Characterizations of complex dimensions for attractors of iterated function schemes

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Abstract

We will consider Cantor sets in the unit interval $[0, 1]$ which occur as the limit sets of C^2 cookie cutter (or iterated function schemes satisfying the strong separation condition). We will be particularly interested in the meromorphic domains of the associated complex functions. The first is the spectral zeta function (defined in terms of the lengths of the complimentary intervals) introduced by Lapidus. The second is the fixed point zeta function (defined, as the name suggests, in terms of fixed points of maps). Finally, we introduce a third complex function, the Poincaré series. Each of these tend to have meromorphic extensions to different half-planes, but when two overlap the associated poles are usually common to both. This suggests the definition of additional "complex dimensions" (i.e., poles of the spectral zeta function) for more general cookie cutters in terms of poles of these other complex functions.

Contents

1 Introduction

There has been considerable interest in recent years in finding analogues of concepts and results for Riemannian manifolds in the context of fractal geometry. A particularly simple setting is the following.

1.1 Iterated function schemes and their attractors

We want to consider a construction of particular Cantor sets $X \subset [0,1]$. Let $T_i : [0,1] \to [0,1]$ (for $i = 1, \cdots, k$ with $k \geqslant 2$) be C^2 contractions with

$$T_i([0,1]) \cap T_j([0,1]) \neq \emptyset \text{ for } i \neq j$$

i.e., an iterated function scheme satisfying the strong separation condition.

We recall Hutchinson's classical result (cf [3]) that there is a unique closed non-empty set $X \subset [0,1]$ such that $X = \cup_{i=1}^k T_i(X)$.

Definition 1.1 *The set X in this case is called a* cookie cutter.

When $X \subset [0,1]$ is the Cantor set for an iterated function scheme then its Box dimension D exists and is equal to its Hausdorff dimension.

In the special case that the contractions are affine this reduces to the self-similar Cantor set cases already comprehensively studied in [7] and [8].

Example 1.2 (Self-affine Cantor sets) *Consider affine contractions $T_j : [0,1] \to [0,1]$ defined by*

$$T_j(x) = xr_j + b_j \textit{ where } 0 < |r_j| < 1 \quad (j = 1, \ldots, k)$$

with the strong separation condition. The resulting Cantor set X is called self-affine.

There are many other classical examples which are not self-affine but are covered by the setting of the more general iterated function schemes. One of the best known is the following.

Example 1.3 (Continued fraction digits $E_{n,m}$) *We can let $T_1(x) = 1/(n+x)$ and $T_2(x) = 1/(m+x)$ where $m > n \geqslant 1$ are natural numbers. The resulting set is denoted $X = E_{n,m}$ and consists of those numbers whose infinite continued fraction expansions contains only the digits n and m.* [1]

1.2 A tale of two zeta functions (and a Poincaré series)

Consider a Cantor set $X \subset \mathbb{R}$ in the real line. Let $\mathcal{L} = \{\ell_j\}_{j \geqslant 1}$ be the lengths of the countable family of maximal bounded intervals (a_j, b_j) $(j \geqslant 1)$ in the compliment $\mathbb{R} \backslash X$, i.e., $\ell_j = b_j - a_j$. By convention we can choose a scaling such that $0 = \inf(X)$ and $1 = \sup(K)$ and the labelling of the lengths of the intervals such that

$$\ell_1 \geqslant \ell_2 \geqslant \ell_3 \geqslant \cdots \geqslant \ell_n \geqslant \cdots\cdots$$

Following ([8], p. 17), we can combine these length data into a complex function $\zeta_{\mathcal{L}}(s)$ of a single variable $s \in \mathbb{R}$.

[1] There has been interest in computing the Hausdorff dimension of $E_{n,m}$ and, in particular, $E_{1,2}$ [6]

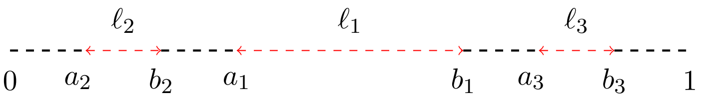

Figure 1: Intervals removed ordered by lengths $\ell_1 \geqslant \ell_2 \geqslant \ell_3 \geqslant \cdots$

1.2.1 The spectral zeta function

We begin with the first zeta function we want to consider.

Definition 1.4 *We formally define the* spectral zeta function $\zeta_{\mathcal{L}}(s)$ *to be the function of a single complex variable formally given by the Dirichlet series*

$$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} \ell_j^s \text{ for } s \in \mathbb{C}.$$

It is easy to see that the function $\zeta_{\mathcal{L}}(s)$ converges to an analytic function for $Re(s) > 1$. Moreover, the abscissa of convergence of $\zeta_{\mathcal{L}}(s)$

$$\delta = \inf\{\sigma > 0 : \zeta_{\mathcal{L}}(\sigma) < +\infty\}$$

is the (upper) box dimension of X (denoted $\overline{\dim}_B(X)$) (see [8], Theorem 1.10, p.17).

The novelty of the next result is the generality of the contractions allowed in the associated iterated function scheme.

Theorem 1.5 *Let X be the attractor associated to a C^2 cookie cutter. There exists $0 < a < \delta$ such that $\zeta_{\mathcal{L}}(s)$ has a meromorphic extension to the larger half-plane $\{s \in \mathbb{C} : Re(s) > a\}$.*

The motivation for this result comes from the locally constant setting.

Example 1.6 (Semi-similar Cantor sets (Lapidus)) *In the particular case that the $\{T_i\}$ were affine contractions then $\zeta_{\mathcal{L}}(s)$ would have a meromorphic extension to $\mathbb{C}$.*

Thus the price we pay for the greater generality of the setting in Theorem 1.5 is the reduction in the domain (compared to the extension to $\mathbb{C}$ for affine contractions in Example 1.6).

1.2.2 A fixed point zeta function

A second useful zeta function associated to the limit set of an iterated function scheme (with contractions $T_j : [0,1] \to [0,1]$ for $1 \leqslant j \leqslant k$) is defined using the weights

$$w(i_1, \cdots, i_n) := |(T_{i_1} \circ \cdots \circ T_{i_n})'(x_{i_1,\cdots,i_n})|$$

for $i_1, \cdots, i_n \in \{1, \cdots, k\}$ where

$$x_{i_1,\cdots,i_n} = (T_{i_1} \circ \cdots \circ T_{i_n})(x_{i_1,\cdots,i_n})$$

is an associated fixed point for a composition of contractions.

Definition 1.7 *We can formally define a (Ruelle-type)* fixed point zeta function *by*

$$R_{\mathcal{W}}(s) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{i_1,\cdots,i_n} w(i_1, \cdots, i_n)^s\right)$$

For Cantor sets X arising from general C^2 iterated function schemes that we have the following result, which is essentially due to Morita [10](see also McMonagle [9]).

The novelty of the next result is the generality of the contractions in the associated iterated function scheme and the surprising size of the extension. .

Proposition 1.8 (after Morita) *For C^2 iterated function schemes the zeta function $R_{\mathcal{W}}(s)$ has an analytic extension to the larger half-plane*

$$\{s \in \mathbb{C} : Re(s) > 0\}.$$

Furthermore, the poles for $\zeta_{\mathcal{L}}(s)$ coincide with the poles for $R_{\mathcal{W}}(s)$ provided $Re(s) > a$.

Thus for C^2 attractors the fixed point zeta function $R_{\mathcal{W}}(s)$ has the advantage over $\zeta_{\mathcal{L}}(s)$ of having a uniform meromorphic extension to the right half plane. With greater regularity it is known we can obtain a larger extension

Proposition 1.9 (after Tangerman) *For C^∞ iterated function schemes $R_{\mathcal{W}}(s)$ has a non-zero meromorphic extension to the entire complex plane $\mathbb{C}$.* [2]

In the case of C^ω functions this is well known [17]. In the special case of self-similar Cantor sets not only is it known that $R_{\mathcal{W}}(s)$ has a meromorphic extension to $\mathbb{C}$, but there is simple closed form.

