

# Anosov Flows and the Fundamental Group

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## 1 Introduction

Paraphrasing a problem mentioned by Vaibhav Gadre in Luminy in December 2023: *Let  $Y \subset M$  be a small transverse section to a transitive Anosov flow  $\varphi_t : M \rightarrow M$  then show that the orbit segments from  $Y$  to itself correspond to generators for the fundamental group  $\pi_1(M, Y)$ .*

A solution to this problem comes from adapting an old approach of Adachi [1] and Adachi-Sunada [2]. This involves first proving an analogous problem for graphs and then applying this to the flow using symbolic dynamics. <sup>1</sup>

## 2 Graphs

Let  $\mathcal{G}$  be a graph with edge set  $\mathcal{E}$  and vertex set  $\mathcal{V}$ . Every edge  $e \in \mathcal{E}$  connects a pair of vertices. In the case of an undirected graph  $\mathcal{G}$  no ordering is associated to this pairs and  $\mathcal{G}$  can be viewed as a CW-complex.

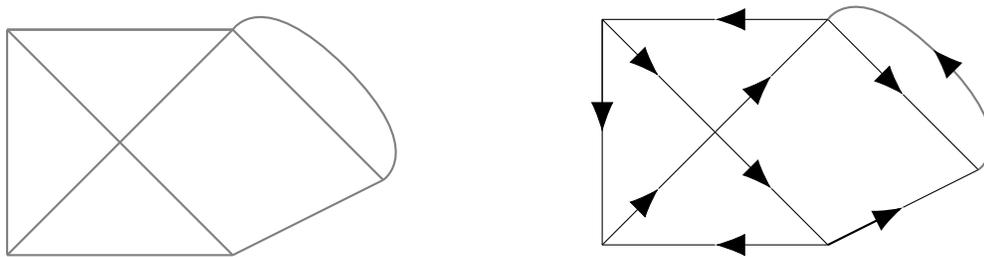


Figure 1: (a) An undirected graph  $\mathcal{G}$ ; (b) A directed graph  $\widehat{\mathcal{G}}$

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<sup>1</sup>I am grateful to Richard Sharp for reminding me of this result.

**Definition 2.1.** *In the case of the undirected graph we can consider paths  $\gamma$  in  $\mathcal{G}$  which can be realised by a sequence of neighbouring edges  $e_1, e_2, \dots, e_n$  whereby  $\mathcal{V} \ni e_i \cap e_{i+1} \neq \emptyset$  (for  $i = 1, \dots, n - 1$ ).*

In the case of directed graphs  $\widehat{\mathcal{G}}$  we have additionally maps  $o, t : \mathcal{E} \rightarrow \mathcal{V}$  which associate to each edge  $e$  an *original* vertex  $o(e)$  and a *terminal* edge  $t(e)$  joined by the edge  $e$ . This assigns a natural orientation to each edge  $e \in \mathcal{E}$  which is more easily visualized by an arrow on the edge in the graph.<sup>2</sup> There is a natural map  $\pi : \widehat{\mathcal{G}} \rightarrow \mathcal{G}$  which just involves forgetting the orientations.

We say that the directed graph is *irreducible* if for any two vertices  $v, v' \in \mathcal{G}$  there are directed paths beginning at  $v$  and ending at  $v'$ .

**Definition 2.2.** *In the case of an irreducible directed graph  $\mathcal{G}$  a (directed) path  $\tau$  additionally requires that the sequence  $e_1, e_2, \dots, e_n$  satisfies  $e_{i-1} \cap e_i = o(e_i) = t(e_{i-1})$  (for  $i = 1, \dots, n$ ).*

This can be visualized as the path taking the same direction as the orientation of the edges. Clearly a path  $\tau$  in a directed graph  $\mathcal{G}$  gives rise to a path  $\gamma = \pi(\tau)$  (corresponding to dropping the orientation of edges). Of course, the converse is not necessarily true: a path  $\gamma$  in the undirected graph may not correspond to a path  $\tau$  in the directed graph.

