Kneading sequences of triangles and tetrahedra

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March 11, 2021

1 Introduction

Given a triangle in the plane, there are variety of simple approaches to generate iteratively a sequence triangles $(T_n)_{n=0}^{\infty}$. If we are only interested in the shape of the triangles (up to isometry and scaling) then the nth triangle T_n can be represented by its internal angles $(\alpha_n, \beta_n, \gamma_n)$ and we can consider the convergence of the T_n in terms of the convergence of these triples of real numbers.

Historically, there have been a number of different constructions that have attracted attention. One simple approach is to associate to each triangle T_n the pedal triangle T_{n+1} whose vertices are the pedal points of T_n. This was studied by Kingston and Synge [3] and for typical choices of initial triangles T_0 the sequence doesn't converge. A second approach is to choose at random one of the six triangles given by the the barycentric subdivision. Iterating this construction Bárány, Beardon and Carne showed that limiting triangles are typically degenerate [1]. Finally, a third simple approach is to associate to the triangle T_n a new triangle T_{n+1} whose vertices are where the incircle of T_n touches the sides of that triangle. In this case it is a nice exercise to show that the shapes of the triangles $(T_n)_{n=0}^{\infty}$ converge to an equilateral triangle [2].

We will add to this list yet another type of sequence of triangles. We would like to consider a sequence of pairs of triangles and base sides $(T_n, E_n)_{n=0}^{\infty}$, where T_{n+1} is derived from T_n by shearing and then rotating, a process which we will call kneading.

Kneading triangles. The triangle T_n at each step is sheared by translating the opposite vertex parallel to the base E_n so as to minimize the total lengths of the sides, and the resulting triangle is T_{n+1}. This is followed by a rotation of the triangle to get another side E_{n+1} to be the base for T_{n+1} (see Figure 1).

In part, the motivation for this new construction comes from the topical study of the action of $SL(2, \mathbb{R})$ on polygons in the plane and the theory of translation surfaces (see

*The second author is partly supported by ERC-Advanced Grant 833802-Resonances and EPSRC grant EP/T001674/1

[4],§3). The effect of this action is to rotate, flatten or shear the polygon. We are therefore considering the special case of triangles and using only shearing and rotating, which are represented by the matrices $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$ (with $t \in \mathbb{R}$ and $\theta \in [0, 2\pi)$), respectively, multiplying vectors on $\mathbb{R}^2$.

In the next section we formulate our main theorem (Theorem 2.2) for sequences of triangles and provide a simple proof. In section 3 we consider the natural analogue of this construction for three dimensional tetrahedra.

2 Kneading triangles

It is convenient to normalize the triangles by taking T_0 to have unit area. Let $\mathcal{T}$ be the set of area 1 triangles up to rotation and translation. Given a triangle T_0 in the plane with a choice of edge E_0 consider the following sequence of unit area triangles and edges $(T_n, E_n)_{n=0}^{\infty}$ defined iteratively by the following kneading sequences which consist of a shearing operation followed by a rotation operation.

(a) Shearing: Given a triangle T_n with a distinguished edge E_n (the base), translate the vertex (the apex) opposite to the base along the line parallel to the base to get an isosceles triangle T_{n+1} with the two non-base edges of equal length.

(b) Rotation: Now let the next counter-clockwise edge E_{n+1} with respect to the current base E_n be the base of the new triangle.

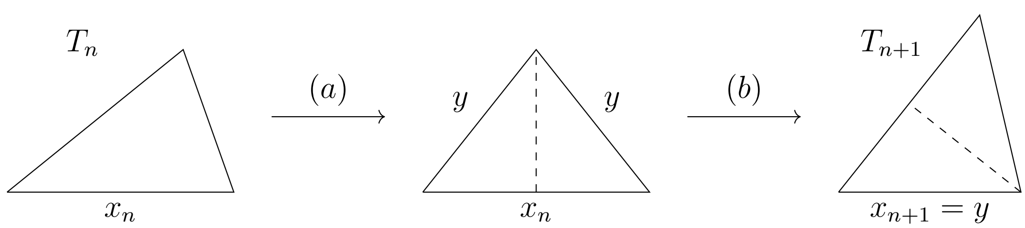

Figure 1: (a) The triangle is sheared so that apex lies above the midpoint of the base and the resultant triangle is isosceles; (b) The new isosceles triangle is rotated clockwise so that the next counter-clockwise edge with respect to the old base is the new base.

Of course, these two operations are area preserving. We denote by x_n the length of the base E_n. Using that the intermediate triangle is isoceles and of unit area we have the following simple relationship between x_{n+1} and x_n.