Example 1.10 (Self-affine Cantor sets) *As is well known, we can write $R_{\mathcal{W}}(s) = 1/\det(I - P^s)$ where P is a finite matrix and P^s has the entries raised to the power $s \in \mathbb{C}$.*

Remark 1.11 (Selberg type zeta functions) *There is a variant on the Ruelle type zeta functions might also have been considered. We can formally define*

$$S_{\mathcal{W}}(s) = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \sum_{i_1,\cdots,i_n} \frac{w(i_1,\cdots,i_n)^s}{1 - w(i_1,\cdots,i_n)}\right)$$

and then note that this is intimately related to the Ruelle type zeta functions by $R_{\mathcal{W}}(s) = S_{\mathcal{W}}(s+1)/S_{\mathcal{W}}(s)$. In particular, the zeros of the Selberg zeta function formally occur in the same place as the poles of $R_{\mathcal{W}}(s)$ on $Re(s) > \delta - 1$, whenever s lies in the meromorphic extension of both functions. The definition of $S_{\mathcal{W}}(s)$ occurs naturally as a determinant of a transfer operator in the special case that the cookie cutter is real analytic.

1.2.3 Poincaré series

Finally, we can introduce a third complex function associated to an iterated function scheme (with contractions $T_j : [0,1] \to [0,1]$ for $1 \leqslant j \leqslant k$).

Definition 1.12 *If we fix a value $x_0 \in [0,1]$ then we can formally define a* Poincaré series

$$\eta_{x_0}(s) = \sum_{n=1}^{\infty} \sum_{i_1,\cdots,i_k} |(T_{i_1} \circ \cdots \circ T_{i_n})'(x_0)|^s,$$

for $s \in \mathbb{C}$.

This converges to an analytic function for $Re(s) > \delta$. Moreover, there is a meromorphic extension given by the following result.

Theorem 1.13 *The Poincaré series $\eta_{x_0}(s)$ has a meromorphic extension to $Re(s) > 0$. Moreover, for $Re(s) > 0$ the poles for $\eta_{x_0}(s)$ correspond to the poles for $R_{\mathcal{W}}(s)$, and for $Re(s) > a$ the poles agree with the poles for $\zeta_{\mathcal{L}}(s)$.*

Whereas the Poincaré series $\eta_{x_0}(s)$ itself depends specific on the choice of x_0 the poles are common to all choices. The result follows from the common use of the transfer operator to extend the complex functions and the interpretation of the poles as where the operator has 1 as an eigenvalue.

Thus for C^2 attractors the Poincaré series $\eta_{x_0}(s)$ also has the advantage over $\zeta_{\mathcal{L}}(s)$ of having a uniform meromorphic extension to the right half plane $Re(s) > 0$.

With greater regularity it is well known we can obtain a larger extension.

[2]The result of Tangerman essentially appeared in his (unpublished?) PhD thesis. But it follows from subsequent work of Ruelle [17]

Proposition 1.14 *For C^∞ iterated function schemes $\eta_{x_0}(s)$ has a non-zero meromorphic extension to the entire complex plane $\mathbb{C}$.*

In the special case of self affine Cantor sets it is easy to see there is a meromorphic extension to $\mathbb{C}$.

Example 1.15 (Self affine Cantor sets) *For self affine Cantor sets one can see that the meromorphic $\eta_{x_0}(s)$ has a meromorphic extension to $\mathbb{C}$ with an explicit closed form.*

1.3 Complex dimensions

Following Lapidus, the poles of $\zeta_{\mathcal{L}}(s)$ give rise to an interesting definition of (more) notions of dimensions for X ([8], Definition 1.12, p.19). More precisely, there is the following defintion

Definition 1.16 *The* complex dimensions *are the countable set of poles of $\zeta_{\mathcal{L}}(s)$ in the meromorphic extension and are denoted by $\mathcal{D} \subset \mathbb{C}$.*

This naturally generalizes the box dimension δ being a pole for $\zeta_{\mathcal{L}}(s)$. [3] The original setting for the definition of complex dimensions was for self-similar Cantor sets.

Example 1.17 (Complex dimensions for self-affine Cantor sets) *In the case of self-affine attractors X in the real line, the zeta functions $\zeta_{\mathcal{L}}(s)$ are seen to have a meromorphic extension to $\mathbb{C}$ [8]. The corresponding zeta function $\zeta_{\mathcal{L}}(s)$ has an explicit form (cf. Theorem 2.4 on p. 38 of [8]) and the complex dimensions correspond to the set of solutions:*

$$\mathcal{D} = \left\{ s \in \mathbb{C} : \sum_{j=1}^{k} r_j^s = 1 \right\}$$

(see Corollary 2.5 on p. 39 of [8]). Many examples are illustrated in ([8], pp. 43-54 and pp.66-71).

In light of the complex zeros originally being defined as zeros for $\zeta_{\mathcal{L}}(s)$ for self-similar Cantor sets, then a natural generalization of the complex dimensions to more general limit sets X can be interpreted as the poles of any of the zeta functions or Poincaré series.

	$\zeta_{\mathcal{L}}(s)$	$R_{\mathcal{W}}(s)$	$\eta_{x_0}(s)$
Self similar	$s \in \mathbb{C}$	$s \in \mathbb{C}$	$s \in \mathbb{C}$
C^2	$Re(s) > a$	$Re(s) > 0$	$Re(s) > 0$
C^∞	$Re(s) > a$	$s \in \mathbb{C}$	$s \in \mathbb{C}$

Table 1: The meromorphic domains of complex functions for iterated function schemes with various regularities

Summary. In the general setting of limit sets X for iterated function schemes of C^2 contractions the fixed point zeta function $R_{\mathcal{L}}(s)$ and Poincaré series $\eta_{x_0}(s)$ have appropriate domains of meromorphic extension and so provide information on their poles, or complex dimensions (see Table 1).

For C^∞ contractions we have no additional information on the meromorphic domain of the spectral zeta function $\zeta_{\mathcal{L}}(s)$. However, there are stronger results for $R_{\mathcal{L}}(s)$ and $\eta(s)$.

Theorem 1.18 *If the contractions are C^∞ then $R_{\mathcal{L}}(s)$ and $\eta_{\mathcal{L}}(s)$ have a meromorphic extension to $\mathbb{C}$.*

Thus although the complex dimensions were defined via the the zeta function $\zeta_{\mathcal{L}}(s)$ these, and potentially far more values besides, can be seen via the new functions $R_{\mathcal{L}}(s)$ and $\eta_{x_0}(s)$.

[3] In [8] there is an interesting characterization of the fractal nature ("fractality") in terms of the poles in the meromorphic extension.

2 Examples

We want to illustrate the results with some natural examples. We begin with a simplified case of the original setting of self-affine Cantor sets considered by Lapidus.

2.1 Self-affine Cantor sets

Let $0 < r_1, r_2 < 1$ with $r_1 + r_2 < 1$. We can then define $T_1, T_2 : [0,1] \to [0,1]$ by

$$T_1(x) = r_1 x \text{ and } T_2(x) = r_2 x + (1 - r_2)$$

then they satisfy the strong separation condition.

As is well known, the Ruelle zeta function in this case takes a simple form

Lemma 2.1 *We can write the Ruelle zeta function in the closed form*

$$R_{\mathcal{W}}(s) = \frac{1}{1 - r_1^s - r_2^s} \qquad \text{for } s \in \mathbb{C}.$$

For easy of notation, we can write $\alpha = |\log r_1|, \beta = |\log r_2| > 0$ and then we want to understand why in some cases the complex zeros s, which occur as the solutions $s = \sigma + it$ to the equation

$$r_1^s + r_2^s = e^{-s\alpha} + e^{-s\beta} = 1 \tag{1}$$

are sufficiently regular so as to appear to lie on curves. This should be related to the diophantine properties of the ratio $\beta/\alpha > 0$, as illustrated in the following examples. In particular, in these examples we chose α and β for which the plots of zeros are particularly distinctive.