**Proposition 2.3** (after Adachi-Sunada [2]). *Let  $v_0 \in \mathcal{V}$ . The fundamental group  $\pi_1(G, v_0)$  of the undirected graph is generated by closed paths in the directed graph (based at  $v_0$ ).*

*Proof.* The proof in [2] needs adapting and we take the opportunity to give a more geometric interpretation. The proposition also appears as Lemma 5.2 in [3] (where it appears without proof).

To proceed it is convenient to introduce the following notion. We can associate to a path  $\tau = e_1, e_2, \dots, e_n$  in the directed graph  $\widehat{\mathcal{G}}$  a *reversed path*  $\tau^{-1} = e_n, e_{n-1}, \dots, e_1$  in the undirected graph.<sup>3</sup>

For every vertex  $v \in \mathcal{G}$  we can use the irreducibility to choose two directed paths:

- a directed “in-path”  $\tau(v)$  in  $\widehat{\mathcal{G}}$  which starts at  $v$  and end at  $v_0$ ; and
- a directed “out-path”  $\bar{\tau}(v)$  in  $\widehat{\mathcal{G}}$  which starts at  $v_0$  and ends  $v$ .

**Step 1.** Given any closed path  $\gamma = e_1 e_2 \dots e_n$  in the undirected graph  $\mathcal{G}$  which starts and finishes at  $v_0$  (i.e.,  $o(e_1) = t(e_n) = v_0$ ) we can write it out in terms of a concatenation of alternating directed or reversed paths in the directed graph  $\widehat{\mathcal{G}}$ . There are four basic cases we need to consider corresponding to:

1. The first and final edges of  $\gamma$  leave the vertex  $v_0$ , i.e.,  $o(e_1) = v_0$  and  $o(e_n) = v_0$  (and  $m$  is even),

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<sup>2</sup>For simplicity we will assume that for any  $e \in \mathcal{E}$  we have  $t(e) \neq o(e)$ , i.e., no edges are loops starting and finishing at the same vertex.

<sup>3</sup>If one fancifully thought of the directed graph as a street map of one way streets then  $\tau$  would be a legal route and  $\tau^{-1}$  would be the path given by driving in reverse along what would be a legal route if driving forwards

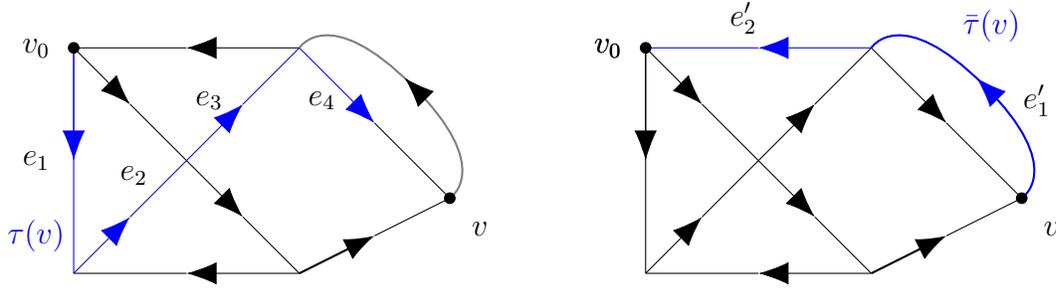


Figure 2: (a) An “out-path”  $\tau(v) = e_1 e_2 e_3 e_4$  from  $v_0$  to  $v$  in  $\mathcal{G}$ ; (b) An “in-path”  $\bar{\tau}(v) = e'_1 e'_2$  from  $v$  to  $v_0$  in  $\mathcal{G}$

2. The first edge of  $\gamma$  leaves and the final edge of  $\gamma$  enters the vertex  $v_0$ , i.e.,  $o(e_1) = v_0$  and  $t(e_n) = v_0$  (and  $m$  is odd),
3. The first and final edge of  $\gamma$  enter the vertex  $v_0$ , i.e.,  $t(e_1) = v_0$  and  $t(e_n) = v_0$  (and  $m$  is even),
4. The first edge of  $\gamma$  enters and the final edge of  $\gamma$  leaves the vertex  $v_0$ , i.e.,  $t(e_1) = v_0$  and  $o(e_n) = v_0$  (and  $m$  is odd).

