of negative curvature g_0 the solution $(g_t)_{t\geq 0}$ exists and converges to a surface of constant negative curvature which we can denote by g_∞, say ([6], Theorem 3.3). Moreover, the metrics g_t all have the same area as the initial metric g_0. Manning showed that the topological entropy $h_{top}(\phi^{g_t})$ (of the associated geodesic flow ϕ^{g_t} decreases monotonically along solutions of the Ricci equation to $h_{top}(g_\infty) = 2\pi|\chi(M)|$ [12].

Given $\alpha \in \mathcal{B}(\phi^{g_t})$ we can now ask about the local behaviour of $h(g_t, \alpha)$ along the orbits of the Ricci flow. The following could be viewed as a partial variant of Manning's result.

Theorem 7.3. *Assume that $\alpha \in \operatorname{int}(\mathcal{B}(\phi^{g_t}))$ then proving $h(\phi^{g_t}, \alpha) > h_{top}(\phi^{g_\infty})$ then $h(\phi^{g_t}, \alpha)$ is monotone decreasing.*

We begin with the observation that from the definition of the Ricci flow and the formula for the first derivative we have that $\frac{\partial h(\phi^{\rho_\lambda}, \alpha)}{\partial \lambda} < 0$ precisely when

$$\left| \int \kappa_{g_t}(v) d\mu_\alpha^{g_t}(v) \right| > h_{top}(g_\infty)^2 \tag{7.1}$$

If we denote by χ the Lyapunov exponent for the geodesic flow ϕ^{g_t} and the (ergodic) measure μ_α and then

$$\chi = \int a(v) d\mu_\alpha(v).$$

where a is a solution to the Riccatti equation

$$\frac{da(\phi_u^{g_t} v)}{du}|_{u=0} = -a(v)^2 - \kappa(v), \text{ for } v \in (SM)_{g_t},$$

along the orbit of $\phi^{g_t}(v)$. Integrating this against $\mu_\alpha^{g_t}$ gives

$$\int a^2(v) d\mu_\alpha^{g_t}(v) = \left| \int \kappa(v) d\mu_\alpha^{g_t}(v) \right| \tag{7.2}$$

Furthermore, using Cauchy-Schwartz we can bound

$$\int a(v) d\mu_\alpha^{g_t}(v) \leq \left(\int a(v)^2 d\mu_\alpha^{g_t}(v) \right)^{\frac{1}{2}} \tag{7.3}$$

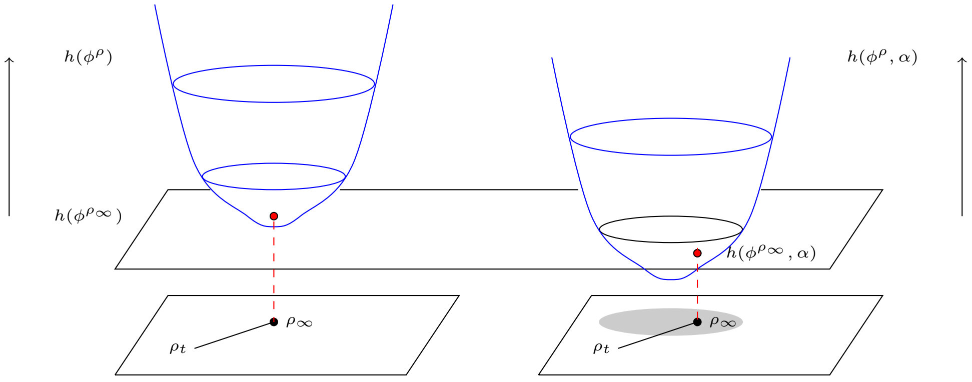
and using the Ruelle inequality (applied to the measure μ_α) we can bound

$$h_{\mu_\alpha^{g_t}}(\phi^{g_t}) = h(\phi^{g_t}, \alpha) \leq \int a(v) d\mu_\alpha^{g_t}(v). \tag{7.4}$$

Comparing (7.2), (7.3) and (7.4) gives that

$$h(\phi^{g_t}, \alpha)^2 \leq \left| \int \kappa_{g_t}(v) d\mu_\alpha^{g_t}(v) \right|.$$

Thus a sufficient condition for the inequality (7.1) to hold is that $h(\phi^{g_t}, \alpha) > h_{top}(\phi^{g_\infty})$.



Ricci flow on metrics

Ricci flow on metrics

Figure 1: (i) The topological entropy decreases along orbits of the normalized Ricci flow; (ii) The homological entropy decreases along orbits of the normalized Ricci flow provided it is greater than the topological entropy of the geodesic flow for the limiting metric ρ_∞

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