Example 2.2 *Consider $\beta = 1$ and $\alpha = \pi$ (or equivalently $r_1 = e^{-1}$ and $r_2 = e^{-\pi}$) then*

$$\frac{\beta}{\alpha} = \frac{7}{22} + \delta$$

where $\delta = 0.000128068\cdots$ (which is small compared with $1/22 = 0.04545454545\ldots$)

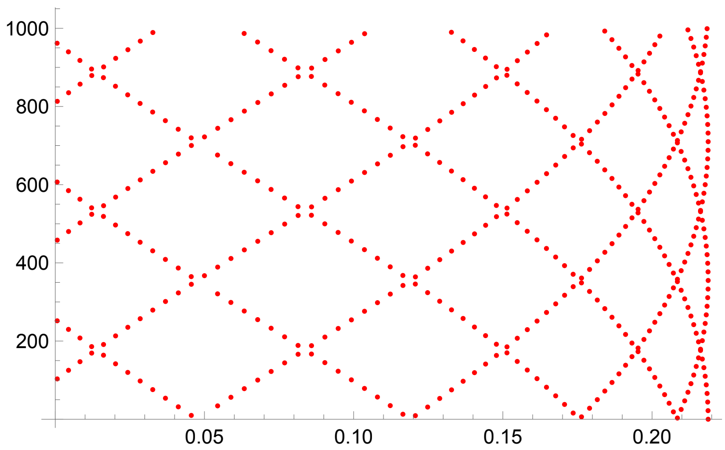

Figure 2: The complex dimensions for Example 2.1 with $0 < Re(s) < 0.18$ and $0 \leqslant Im(s) \leqslant 1000$

Example 2.3 *Consider $\beta = \log((\sqrt{53} - 7)/2)$ and $\alpha = \log((\sqrt{85} - 9)/2)$* [4] *then*

$$\frac{\beta}{\alpha} = 0.8897288842\ldots = \frac{8}{9} + \delta$$

where $\delta = 0.000839995\cdots$ (which is small compared with $1/9 = 0.1111111111\cdots$).

In an appendix we give a heuristic argument explaining the patterns observed in these two self-affine examples.

[4] The curious choice of values is explained in the next subsection

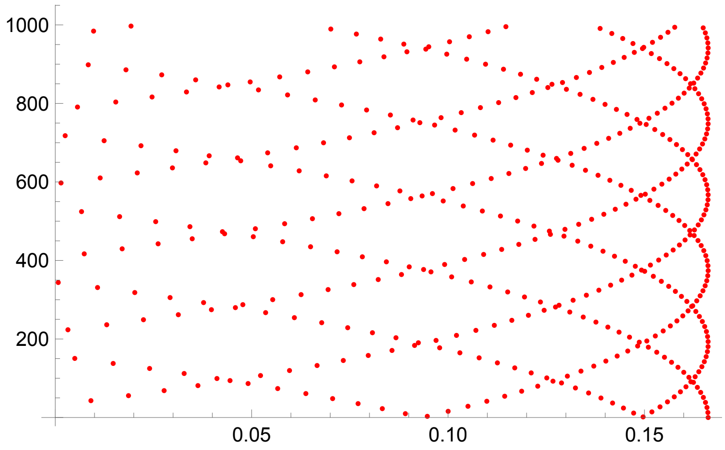

Figure 3: The complex dimensions for Example B.2 with $0 < Re(s) < 0.18$ and $0 \leqslant Im(s) \leqslant 1000$

2.2 Continued fraction Cantor sets $E_{n,m}$

We can consider Cantor sets $E_{n,m}$ which are limit sets of contractions $T_n, T_m : [0,1] \to [0,1]$, where $n, m \in \mathbb{N}$ with $n \neq m$, defined by

$$T_n(x) = \frac{1}{x+n} \text{ and } T_m(x) = \frac{1}{x+m}.$$

Each of these contractions will have distinct fixed points $T_n(x_n) = x_n$ and $T_m(x_m) = x_m$ where $0 < x_n < 1$, for example, has the continued fraction expansion consisting only of the digit n, i.e.,

$$x_n = [n, n, n, \ldots] = \frac{1}{n + \frac{1}{n + \ldots}}.$$

We can denote the absolute value of the derivatives of these maps at their respective fixed points by

$$r_1 = |T_m'(x_m)| \text{ and } r_2 = |T_n'(x_n)|$$

Unfortunately, there is no closed form for the Ruelle zeta function and so some approximation is required.

2.2.1 Another zeta function more appropriate for computation

In particular, for approximating the poles of the Ruelle zeta function it is more convenient to briefly a third type of zeta function whose zeros help to locate the complex dimensions for $E_{n,m}$.

Definition 2.4 *We formally define a* Selberg type *zeta function*

$$S_{\mathcal{W}}(s) = \exp\left(- \sum_{n=1}^{\infty} \frac{1}{n} \sum_{i_1, \cdots, i_n} \frac{|(T_{i_1} \circ \cdots \circ T_{i_n})'(x_{i_1, \cdots, i_n})|^s}{1 - (T_{i_1} \circ \cdots \circ T_{i_n})'(x_{i_1, \cdots, i_n})} \right)$$

since its zeros in $\delta - 1 < Re(s) < \delta$ agree with the poles of $R_{\mathcal{W}}(s)$ in the same region. This converges for $Re(s) > \delta$. The definition is inspired by the classical Selberg zeta function for closed geodesics on a Riemann surface [18], [5]. Formally

$$R_{\mathcal{W}}(s) = \frac{S_{\mathcal{W}}(s+1)}{S_{\mathcal{W}}(s)}$$

To understand the advantages of this zeta function we can introduce a (dummy) variable $z \in \mathbb{C}$ and define

$$S_{\mathcal{W}}(z, s) = \exp\left(- \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{i_1, \cdots, i_n} \frac{|(T_{i_1} \circ \cdots \circ T_{i_n})'(x_{i_1, \cdots, i_n})|^s}{1 - (T_{i_1} \circ \cdots \circ T_{i_n})'(x_{i_1, \cdots, i_n})} \right)$$

and carry out the formal expansion

$$S_{\mathcal{W}}(z,s) = 1 + \sum_{k=1}^{\infty} a_k(s) z^k.$$

The following is well known (and dates back to the original work of Ruelle [17] and Fried [4], and earlier work of Grothendieck).

Lemma 2.5 *There exists $0 < \rho < 1$ such that for s in a bounded region one can choose $C > 0$ and such that $|a_n(s)| \leqslant C\rho^{n^2}$.*

In particular,

$$S_{\mathcal{W}}(s) = S_{\mathcal{W}}(1,s) = 1 + a_1(s) + \sum_{k=2}^{\infty} a_k(s)$$

converges to analytic function for all $s \in \mathbb{C}$. This corresponds to the case of a self-similar Cantor set.

2.2.2 Plotting complex dimensions using $S_{\mathcal{W}}(s)$.

The expansion in Lemma 2.5 provides a route to approximating plots of the zeros of $S_{\mathcal{W}}(s)$. In particular, by truncating the series we obtain a finite Dirichlet series whose zeros can be more easily estimated and empirically are seen to lie close to those of the original function $S_{\mathcal{W}}(s)$. For example, taking just the first terms $1 + a_1(s)$ of $S_{\mathcal{W}}(s)$ takes the simple form [5]

$$1 + a_1(s) = 1 + \frac{\alpha_1^{2s}}{1+\alpha_1^2} + \frac{\alpha_2^{2s}}{1+\alpha_2^2}.$$

It is interesting to compare the zeros of $S_{\mathcal{W}}(s)$ and $1 + a_1(s)$.