We can concentrate on the first case, the other three being similar. In particular, let us write  $\gamma$  in the form:

$$\gamma = \underbrace{e_1 \cdots e_{k_1}}_{\tau_1} \underbrace{e_{k_1+1} \cdots e_{k_2}}_{\bar{\tau}_2} \cdots \underbrace{e_{k_{m-2}+1} \cdots e_{k_{m-1}}}_{\tau_{m-1}} \underbrace{e_{k_{m-1}+1} \cdots e_n}_{\bar{\tau}_m}$$

where  $\tau_i$  and  $\bar{\tau}_i$  represent maximal directed and reversed paths, respectively. We further denote the vertices where these subpaths meet as

$$e_{k_1} \cap e_{k_1+1} = v_1, e_{k_2} \cap e_{k_2+1} = v_2, \dots, e_{k_{m-1}} \cap e_{k_{m-1}+1} = v_{m-1}.$$

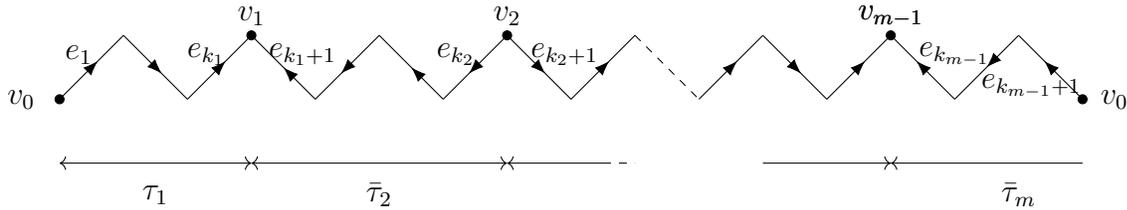


Figure 3: The vertices  $v_i$  are where the maximal directed subpaths and reversed subpaths meet.

and where

$$\begin{aligned} t(e_j) &= o(e_{j+1}) \text{ for } 1 \leq i \leq k_1 - 1 \text{ (for the directed path } \tau_1) \\ o(e_j) &= t(e_{j+1}) \text{ for } k_1 \leq i \leq k_2 - 1 \text{ (for the reversed path } \bar{\tau}_2). \\ &\vdots \\ o(e_j) &= t(e_{j+1}) \text{ for } k_{m-1} \leq i \leq n - 1 \text{ (for the reversed path } \bar{\tau}_m). \end{aligned}$$

**Step 2.** We can then add at each vertex  $v_i$  either the in-path  $\tau(v_i)$  to  $v_0$  if  $t(e_{k_i+1}) = v_i$  (i.e., when  $i$  is odd) or the out-path  $\bar{\tau}(v_i)$  from  $v_0$  to  $v_i$  if  $o(e_{k_i+1}) = v_i$  (i.e., when  $i$  is even). Thus up to homotopy in  $\mathcal{G}$  the original path  $\gamma$  in the undirected graph  $\mathcal{G}$  is equivalent to

$$(\tau_1\tau(v_1)) (\tau(v_1)^{-1}\bar{\tau}_2\bar{\tau}(v_2)) (\bar{\tau}(v_2)^{-1}\tau_3\tau(v_3)) \cdots (\tau(v_{m-1})^{-1}\bar{\tau}_m)$$

where we replace  $\bar{\tau}_i$  by  $\tau(v_{i-1})^{-1}\bar{\tau}_i\bar{\tau}(v_i)$  if  $1 \leq i \leq m-1$  is even or  $\tau_i$  by  $\bar{\tau}(v_{i-1})^{-1}\tau_i\tau(v_i)$  if  $1 \leq i \leq m-1$  is odd.