Lemma 2.1. *The length x_{n+1} of the base of the triangle T_{n+1} is related to the length of the base x_{n+1} of the triangle T_{n+1} by*

$$x_{n+1} := \sqrt{\left(\frac{x_n}{2}\right)^2 + \left(\frac{2}{x_n}\right)^2}.$$

Proof. If T_n has height h then the area is $1 = \frac{1}{2}hx_n$. Moreover, by Pythagorus' theorem $(x_n/2)^2 + h^2 = y^2 = x_{n+1}^2$ and the result follows. $\square$

It is an easy exercise to show that a unit area isosceles triangle has the smallest perimeter for triangles of a specified base and height. Furthermore, a simple calculation shows an equilateral triangle Δ of unit area has sides of length $2/\sqrt[4]{3}$.

We are ready to state and prove the main theorem for triangles.

Theorem 2.2. *For any choice of initial triangle T_0 and base E_0, the associated kneading sequence $(T_n, E_n)_{n=0}^{\infty}$ is such that T_n converges to the unique unit area equilateral triangle Δ (i.e., the internal angles $\alpha_n, \beta_n, \gamma_n$ of T_n converge to $\frac{\pi}{3}$).*

Proof. The theorem follows from the dynamics of the map of the positive real numbers given by

$$f : (0, +\infty) \to (0, +\infty) \text{ given by } f(x) = \sqrt{\left(\frac{x}{2}\right)^2 + \left(\frac{2}{x}\right)^2}.$$

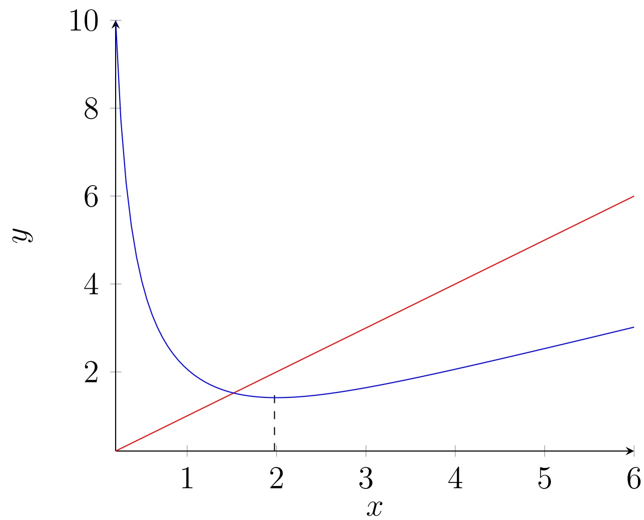

Figure 2: Plots of $y = f(x)$ and $y = x$.

If we denote $\alpha = 2/\sqrt[4]{3} = 1.51967\ldots > \beta = 2\sqrt[4]{2/\sqrt{3} - 1} = 1.2543\ldots$ then the following properties are easily checked:

1. the unique solution (or fixed point) for $f(x) = x$ occurs at $x = \alpha$;
2. the unique critical point for $f(x)$ occurs at $x = 2$ (i.e., $f'(2) = 0$) and $f(2) = \sqrt{2} = 1.41421\ldots$;
3. $f'(\beta) = -1$ and $|f'(x)| < 1$ for all $x \in (\beta, \infty)$; and
4. the image of $(0, +\infty)$ under f is contained in the interval (β, ∞) i.e.,

$$f((0, +\infty)) = [f(2), +\infty) = [\sqrt{2}, +\infty) \subset (\beta, \infty).$$

Since for all $n \geq 0$,

$$|x_{n+1} - \alpha| = |f(x_{n+1}) - f(\alpha)| < |x_{n+1} - \alpha|$$

we see that $x_n = f^n(x_0)$ converges to the unique fixed point α for all $x > 0$.

Using the fact that $(x_n)_{n=0}^{\infty}$ converges to α for all x, we see that each of the lengths of the edges of the unit area triangles T_n converge to α and hence deduce that $(T_n)_{n=0}^{\infty}$ converges to Δ. □

One can also see from the proof that the convergnce is exponential.

3 Kneading tetrahedra

We want to describe an analogous result for tetrahedra. Let $\mathcal{T}_1$ denote the space of unit volume tetrahedra considered up to isometry. The shear operation will be defined with respect to one of the faces of the tetrahedron, which we will refer to as the base of the tetrahedron.