Example 2.6 (**$E_{n,m}$ with $n = 7$ and $m = 9$**) *As a first approximation we might try to compare the complex dimensions for the Cantor set $E_{7,9}$ (i.e., the limit set associated to $T_7, T_9 : [0,1] \to [0,1]$)* [6] *with the zeros of the simpler function $1 + a_1(s)$.*

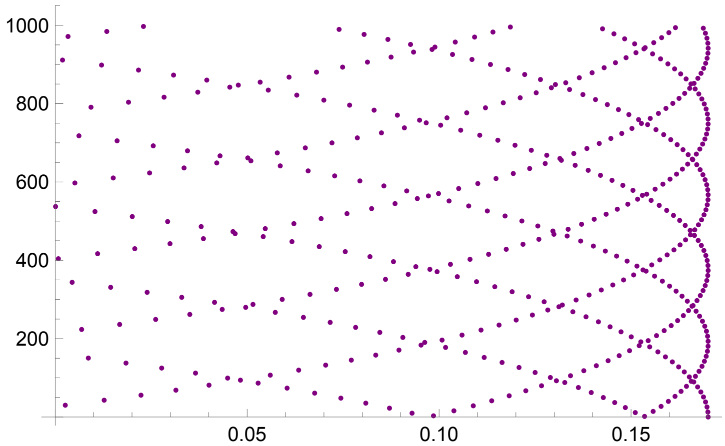

Figure 4: The poles of $1 + a_1(s)$ for $0 < Re(s) < 1.8$ and $0 \leqslant Im(s) \leqslant 1000$

[5]The positive signs in the denominator come from T_n and T_m being orientation reversing

[6]The derivatives at the two fixed points are $r_1 = (\sqrt{53}-7)/2$ and $r_2 = (\sqrt{53}-7)/2$, explaining the choice of values for the self-affine example in the precious subsection

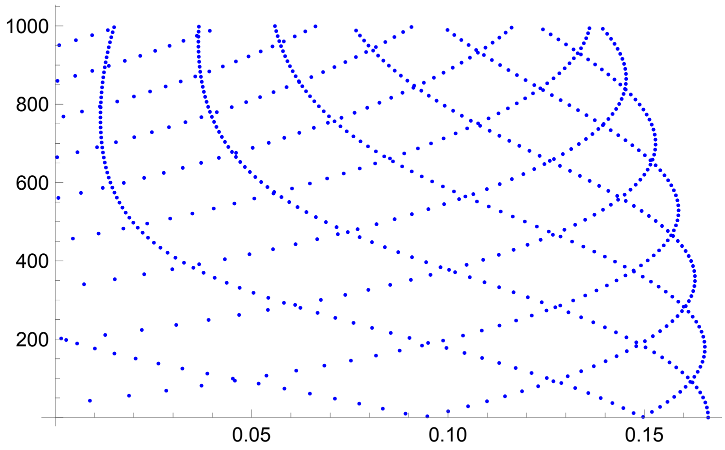

Figure 5: The complex dimensions for E_{79} for $0 < Re(s) < 1.8$ and $0 \leqslant Im(s) \leqslant 1000$

We can bound the difference

$$|S_{\mathcal{W}}(s) - (1 + a_1(s))| = \sum_{n=2}^{\infty} |a_n(s)|$$

Furthermore, we can bound the infinite series using Lemma 2.5, for example, or the more elaborate bounds in [6]. Empirically, this naive approach works surprisingly well providing $Im(s)$ is not too large.

However, a basic approach to compare the zeros of $S_{\mathbb{W}}(s)$ and $1 + a_1(s)$ is to use Rouché's theorem as follows.

Proposition 2.7 *If s_n is a zero for $1 + a_1(s)$ and for some $\epsilon > 0$ we have*

$$|S_{\mathcal{W}}(s) - (1 + a_1(s))| < |1 + a_1(s)|$$

whenever $s \in \Gamma \in \{s \in \mathbb{C} : |s - s_n| = \epsilon\}$ then there exists a zero s_n' for $S_{\mathcal{W}}(s)$ with $|s_n - s_n'| < \epsilon$.

3 The proofs of Theorems 1.5, 1.9 and 1.13

In the first subsection we introduce one of the principle tools in the proofs of Theorems 1.9 and 1.13.

3.1 Transfer operators and their spectra

We need to study the family of associated transfer operators. These are a class of bounded linear operators on functions (on the attractor) which has proved very useful in dynamical systems. To work at the level of generality of C^2 iterated function schemes we will follow the approach of Morita [10].

Definition 3.1 *Consider the Banach space of Lipschitz functions*

$$C^{Lip}([0,1]) = \{f : [0,1] \to \mathbb{R} : \|f\|_{Lip} < +\infty\}$$

with norm $\|f\| = \|f\|_{Lip} + \|f\|_\infty$ where

$$\|f\|_{Lip} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|} \text{ and } \|f\|_\infty = \sup_x |f(x)|.$$

We are particularly interested in the following family of linear operators.

Definition 3.2 *Given $s \in \mathbb{C}$, we can define the family of* transfer operators $\mathcal{L}_s$: *$C^{Lip}([0,1]) \to C^{Lip}([0,1])$ for $Re(s) > 0$ by*

$$\mathcal{L}_s f(x) = \sum_{i=1}^{k} |T_i'(x)|^s f(T_i x).$$

The *spectral radius* $\rho(\mathcal{L}_s)$ of $\mathcal{L}_s$ is given by the standard spectral radius theorem:

$$\rho(\mathcal{L}_s) = \lim_{n\to+\infty} \|\mathcal{L}_s^n\|^{1/n}$$

where $\|\mathcal{L}_s^n\| = \sup\{\|\mathcal{L}_s f\| : \|f\| \leqslant 1\}$ is the operator norm.

Definition 3.3 *The essential spectral radius is defined to be of the form*

$$\rho_e(\mathcal{L}_s) = \lim_{n\to+\infty} \inf\{\|\mathcal{L}_s^n + K\|^{1/n} : K = \text{compact operator on } C^{Lip}([0,1])\}.$$

In particular, for any $\epsilon > 0$ the intersection of the spectrum of the operator $\mathcal{L}_s$ with the region $|z| > \rho_e(\mathcal{L}_s) + \epsilon$ has only finitely many eigenvalues of finite multiplicity.

We require the following result of Morita (the interesting proof of which we recall and summarize in an appendix).

Proposition 3.4 (Morita) *If $Re(s) > 0$ then the essential spectral radius $\rho_e(\mathcal{L}_s)$ of $\mathcal{L}_s : C^{Lip}([0,1]) \to C^{Lip}([0,1])$ is strictly smaller than* 1.

Remark 3.5 *In fact, in Morita's result Proposition 3.4 the key technical hypothesis is that $C > 0$ and $0 < \kappa < 1$ such that*

$$\sum_{I\in\mathcal{I}_n} \mathrm{diam}(I) \leqslant C\kappa^n,$$

where $\mathcal{I}_n := \{T_{i_1} \cdot \dots \cdot T_{i_n}[0,1]\}$ are nth level intervals and then the actual conclusion is that for any $\rho > 1$ sufficiently small that $\rho\kappa < 1$ then

$$\rho_e(\mathcal{L}_s) \leqslant \rho\kappa < 1.$$

The proofs of Theorems 1.9 and 1.13 now follow relatively easily from this result on the transfer operator.

3.2 The proof of Theorem 1.9

This follows almost immediately from the work of Morita [10] and McMonagle ([9], §3.1.5). The zeta function $Z_{\mathcal{L}}$ can be related to the spectrum of the transfer operator. From the proof it is known that s is a pole for $Z_{\mathcal{L}}(s)$ whenever $\mathcal{L}_s$ has 1 as an eigenvalue.

3.3 The proof of Theorem 1.13

This result is also a straightforward result given Morita's result (Proposition 3.4) on the spectrum of the operator $\mathcal{L}_s$.