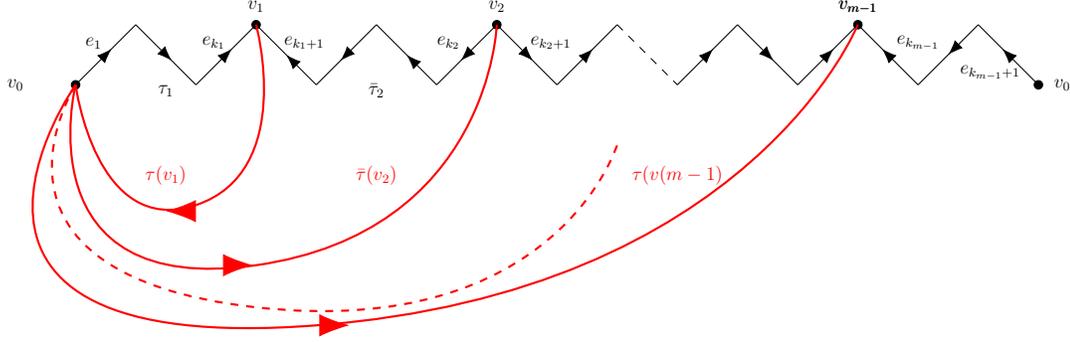


Figure 4: The curve  $\gamma$  is homotopic to alternating directed paths and reversed paths

**Step 3.** We observe that

- (a) The terms  $\tau_1\tau(v_1)$  and  $\tau(v_{i-1})^{-1}\bar{\tau}_i\bar{\tau}(v_i)$  for  $i$  even are directed closed loops in  $\widehat{\mathcal{G}}$  starting finishing at  $v_0$ .
- (b) The terms  $\tau(v_{m-1})^{-1}\bar{\tau}_m$  and  $\bar{\tau}(v_{i-1})^{-1}\tau_i\tau(v_i)$  for  $i$  odd are reversed loops in  $\widehat{\mathcal{G}}$  starting finishing at  $v_0$ .

Thus the homotopy class  $[\gamma]$  can be written in terms of the concatenation of alternating homotopy classes of directed closed loops or their inverses.  $\square$

*Remark 2.4.* In the other cases analogous arguments give the following.

- 2) The path  $\gamma = \tau_1\bar{\tau}_2 \cdots \bar{\tau}_{m-1}\tau_m$  is homotopic to

$$(\tau_1\tau(v_1))(\tau(v_1)^{-1}\bar{\tau}_2\bar{\tau}(v_2))(\tau(v_2)^{-1}\tau_3\tau(v_3)) \cdots (\bar{\tau}(v_{m-1})\tau_m)$$

- 3) The path  $\gamma = \bar{\tau}_1\tau_2 \cdots \bar{\tau}_{m-1}\tau_m$  is homotopic to

$$(\bar{\tau}_1\bar{\tau}(v_1))(\bar{\tau}(v_1)^{-1}\tau_2\tau(v_2))(\bar{\tau}(v_2)^{-1}\bar{\tau}_3\bar{\tau}(v_3)) \cdots (\bar{\tau}(v_{m-1})\tau_m)$$

- 4) The path  $\gamma = \bar{\tau}_1\tau_2 \cdots \tau_{m-1}\bar{\tau}_m$  is homotopic to

$$(\bar{\tau}_1\bar{\tau}(v_1))(\bar{\tau}(v_1)^{-1}\tau_2\tau(v_2))(\bar{\tau}(v_2)^{-1}\bar{\tau}_3\bar{\tau}(v_3)) \cdots (\tau(v_{m-1})\bar{\tau}_m)$$

### 3 Anosov flows

The application of Proposition 2.3 for graphs to Anosov flows uses symbolic dynamics. Let  $M$  be a compact  $C^\infty$  manifold.