Given an initial tetrahedron $\mathbb{T}_0 \in \mathcal{T}_1$ and a choice of face F_0 for $\mathbb{T}_0$ we can define a sequence $((\mathbb{T}_n, F_n))_{n=1}^{\infty}$ where $\mathbb{T}_n \in \mathcal{T}_1$ and F_n is one of the four faces of $\mathbb{T}_n$ designated as its base. We proceed iteratively as follows.

(a) Shearing: Given a tetrahedron $\mathbb{T}_n$ with a distinguished face F_n (the base), translate the opposite vertex in the plane parallel to F_n until we arrive at the tetrahedron $\mathbb{T}_{n+1}$ which minimizes the total surface area of such tetrahedra (Figure 3); and

(b) Rotation: We choose a different face F_{n+1} to be the base and let $\mathbb{T}_{n+1}$ be the corresponding tetrahedron.

When we constructed the sequences of kneading triangles, we rotated the triangle clockwise, or equivalently, selected the edge counter-clockwise to the base to be the new base. For tetrahedra there are now three possible edges that we could choose to be the new base. But we can simply choose to cycle through these peridocially, say, although other choices would also work.

We would like to consider the limit of the sequence $(\mathbb{T}_n)_{n=0}^{\infty}$, where convergence will correspond to the angles in the faces converging.

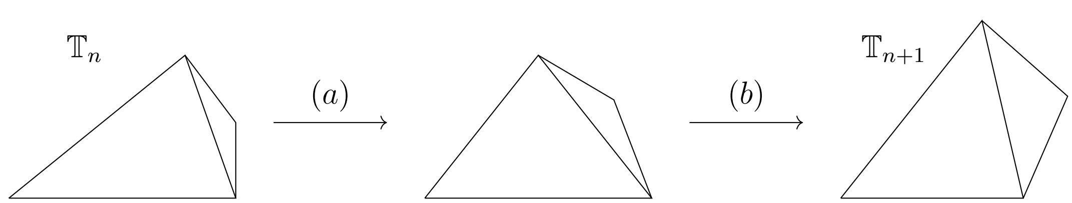

Figure 3: (a) The tetrahedron $\mathbb{T}_n$ is sheared to $\mathbb{T}_{n+1}$ so as me minimize its surface area; (b) The tetrahedron $\mathbb{T}_{n+1}$ is rotated so as to make another face F_{n+1} the base for T_{n+1}

Let $A : \mathcal{T}_1 \to \mathbb{R}^+$ be the function which associates the surface area of a unit volume tetrahedron. Let $\Delta \in \mathcal{T}_1$ denote the unique regular tetrahedron whose faces are all equilateral triangles. Then this is easily seen to give the minimum for $A(\cdot)$ on $\mathcal{T}_1$.

Remark 3.1. For $\mathbb{T} \in \mathcal{T}_1$ one has $\sqrt[3]{216\sqrt{3}} \leq A(\mathbb{T})$, with equality if and only if $\mathbb{T}$ is a regular tetrahedron.

In step (a) we choose $\mathbb{T}_{n+1}$ to minimize $A(\cdot)$ over the sheared tetrahedra arising from $\mathbb{T}_n$ with fixed face F_n. The next result shows that this occurs at the the tetrahedron whose apex lies above the incenter of the face F_n.

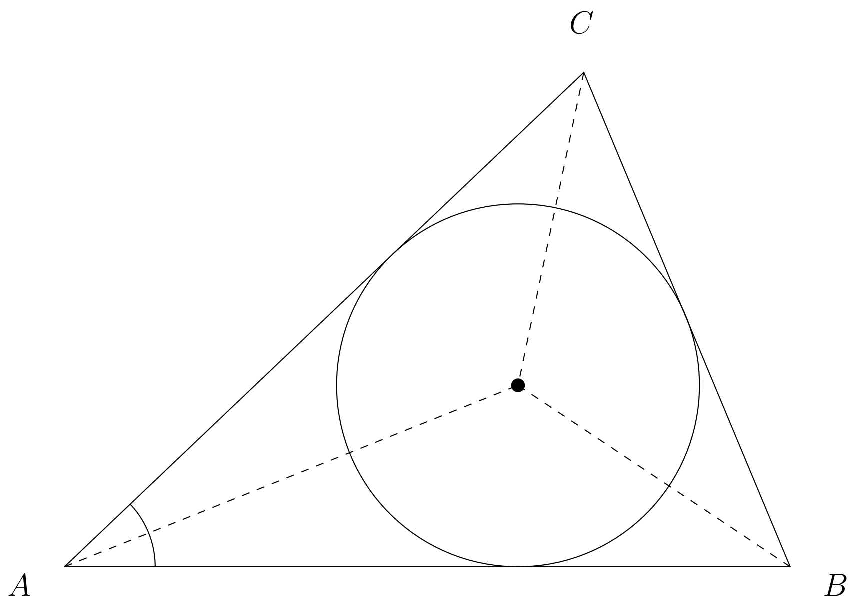

Figure 4: A triangle with its incenter represented by a black dot. The incenter is equidistant from each of the triangle's edges and the lines which connect the incenter to the vertices bisect the angle at the vertices.