The Poincaré series $\eta_{\mathcal{L}}(s)$ can be written in terms of $\mathcal{L}_s$ (for $Re(s) > \delta$, say) as the convergent series

$$\eta(s) = \sum_{n=1}^{\infty} (\mathcal{L}_s^n 1)(x_0) = \mathcal{L}_s(1-\mathcal{L}_s)^{-1}1(x_0)$$

where $1(x)$ denotes the constant function 1. It follows from Proposition 3.4 that for $Re(s) > 0$ the essential spectral radius of $\mathcal{L}_s$ is (strictly) smaller than 1 and thus we can write

$$\mathcal{L}_s^n = \sum_{i=1}^{k} P_{i,s}^n + U_{s,n}$$

where $P_{i,s}$ are eigenprojections and $\lim_{n\to+\infty} \|U_{s,n}\|^{1/n} < 1$ uniformly in s on bounded domains. We can then write that

$$\eta(s) = \sum_{i=1}^{k} \left(P_{i,s}(I - P_{i,s})^{-1}\right) 1(x_0) + \left(U_{n,s}(I - U_{n,s})^{-1}\right) 1(x_0).$$

The last term is analytic for $Re(s) > 0$. The first term contributes a meromorphic function on the same domain $Re(s) > 0$ where the poles for $\eta(s)$ on $Re(s) > 0$ come from where 1 is an eigenvalue associated to $P_{i,s}$.

3.4 The proof of Theorem 1.5

The proof of this result relies on symbolic dynamics. In particular, we need to replace Proposition 3.4 by the corresponding result on the transfer operator for Hölder functions on subshifts of finite type.

We begin with some useful notation.

Definition 3.6 *We want to consider the sequence space* $\Sigma_2 = \{1, \cdots, k\}^{\mathbb{N}_0}$ *and the shift map* $\sigma : \Sigma \to \Sigma$ *defined by* $\sigma(x_n)_{n=0}^{\infty} = (x_{n+1})_{n=0}^{\infty}$. *Given a finite word* $\underline{i} = (i_1, \ldots, i_n) \in \{1, \cdots, k\}^n$, *for some* $n \geqslant 1$, *we can associate the maps* $T_{\underline{i}} = T_{i_1} \circ \cdots \circ T_{i_n} : [0,1] \to [0,1]$.

The following basic lemma is easily seen to hold (and the basic idea is illustrated in Figure 4).

Lemma 3.7 *The locally maximal intervals* [7] (a_j, b_j), *for* $j \geqslant 1$, *correspond to points* $a_j = T_{\underline{i}}y$ *and* $b_j = T_{\underline{i}}x$, *with* $x = T_1(0)$ *and* $y = T_2(1)$, *for suitable* $\underline{i} \in \{1, \cdots, k\}^n$.

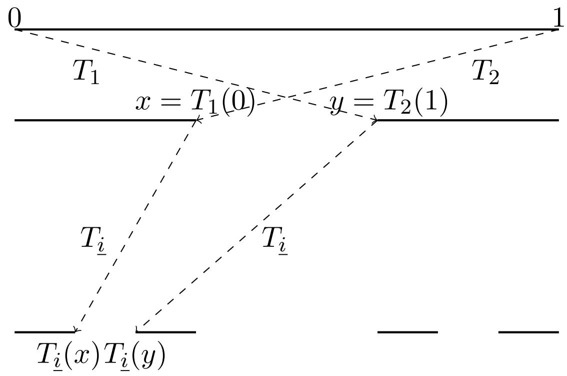

Figure 6: An illustration of the proof of Lemma 3.7

It will prove more convenient to work with infinite sequences rather than finite sequences.

Definition 3.8 *We can extend the sequence space* $\{1, \cdots, k\}^{\mathbb{N}_0}$ *to*

$$\Sigma_A = \left\{ (x_n)_{n=0}^{\infty} \in \{0, 1, \cdots, k\}^{\mathbb{N}_0} \ : \ A(x_n, x_{n+1}) = 1 \text{ for } n \in \mathbb{N}_0 \right\}$$

with transition matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

and again we define $\sigma : \Sigma_A \to \Sigma_A$ *by* $\sigma(x_n)_{n=0}^{\infty} = (x_{n+1})_{n=0}^{\infty}$.

Given a finite word $\underline{i} = (i_1, \cdots, i_n)$ we can denote $\underline{i}\dot{0} = (i_1, \cdots, i_n, 0, 0, \cdots) \in \Sigma_A$. In particular, given $\underline{i}\dot{0} = (i_1, \cdots, i_n, 0, 0, \ldots)$ we have $\sigma\underline{i}\dot{0} = (i_2, i_3 \cdots, i_n, 0, 0, \ldots)$. We can define a metric d on Σ_A by

$$d\left((x_n)_{n=0}^{\infty}, (y_n)_{n=0}^{\infty}\right) = \begin{cases} |(T_{x_1} \circ \cdots \circ T_{x_N})'(0)| & \text{if } x_n = y_n \text{ for } 1 \leqslant n \leqslant N \text{ and } x_N \neq y_N \\ 0 & \text{if } (x_n) = (y_n). \end{cases}$$

Finally, we can denote the set of all finite words by $\Sigma^* = \cup_{n=1}^{\infty} \{1, \cdots, k\}^n$ and we write $|\underline{i}| = n$.

A key technical lemma is now the following.

Lemma 3.9 *There exists a Hölder continuous function* $f : \Sigma_A \to \mathbb{R}$ *such that*

$$\mathcal{L} = \left\{ \exp\left(-\sum_{k=0}^{n-1} f(\sigma^k \underline{i}\dot{0}) \right) \ : \ \underline{i} \in \Sigma^* \right\}.$$

[7] We assume for simplicity that the maps T_i are orientation preserving. The other cases are similar

Proof. For a word $\underline{i} \in \Sigma^*$ we can clearly write $\{(a_j, b_j) : j \geqslant 1\} = \left\{(T_{\underline{i}}(x), T_{\underline{i}}(y)) : \underline{i} \in \Sigma^*\right\}$. In particular, to obtain all of the values ℓ_j ($j \geqslant 1$) is equivalent to obtaining all the values $(T_{\underline{i}}(x) - T_{\underline{i}}(y))$ ($\underline{i} \in \Sigma$). Given $\underline{i} \in \Sigma^* \cup \{\dot{0}\}$ we can define

$$f(\underline{i}\dot{0}) = \begin{cases} \log\left(\frac{(T_{\underline{i}}(x) - T_{\underline{i}}(y))}{(T_{\sigma \underline{i}}(x) - T_{\sigma \underline{i}}(y))}\right) & \text{if } |\underline{i}| \geqslant 2 \\ \log\left((T_{\underline{i}}(x) - T_{\underline{i}}(y))\right) & \text{if } |\underline{i}| = 1 \end{cases}$$

and in addition set $f(\dot{0}) = 0$. Then if $|\underline{i}| = n$ we can write

$$\sum_{k=0}^{n-1} f(\sigma^k \underline{i}\dot{0}) = \log(T_{\underline{i}}(x) - T_{\underline{i}}(y)).$$

We need to show that above function extends as a H\'older continuous function to Σ_A. By the Mean Value Theorem we can associate to $\underline{i} \in \Sigma_*$ a value $\xi = \xi_{\underline{i}} \in [T_{\sigma \underline{i}}(x), T_{\sigma \underline{i}}(y)]$ such that $f(\underline{i}\dot{0}) = \log |T'_{i_1}(\xi)|$. If $\underline{i}, \underline{j}$ agree in the first m places then we expect that $|\xi_{\underline{i}} - \xi_{\underline{j}}| \leqslant C\theta^m$ where $\theta = \max_i \sup_{0 \leqslant \xi \leqslant 1} |T'_i(\xi)| < 1$. Thus

$$\begin{aligned} |f(\underline{i}\dot{0}) - f(\underline{j}\dot{0})| &= |\log |T'_{i_1}(\xi_{\underline{i}})| - \log |T'_{i_1}(\xi_{\underline{j}})|| \\ &\leqslant \sup_{0 \leqslant x \leqslant 1} \left| \frac{T''_{i_1}(x)}{T'_{i_1}(x)} \right| .|(\xi_{\underline{i}} - \xi_{\underline{j}})| \leqslant D\theta^m. \end{aligned}$$

We thus deduce that the function f (restricted to the dense set of sequences in Σ_A) is α-Hölder continuous (with exponent $\alpha = \log \theta / \log 2$). This completes the proof.