**Definition 3.1.** *We say that a  $C^1$  flow  $\varphi : M \rightarrow M$  is Anosov if there exists a continuous splitting  $TM = E^s \oplus E^u$  and constants  $C > 0$  and  $\lambda > 0$  such that*

$$\|D\varphi_t|E^s\| \leq Ce^{-\lambda t} \quad (t \geq 0) \quad \text{and} \quad \|D\varphi_{-t}|E^u\| \leq Ce^{-\lambda t} \quad (t \geq 0).$$

*We say that  $\varphi$  is transitive if there exists a point  $x \in M$  such that closure of its  $\varphi$ -orbit is  $M$ .*

For any  $\varepsilon > 0$  we can use the work of Bowen and Ratner to choose a finite set of Markov sections  $\{T_i\}_{i=1}^N$  to the flow of size less than  $\varepsilon$ . There exists a “return time function”  $r : \cup_{i=1}^N T_i \rightarrow \mathbb{R}^+$  such that the parallelpipeds  $\varphi_{[0,r]}(T_i) = \{\varphi(x) : x \in T_i, 0 \leq t \leq r(x)\}$  ( $1 \leq i \leq N$ ) have disjoint interiors and cover  $M$ . (This picture is slightly complicated by the possibility that the sections  $T_i$  may not be connected and may have fractal boundaries, but were constructed to be contained in arbitrarily small transverse disks  $D_i$  ( $1 \leq i \leq N$ ), say).

We can associate to the Markov sections a directed graph  $\widehat{\mathcal{G}}$  whose vertices  $v_i$  correspond to the sections  $T_i$  and where an edge from vertex  $v_i$  to vertex  $v_j$  corresponds to the existence of a  $\varphi$ -orbit segment from the interior of  $T_i$  to the interior of  $T_j$  without intersecting the interior of a third section.

Any closed loop in  $M$  based at  $x_0$  and passing through a given section  $T_j$  can be isotoped to consist of the concatenation of (unoriented) orbit segments and paths within the disks containing the Markov sections. It then corresponds to a loop in the undirected graph  $\mathcal{G}$  based at a vertex  $v_0$ , say, corresponding to a section  $T_{i_0}$ , say. By the result for graphs (Proposition 2.3) this is homotopy equivalent to a concatenation of the homotopy class of closed directed paths and their inverses. We now appeal to the following simple lemma.

**Lemma 3.2.** *If  $e_1 e_2 \cdots e_n$  is a directed closed path in  $\widehat{\mathcal{G}}$  based at  $x_0$  (i.e.,  $o(e_1) = t(e_n) = x_0$ ) then there is an orbit segment from  $T_{i_0}$  back to itself.*

Given any sufficiently small section  $Y$  to the Anosov flow we can choose the family of Markov sections  $\{T_i\}_{i=1}^N$  and a distinguished section  $T_{i_0}$ , say, with  $T \subset \text{int}(T_{i_0})$ . This suffices to show the result.

*Remark 3.3.* The above proof using symbolic dynamics may be applicable more generally. For example, in the context of pseudo-Anosov flows which play an interesting role in the context of three manifolds.

## References

- [1] T. Adachi, Closed orbits of an Anosov flow and the Fundamental Group, Proc. Amer. Math. Soc. 100 (1987) 595-598.

- [2] T. Adachi and T. Sunada, 2. T. Adachi and T. Sunada, Homology of closed geodesics in a negatively curved manifold, *J. Differential Geom.* 26 (1987) 81-99.
- [3] M. Bell, V. Delecroix, V. Gadre, R. Gutiérrez-Romo and S. Schleimer, Diagonal flow detects topology of strata (<https://arxiv.org/abs/2101.12197>)