Lemma 3.2. *Consider a tetrahedron $\mathbb{T}$ with base ABC and apex X. Let p denote the orthogonal projection of X to the plane containing ABC and let h denote the height of the tetrahedron, i.e. the distance from X to p. Then the following hold*

1. *The surface area of tetrahedra with base ABC and height h is minimised when p is the incenter of ABC.*

2. *The only tetrahedron which has all of its vertices above their opposite face's incenter is the regular tetrahedron.*

Proof. Let α, β, γ denote the lengths of edges BC, AC and AB, respectively. Let a, b, c denote the distances from p to BC, AC and AB, respectively.

For part 1, first note that the area of the triangular face XBC is given by $\frac{1}{2}\alpha\sqrt{a^2+h^2}$. Using similar formulae for the other faces of $\mathbb{T}$, the area of the union of the faces $XBC \cup XAC \cup XAB$ is given by

$$\frac{1}{2}\alpha\sqrt{a^2+h^2} + \frac{1}{2}\beta\sqrt{b^2+h^2} + \frac{1}{2}\gamma\sqrt{c^2+h^2}. \tag{1}$$

By applying the Minkowski inequality to (1), we see that this area is greater than or equal to

$$\frac{1}{2}\sqrt{h^2(\alpha+\beta+\gamma)^2+(a\alpha+b\beta+c\gamma)^2}, \tag{2}$$

with equality if and only if $a=b=c$, i.e. p is the incenter of ABC.

Observe that the expression $h^2(\alpha+\beta+\gamma)^2$ in (2) is independent of X and $a\alpha+b\beta+c\gamma$ is twice the sum of the areas of the triangles pBC, pAC and pAB in the base face. It is easy to see that the projection of these faces cover ABC, hence $(a\alpha+b\beta+c\gamma)$ is greater than or equal to twice the area of the base ABC, $\text{Area}(ABC)$, with equality if and only if p lies inside ABC.

Finally, combining these inequalities, we see that the total surface area of a tetrahedron with base ABC and height h is greater than or equal to

$$\frac{1}{2}\sqrt{h^2(\alpha+\beta+\gamma)^2+(2\text{Area}(ABC))^2}+\text{Area}(ABC),$$

with equality if and only if p is the incenter of ABC.

For part 2, recall that the incenter of a triangle is the unique point on the triangle such that the lines connecting the incenter to the triangle's vertices biscect the angles at the vertices. Consequently, if a tetrahedron is such that each vertex lies above the opposite face's incenter, then the three angles around any vertex will be equal. This can only hold if the tetrahedron is regular. □

We can now prove the following analogue for tetrahedra of Theorem 2.2 for triangles.

Theorem 3.3. *For any $\mathbb{T}_0$ and face F_0 the sequence $((\mathbb{T}_n, F_n))_{n=0}^{\infty}$ is such that $\mathbb{T}_n$ converges to the unique unit volume tetrahedron Δ.*

Proof. Let $A_n := A(\mathbb{T}_n)$ for all $n \geq 1$. By construction, $A_{n+1} \leq A_n$ and so the bounded non-increasing sequence $(A_n)_{n=1}^{\infty}$ of areas converges to some value $A^* > 0$. Moreover, the sequence $(\mathbb{T}_n)_{n=0}^{\infty}$ of tetrahedra are bounded and $A : \mathcal{T}_1 \to \mathbb{R}$ is continuous. Therefore we can choose a convergent subsequence $\mathbb{T}_{n_k} \to \mathbb{T}^* \in \mathcal{T}_1$ and observe that $A(\mathbb{T}^*) = A^*$.

However, by part 2 of Lemma 3.2, this implies that the each vertex of $\mathbb{T}^*$ lies above its opposite face's incenter. Consequently, by part 1 of Lemma 3.2, $\mathbb{T}^* = \Delta$ We therefore conclude that $\mathbb{T}_{n_k} \to \Delta$ for any subsequence, and thus the original sequence $\mathbb{T}_n \to \Delta$ as $n \to +\infty$. □

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