In particular, from the definition of $\zeta_{\mathcal{L}}(s)$ we have the following.

Corollary 3.10 *We can write*

$$\zeta_{\mathcal{L}}(s) = \sum_{n=1}^{\infty} \sum_{|\underline{i}|=n} \exp\left(-s \sum_{k=0}^{n-1} f(\sigma^k \underline{i}\dot{0})\right) \tag{3.1}$$

for $Re(s) > \delta$, say.

We can introduce the symbolic transfer operators associated to $\sigma : \Sigma_A \to \Sigma_A$. Fix $\alpha > 0$ and let $C^\alpha(\Sigma_A)$ denote the Banach space of α-Hölder continuous functions $g : \Sigma_A \to' \Sigma_A$ with the norm

$$\|g\| = \sup_{x \in \Sigma_A} |g(x)| + \sup_{x, y \in \Sigma_A} \frac{|g(x) - g(y)|}{d(x, y)^\alpha}$$

Definition 3.11 *Given $s \in \mathbb{C}$ we can define $\mathcal{L}_s : C^\alpha(\Sigma_A) \to C^\alpha(\Sigma_A)$ by*

$$\mathcal{L}_s g(x) = \sum_{\sigma y = x} e^{-sf(y)} g(y), \quad \text{for } g \in C^\alpha(\Sigma_A) \text{ and } x \in X_A.$$

We require the more classical version on the spectrum of symbolic transfer operators:

Lemma 3.12 *Let $s = \sigma + it$ and denote by $e^{P(\sigma)}$ spectral radius $\mathcal{L}_\sigma$. Then the essential spectral radius of $\mathcal{L}_s : C^\alpha(\Sigma_A) \to C^\alpha(\Sigma_A)$ is at most $\theta^\alpha e^{P(\sigma)}$.*

Let $h > 0$ be such that $e^{P(h)} = 1$. It follows from Lemma 3.12 that there exists $\epsilon > 0$ such that $\theta^\alpha e^{P(\sigma)} < 1$ for $Re(s) > h - \epsilon$. In particular, for $Re(s) > h - \epsilon$ the essential spectral radius of $\mathcal{L}_s$ is (strictly) smaller than 1 and thus we can write

$$\mathcal{L}_s^n = \sum_{i=1}^{k} P_{i,s}^n + U_{s,n}$$

where $P_{i,s}$ are eigenprojections associated to isolated eigenvalues and $\lim_{n \to +\infty} \|U_{s,n}\|^{1/n} < 1$ uniformly in s on bounded domains.

This would be a classical result if A were aperiodic. However, the spectrum for $\mathcal{L}_s : C^\alpha(\Sigma_A) \to C^\alpha(\Sigma_A)$ is equal to that of the restriction $\mathcal{L}_s : C^\alpha(\Sigma_2) \to C^\alpha(\Sigma_2)$ where $\Sigma_2 = \{0, 1\}^{\mathbb{Z}_0}$ is a full shift on two symbols cf. [14]. To understand this, we begin with the following well known result.

Lemma 3.13 *There exists $C > 0$ such that*

$$\|\mathcal{L}^n f\| \leqslant C\|f\|_\infty = C\|f\|_\infty + (\theta^\alpha)^n \|f\|$$

for $f \in C^\alpha(X_A)$ and $n \geqslant 1$.

We can consider the restriction $\overline{f} = f|\Sigma_2$ then the restriction has the the same pressure as the original function providing $P(f) \geqslant f(\dot{0}) = 0$. To establish the essential spectral radius it suffices to consider the locally constant projections $E_n : C^\alpha(\Sigma_A) \to C^\alpha(\Sigma_A)$ such that

$$E_n(f) = \sum_{|\underline{i}|=n} f(x_{\underline{i}})\chi_{[\underline{i}]}$$

where we make the choice $x_{\underline{i}} \in [\underline{i}]$. Then by applying Lemma 3.13 we can deduce that $\|L^n E_n(f))\| \leqslant C\theta^\alpha$.

Lemma 3.14 *The eigenvalues of $\mathcal{L}_s : C^\alpha(\Sigma_A) \to C^\alpha(\Sigma_A)$ in $|z| > \theta^\alpha e^{P(\sigma)}$ are equal to those for $\mathcal{L}_s : C^\alpha(\Sigma_2) \to C^\alpha(\Sigma_2)$.*

If λ is an eigenvalue for $\mathcal{L}_s : C^\alpha(\Sigma_A) \to C^\alpha(\Sigma_A)$. with $|\lambda| > \theta^\alpha e^{P(\sigma)}$ then we can let $\overline{h} \in C^\alpha(\Sigma_A)$ be an associated function. We can consider the restriction $h := \overline{h}|\Sigma_2$ to Σ_2. If this restriction is non-zero then we have the eigenvalue equation $\mathcal{L}_s h = \lambda h$ and λ is an eigenvalue for $\mathcal{L}_s$. On the other hand, if $h = 0$ on Σ_2 then since $\Sigma_2 \subset \Sigma_A$ is dense this implies that $\overline{h} = 0$, giving a contradiction.

We can rewrite $\zeta_{\mathcal{L}}(s)$ in terms of $\mathcal{L}_s$ for $Re(s) > h$ as the convergent series

$$\zeta_{\mathcal{L}}(s) = \sum_{n=1}^{\infty} \mathcal{L}_s^n 1(\dot{0}) = \mathcal{L}_s(1 - \mathcal{L}_s)^{-1}1(\dot{0})$$

where $1(x)$ denotes the constant function 1. We can then write that

$$\zeta_{\mathcal{L}}(s) = \sum_{i=1}^{k} \left(P_{i,s}(I - P_{i,s})^{-1}\right) 1(x_0) + \left(U_{n,s}(I - U_{n,s})^{-1}\right) 1(\dot{0}).$$

The last term is analytic for $Re(s) > h - \epsilon$. The first term contributes a meromorphic function on the same domain $Re(s) > 0$ where the poles for $\eta(s)$ on $Re(s) > h - \epsilon$ come from where 1 is an eigenvalue associated to $P_{i,s}$.

4 Location of poles

Having established in the previous section that each of the complex functions has *some* extension to a larger half-plane, we now turn to the second question we want to address:

Question. Where are the poles in the meromorphic extensions?

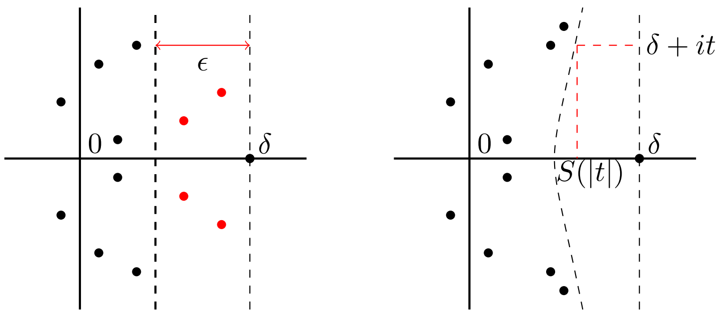

Figure 7: (a) The poles do not accumulate on $Re(s) = \delta$ (and are bounded away by a distance $\epsilon > 0$); and (b) The poles accumulate on $Re(s) = \delta$ (outside of a curve parameterized by $S(|t|) + it$ $(t \in \mathbb{R})$)

For many (but, as we shall soon see, not all) examples there will be a sequence of poles in the extension whose real parts accumulate on $Re(s) = \delta$. These can be viewed as poles of $\zeta_{\mathcal{L}}(s)$, $R_{\mathcal{W}}(s)$ or $\eta_{\mathcal{L}}(s)$ when they are close enough to $Re(s) = \delta$. In this case, we can consider how quickly the poles accumulate in different cases. As we have seen, the answer will be the same for any of the complex functions we have described.

4.1 Case I: When poles may accumulate on $Re(s) = \delta$

We follow Lapidus in the following notation. We begin with a quantification of how poles might accumulate on the line $Re(s) = \delta$.

