Self consistent transfer operators

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Abstract

Self consistent transfer operators appear naturally in the study of mean field interaction coupled expanding maps and generalise the classical linear transfer operators in ergodic theory. We give an abstract setting for a special type of self-consistent transfer operator and show it to have a positive eigenvector with a positive eigenvector.

1 Introduction

In recent years a number of authors have been interested in self-consistent transfer operators cf. [6]. These are non-linear operators in the sense that the operator itself depends in a non-trivial way on the function it is applied to. Let $T : \mathbb{T} \to \mathbb{T}$ be a C^1 expanding map of the circle $\mathbb{T}$ and denote by $B = C^{Lip}(\mathbb{T}, \mathbb{R})$ the Banach space of Lipschitz functions with the usual norm

$$\|v\| := \|v\|_\infty + \sup_{x \neq y} \left\{ \frac{|v(x) - v(y)|}{d(x,y)} \right\}.$$

In the classical case of expanding maps for a fixed choice of weight function $w \in B$ the traditional associated transfer operator is a linear operator $\mathcal{L}_w : B \to B$ of the form

$$\mathcal{L}_w v(x) = \sum_{Ty=x} e^{w(y)} v(y), \text{ for } v \in B \text{ and } x \in X.$$

In this setting the well known Ruelle operator gives the existence of a (unique maximal) positive eigenvalue:

Theorem 1.1 (Ruelle Operator Theorem) *Given $w \in B$ there exists $h \in B$ with $h > 0$ and $\lambda > 0$ satisfying the eigenvalue equation $\mathcal{L}_w h = \lambda h$.*

However, a self consistent transfer operator is a generalization which is no longer linear in the sense that the weight function $w = w(v)$ and the transformation $T = T(v)$ may also both depend on $v \in B$, say, to which it is applied.

We will consider an abstract version of this problem where the results are based on some general hypotheses.

Hypothesis I (on $w(\cdot)$). Assume that $U \subset B$ is nonempty and there are weights $w : U \to B$ that satisfy $D := \sup_{v \in U} \|w(v)\| < +\infty$.

Definition 1.2 *Given a family of weights $w : U \to B$ we can define the* inhomogeneous transfer operator $\mathbb{L}_{w(\cdot)} : U \to B$ *by*

$$\mathbb{L}_{w(v)}(v)(x) = \sum_{T(v)y=x} e^{w(v)(y)} v(y). \tag{0.1}$$

Part of the motivation for the study of self-consistent transfer operators comes from the theory of coupled maps.

Remark 1.3 (Coupled maps [4], [5], [6], [3]) *Consider M points $x_, \cdots, x_M \in \mathbb{T}$ on the unit circle then we can keep track of these using the measure $\mu_0 = \frac{1}{N} \sum_{i=1}^{M} \delta_{x_i}$ on $\mathbb{T}$. Let $T : \mathbb{T} \to \mathbb{T}$ be an expanding map of the circle and consider a mean field interaction $\Phi_{\mu_0} : \mathbb{T} \to \mathbb{T}$ (associated to a measure μ_0 on $\mathbb{T}$) defined by*

$$\Phi_{\mu_0}(x) = x + \pi \left(\delta \int_{\mathbb{T}^N} h(x-y)\mu_0(y) \right)$$

where $\pi : \mathbb{R} \to \mathbb{T}$ is the natural projection, $h : \mathbb{T} \to \mathbb{R}$ is a C^∞ function, and $\delta > 0$ is a coupling constant.

For sufficiently small $\delta > 0$ the map $\Phi_{\mu_0} : \mathbb{T} \to \mathbb{T}$ is a diffeomorphism and writing $F_{\mu_0} = T \circ \Phi_{\mu_0}$ the evolution of the points are given by $\mu_{n+1} = (F_{\mu_n})_* \mu_n$.

If we assume instead that μ_0 is absolutely continuous (i.e., $d\mu_0(x) = f_0 dx$) then the densities $f_n = \frac{d\mu_n}{dx}$ satisfy $f_{n+1} = P f_n$ ($n \geqslant 0$) where

$$Ph(x) = \sum_{T(y)=x} \frac{h(y)}{|T'(y)|}, \quad x \in \mathbb{T}.$$

We denote the (local) inverse branches of $T(v)$ by $\{T_i(v)\}$ ($v \in U$) and denote

$$\theta = \sup_{v \in U} \sup_{x} \max_{i} \|DT_i(v)(x)\|_\infty.$$

To proceed we require some additional assumptions. The first is a uniform bound on the contractions.

Hypothesis II (on $T(\cdot)$). We assume that $\theta < 1$.

The second assumption helps with making iterations of a suitable operator be well defined.

Hypothesis III (on U). We assume that there exists $\epsilon > 0$ such that $U \supset \{v \in B : \|v\| \leqslant \frac{D \cdot e \cdot \theta}{1-\theta} + \epsilon\}$.

In this context we have the following partial generalization of the Ruelle operator theorem.