Definition 4.1 *Given a (monotone decreasing) function $S : \mathbb{R}^+ \to [\delta - \epsilon, \delta]$ we can consider the region*

$$\mathcal{W}_S = \{s = \sigma + it : S(|t|) \leqslant \sigma < \delta\}.$$

In particular, for different attractors K we can ask about such functions S for which $\mathcal{W}_S$ does **not** contain poles. [8]

Before stating a result, we recall a definition from number theory.

Definition 4.2 *A real number Θ is called a* badly approximable number *if there exists $C > 0$ and $\beta > 0$ such that*

$$\left|\Theta - \frac{p}{q}\right| \geqslant C/q^{2+\beta}$$

for all $p \in \mathbb{Z}$ and $q \in \mathbb{Z} - \{0\}$.

For example, any (non-rational) algebraic number is badly approximable. More generally, the set of all badly approximable numbers has Hausdorff Dimension 1.

Theorem 4.3 *Let X be a Cantor set associated to $C^{1+\alpha}$ cookie cutter maps $T_1, T_2 : [0,1] \to [0,1]$. Assume that there are $i_1, \cdots i_n, j_1, \cdots, j_m \in \{1,2\}$ with fixed points*

$$z = T_{i_1} \circ \cdots \circ T_{i_n}(z) \text{ and } y = T_{j_1} \circ \cdots \circ T_{j_m}(y)$$

such that

$$\Theta := \frac{\log |(T_{i_1} \circ \cdots \circ T_{i_n})'(z)|}{\log |(T_{j_1} \circ \cdots \circ T_{j_n})'(y)|}$$

is a badly approximable number. Then there exists $E > 0$ and $\eta > 0$, $\xi > 0$ such that for $S(u) = \max\{\delta - \xi, \delta - E/u^\eta\}$ the only pole in $\mathcal{W}_S$ is D.

Lapidus et al considered the specific class of self-similar Cantor sets.

Example 4.4 *There are examples of (typical self-similar sets) X where $s = \delta$ will be the only pole on $Re(s) = \delta$, but there will be poles $s_n \in \mathcal{D}$ with $Re(s_n) \to \delta$. In fact the only alternative to this would be that the poles are periodic in the imaginary direction.*

As a partcular special case, this recovers some results of Lapidus et al. [9]

Example 4.5 (Self-similar Cantor sets) *Assume that we fix $0 < \alpha, 1 - \beta < 1$ then we can associate the Cantor set by removing from successive intervals I the subinterval $J \subset I$ of length $(1 - \alpha - \beta)|I|$ leaving intervals $\alpha|I|$ and $\beta|I|$.*

We can consider the case that $\alpha/\beta = \frac{\sqrt{5}-1}{2}$, say, which is badly approximable. Then the above Theorem applies since we can take

$$T_1(x) = \alpha x \text{ and } T_2(x) = (1 - \beta) + \beta x$$

and then $z = 0$ and $y = 1$ are the respective fixed points. Thus $|T_1'(0)|/|T_2'(1)| = \alpha/\beta$.

The proof of Theorem 4.3 is based on norm bounds on the transfer operator. In particular, we have the following bound due to Dolgopyat [2].

[8] In the notation of Lapidus S stands for screen (preumably in the american sense) and W for window

[9] These are described in the book of Lapidus, in Chapters 2 and 3.

Lemma 4.6 *Under the diophantine hypotheses there exist constants $t_0 \geqslant 1$, $\tau > 0$, $C_1 > 0$ and $C > 0$ such that whenever $|Im(s)| \geqslant t_0$ and $m \geqslant 1$ then*

$$\|\mathcal{L}_s^{mN}\| \leqslant C|Im(s)|e^{2NmP(Re(s))}\left(1 - \frac{1}{|Im(s)|^\tau}\right)^{m-1}$$

where $N = [C \log |Im(s)|]$ and $e^{P(Re(s))}$ denotes the spectral radius $\mathcal{L}_{Re(s)}$.

For definiteness, we can consider the original spectral zeta function $\zeta_{\mathcal{L}}(s)$, which we can write as

$$\zeta_{\mathcal{L}}(s) = \sum_{n=1}^{\infty} \mathcal{L}_s 1(\dot{0})$$

for those s for which the series converges. We can then write

$$\begin{aligned}|\zeta_{\mathcal{L}}(s)| &\leqslant \sum_{n=1}^{\infty} |\mathcal{L}_s^n 1(\dot{0})| \leqslant \sum_{n=1}^{\infty} \|L_s^n\| \\ &\leqslant \sum_{m=1}^{\infty}\sum_{j=0}^{N-1} \|L_s^{mN+j}\| \\ &\leqslant (1 + \|L_s\| + \cdots + \|L_s^{N-1}\|) \sum_{m=1}^{\infty} \|\mathcal{L}_s^{mN}\|\end{aligned}$$

In particular, we have convergence to an analytic function provided

$$e^{2NP(Re(s))}\left(1 - \frac{1}{|Im(s)|^\tau}\right) < 1.$$

Since $P(\delta) = 0$ and $P(Re(s)) \leqslant C|Re(s) - \delta|$ for $Re(s)$ sufficiently close to δ. In particular, for $\eta < \tau$ the theorem holds.

4.2 Case II: Where poles do not accumulate on $Re(s) = \delta$

There are examples where the poles do not accumulate on $Re(s) = \delta$. For convenience, we present a fairly simple and familiar, but concrete, example. [10]

We can consider iterated function schemes $T_1, \cdots, T_k : [0, 1] \to [0, 1]$ where the maps are Möbius.

Example 4.7 (E_2) *We can consider the case of*

$$E_2 = \left\{ \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}} : a_1, a_2, a_3, \cdots \in \{1, 2\} \right\}$$

the Cantor set consisting of real numbers in the unit intervals whose infinite continued fraction expansion uses only the digits 1 or 2.

More generally, there is the following result.

Theorem 4.8 *Let X be a Cantor set associated to Möbius map contractions $T_1, \cdots, T_k : I \to I$. Then there exists $\epsilon > 0$ such that for $S(u) = \delta - \epsilon$ the only pole in $\mathcal{W}_S$ is δ.*

The proof of Theorem 4.8 is based on a result of Naud [11], which in turn was based on an approach of Dolgopyat, which was used to show that there are no resonances in the strip $\delta - \epsilon < Re(s) < \delta$. But the technical result actually proved is that the associated transfer operator $\mathcal{L}_s$ has spectral radius strictly smaller than 1 uniformly in the strip ([11], Theorem 2.6).

To illustrate the key ingredients, we can concentrate on the case E_2 the general case being similar. We first need that there exists $(\xi_n)_{n=1}^\infty, (\eta_n)_{n=1}^\infty \in \{1, 2\}^{\mathbb{N}}$ and $u_0, v_0 \in [0, 1]$ such that the function $\phi_{\xi,\eta} : [0, 1] \times [0, 1] \to \mathbb{R}$ defined by

$$\phi_{\xi,\eta}(u, v) := 2 \log \left(\prod_{n=1}^{\infty} \frac{[\xi_1, \cdots, \xi_n + u]}{[\xi_1, \cdots, \xi_n + v]} \frac{[\eta_1, \cdots, \eta_n + u]}{[\eta_1, \cdots, \eta_n + v]} \right)$$

[10] In fact, one anticipates that providing the Cantor set isn't self-affine, or smoothly conjugate to such, then this will be the case cf. Avila-Gouezel-Yoccoz [1]

Figure 8: The poles of the zeta function $Z_{\mathcal{L}}(s)$ for E_2

has the property that

$$\frac{\partial \phi_{\xi,\eta}}{\partial u}(u_0, v_0) \neq 0$$

(see [11], Definition 2.1) In the present example(s), such explicit choices are easy to make.