Theorem 1.4 *Assume hypotheses I, II and III. There exists $v_\infty \in C(\mathbb{T})$ with $v_\infty > 0$ and $\lambda > 0$ such that $\mathbb{L}_{w(v_\infty)}(v_\infty) = \lambda v_\infty$.*

Of course, if $w = w(v)$ and $T = T(v)$ are independent of v then this reduces to the linear transfer operator in Theorem 1.1. However, in the more general setting it is no longer necessarily the case that there is a unique solution to the eigenvalue equation [2], [1].

The usual approach to proving the Ruelle operator theorem and some variants involving self consistent transfer operators is to use cones and the Banach metric. However, we will adopt a slightly different approach.

2 Proof of Theorem 1.4

The proof of Theorem 1.4 is based on a couple of simple lemmas. We begin with a definition.

Definition 2.1 *For $C > 0$ we can define*

$$\Lambda_C = \left\{ v \in C(\mathbb{T}, \mathbb{R}) : 0 \leqslant v(x) \leqslant 1, v(x) \leqslant v(y)e^{Cd(x,y)} \text{ with } x, y \in \mathbb{T} \right\}.$$

Every set Λ_C at least contains the constant function 1, and thus is non-empty.

Lemma 2.2 *Λ_C has the following properties.*

1. *Λ_C is $\|\cdot\|_\infty$-closed and convex.*
2. *$\Lambda_C \subset \{v \in B : \|v\| \leqslant Ce\}$.*
3. *Λ_C is $\|\cdot\|_\infty$-compact.*

Proof. The first part is easily observed. For part 2, let $v \in \Lambda_C$ we can write

$$v(x) \leqslant v(y)e^{Cd(x,y)} \leqslant v(y)\left(C \cdot e \cdot d(x,y) + 1\right)$$

and interchanging x and y gives

$$v(y) \leqslant v(x)e^{Cd(x,y)} \leqslant v(x)\left(C \cdot e \cdot d(x,y) + 1\right)$$

using the mean value theorem. Together these give that $|v(x) - v(y)| \leqslant Ced(x,y)$ with the implied constant being uniformly bounded by compactness. Finally, Part 3 follows from the Arzela-Ascoli theorem.

Lemma 2.3 *Let $C = D\theta/(1-\theta)$. Then for a function $v \in \Lambda_C$ which is not identically zero we have $\mathcal{L}_{w(v)}v/\|\mathcal{L}_{w(v)}v\|_\infty \in \Lambda_C$.*

Proof. Let $x, y \in X$. We can use that $v \in \Lambda_C$ and the hypothesis to write

$$\begin{aligned}
&\mathcal{L}_{w(v)}v(x) \\
&= \sum_i e^{w(v)(T_i(v)x)} v(T_i(v)x) \\
&\leqslant \sum_i \left(e^{w(v)(T_i(v)y)} e^{\|w(v)\| d(T_i(v)x, T_i(v)y)}\right) \cdot \left(v(T_i(v)y)e^{Cd(T_i(v)x, T_i(v)y)}\right) \\
&\leqslant \mathcal{L}_{w(v)}v(y)e^{(D+C)\theta d(x,y)} \\
&= \mathcal{L}_{w(v)}v(y)e^{Cd(x,y)}
\end{aligned}$$

since $C = \frac{D\theta}{1-\theta}$ and $v \in U$, by virtue of part 2 of Lemma 2.2 and hypotheses III. Dividing through by $\|\mathcal{L}_{w(v)}v\|_\infty > 0$ gives the result.

Observe that Λ_C contains the constant functions taking values $1/n$ ($n \geqslant 1$). This allows us to make the following definition.

Definition 2.4 *For $n \geqslant 1$ and $v \in B$ we can define*

$$\mathbb{L}_n(v) = \frac{\mathcal{L}_{w(v)}(v+1/n)}{\|\mathcal{L}_{w(v)}(v+1/n)\|_\infty} \text{ for } n \geqslant 1.$$

We have the following corollary to Lemma 2.3.

Corollary 2.5 *For $n \geqslant n_0 := [1/\epsilon] + 1$ we can have that $\mathbb{L}_n(\Lambda_C) \subset \Lambda_C$.*

Proof of Theorem 1.4. By the Schauder fixed point theorem [1] there exists a fixed point $\mathbb{L}_n v_n = v_n \in \Lambda_C$ for each $n \geqslant n_0$. Moreover, one can deduce that if $\sup_x v_n(x) = 1$ then $\inf_x v_n(x) \geqslant e^{-A}$.

By compactness of Λ_C we can choose a convergent subsequence $v_{n_i} \to v_\infty \neq 0$. Then $\mathbb{L}_{w(v_\infty)}(v_\infty) = v_\infty$, i.e., $\mathcal{L}_{w(v_\infty)}v_\infty = \lambda v_\infty$, where $\lambda = \|\mathcal{L}_{w(v_\infty)}v_\infty\|_\infty > 0$.

Remark 2.6 *If hypothesis III on the size of the domain U seems unduly restrictive, we observe that replacing $w(\cdot)$ by $\delta > 0$ replaces D by δD (in Hypothesis I) thus reducing the radius of the ball that U is required to contain in Hypothesis III.*

References

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[1]If $K \subset V$ is a nonempty convex closed subset of a Hausdorff topological vector space and f is a continuous mapping of Kinto itself such that $f(K)$ is contained in a compact subset of K, then f has a fixed point.

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