A second ingredient is that, because of the analyticity, there is also a general argument (see [11] Lemmas 4.3 and the short argument thereafter) which shows that the function

$$g : [1/3, 1/2] \coprod [1/2, 1] \to \mathbb{R}$$

defined by

$$g(x) = \begin{cases} \log(1+x) \text{ for } x \in [1/2, 1] \\ \log(2+x) \text{ for } x \in [1/3, 1/2] \end{cases}$$

is not cohomologous to a locally constant function (which can easily be seen not to hold by considering periodic orbits).

5 Weyl-type Theorems for $E_{n,m}$

A Weyl type theorem is one which gives estimates on the growth of the number of poles in a vertical strip. These appear in the context of poles for the Ruelle zeta function in the setting of Schottky surfaces [12]. [11] Moreover, the name derives from a study of the poles of the zeta function (actually zeros of the Selberg zeta function) for compact Riemann surfaces which are intimately relates to the eigenvalues of the Laplacian on the compact surface, and for which Weyl famously proved an asymptotic formulae in 1916.

In the present setting we will consider the Cantor set $E_{n,m}$ for $n, m \in \mathbb{N}$ $(n \neq m)$ [12] and consider the growth of the number $\pi(T)$ of poles (for either $R_{\mathcal{W}}(s)$ or $\eta_{\mathcal{L}}$) in a strip $\frac{\delta}{2} \leqslant \sigma \leqslant Re(s) \leqslant \delta$, say, with $|Im(s)| \leqslant T$, as $T \to +\infty$. We would similarly like to adapt an approach of Naud for another problem to the present setting.

Proposition 5.1 *For $E_{n,m}$ we can bound $\pi(T) = O(T^{1+\delta})$.*

In point of fact a stronger result holds.

Proposition 5.2 *For $E_{n,m}$ we can bound $\pi(T) = O(T^{1+\tau(\sigma)})$ where $\tau : [\delta/2, \delta] \to [0, \delta]$, with $\tau(\delta/2) = \delta$ and $\tau(\sigma) < \delta$ for $\delta/2 < \sigma \leqslant \delta$.*

We now sketch the argument (following [12]). Let $h > 0$. The transfer operator is defined on a disjoint union of neighbourhoods from $B(E_{n,m}, h)$ with $N(h)$, say, connected components of size at most $O(h)$. The corresponding disks in $\mathbb{C}$ are used to define a Hardy space. We can assume that the associated transfer operator $\mathcal{L}_s$ is trace class.

Lemma 5.3 *There exists $C > 0$, n_0 and $R > 0$ such that for $\sigma_0 \leqslant Re(s) \leqslant \delta$ and $n \geqslant n_0$ we have that $|Im(s)| \geqslant R$ we have*

$$\log|\det(I - \mathcal{L}_s^n)| \leqslant C|Im(s)|^\delta e^{nP(\sigma_0)}$$

Since the operator is trace class we have a bound

$$\log|\det(I - \mathcal{L}_s^n)| \leqslant \sum_{l=0}^{\infty} \|\mathcal{L}_s(e_l)\|$$

where $(e_k^l)_{k \in \mathbb{N}, 0 \leqslant l \leqslant N(h)}$ is a basis for the Hilbert space consisting of orthonormal analytic functions on each of the $N(h)$ disks. Moreover, we can bound $N(h) \leqslant h^{-\delta}$.

[11] There are apparently improvements on this dues to Soares
[12] More general results should hold

By the Cauchy-Schwartz inequality we can bound

$$\log|\det(I-\mathcal{L}_s^n)| \leqslant Ch^{-\delta/2}\sum_{k=0}^{\infty}\left(\sum_{l=1}^{N(h)}\|\mathcal{L}_s^n(e_k^l)\|_{H^2(h)}^2\right)^{\frac{1}{2}}.$$

Moreover, there is a bound

$$\sum_{l=1}^{N(h)}\|\mathcal{L}_s^n(e_k^l)\|_{H^2(h)}^2 \leqslant Ch^{-\delta}\rho^k e^{2nP(\sigma_0)}.$$

Combining these inequalities gives the required bound.

To convert the bound in Lemma 5.3 into a bound on $\pi(T)$ we follow Naud in using a bound essentially due to Littlewood.

Lemma 5.4 *For $\sigma_0 < \delta$ there exists $\nu_0 > 0$, $T_0 > 0$ such that for $\sigma_0 < \sigma < \delta$ there exists $C, C' > 0$ such that for $T \geqslant T_0$ and $n(T) = [\nu \log T]$ with $0 < \nu \leqslant \nu_0$ then*

$$N(T) \leqslant C_\delta \int_{T/2}^{T} \log|\det(I-\mathcal{L}_{\sigma_0+it}^{n(T)})|dt + C'T.$$

Comparing the bounds in Lemma 5.3 and Lemma 5.4 completes the proof of the proposition.

A The estimates of Morita

The content of this appendix is essentially expository. We recall the following definition.

Definition A.1 *Given $s \in \mathbb{C}$, we can define the family of linear transfer operators $\mathcal{L}_s : C^{Lip}([0,1]) \to C^{Lip}([0,1])$ for $Re(s) > 0)$ by*

$$\mathcal{L}_s f(x) = \sum_{i=1}^{k} |T_i'(x)|^s f(T_i x).$$

Given $N \geqslant 1$, we can consider the N-power of the operator $\mathcal{L}_s$ which takes the form

$$\mathcal{L}_s^N h(x) = \sum_{|\underline{i}|=n} |T_{\underline{i}}'(x)|^s h(T_{\underline{i}} x)$$

where $\underline{i} = (i_1, \cdots, i_N) \in \{1, \cdots, k\}^N$ and we write $|\underline{i}| = N$. Moreover, for later convenience we rewrite this as

$$\mathcal{L}_s^N h(x) = \sum_{|\underline{i}|=n} \mathcal{L}_{\underline{i},s} h(x)$$

where $\mathcal{L}_{\underline{i},s} : C^{Lip}([0,1]) \to C^{Lip}([0,1])$ is defined by

$$\mathcal{L}_{\underline{i},s} h(x) = |T_{\underline{i}}'(x)|^s h(T_{\underline{i}} x).$$

A.1 Families of subintervals

Let $I_{\underline{i}} = T_{\underline{i}}([0,1])$ denote the images of the unit interval. The next lemma is a simple consequence of the attractor having dimension less than 1.

Lemma A.2 *There exists $0 < \kappa < 1$ such that* [13]

$$\sum_{|\underline{i}|=n} \mathrm{diam}(I_{\underline{i}}) = O(\kappa^n)$$

Our aim is to bound the essential spectral radius of $\mathcal{L}_s$ using the essential spectral radius formula. To this end, we want to obtain bounds on functions that arise from summing the terms

$$\mathcal{L}_s^n\left((f - f(x_{\underline{i}}))\chi_{I_{\underline{i}}}\right).$$

The next subsection gives a supremum bound on such functions. The subsequent subsection gives a Lipschitz bound. The estimates then follow the lines of those in [10].

[13]We can take $\kappa = e^{P(-1)} < 1$ where $P : \mathbb{R} \to \mathbb{R}$ is the usual pressure function

A.2 Supremum bounds

We can choose for each $|\underline{i}| = n$ the fixed point $x_{\underline{i}} = T_{\underline{i}}(x_{\underline{i}})$. We need the following basic bounds. Let $\rho > 1$.

1. We can choose $\epsilon > 0$ sufficient small that for $Re(s) > -\epsilon$ we have that $\|T_{\underline{i}}'(\cdot)|^s\|_\infty = O(\rho^n)$.
2. Since $|T_{\underline{i}}(x) - x_{\underline{i}}| \leqslant \mathrm{diam}(I_{\underline{i}})$ we can bound $|f(T_{\underline{i}}(x)) - f(x_{\underline{i}})| \leqslant \|f\|_{Lip}\mathrm{diam}(I_{\underline{i}})$ and then write

$$\|\mathcal{L}_s^n\left((f - f(x_{\underline{i}}))\chi_{I_{\underline{i}}}\right)\|_\infty = O\left(\rho^n \mathrm{diam}(I_{\underline{i}})\|f\|_{Lip}\right)$$

Summing over all strings $\underline{i}$ with $|\underline{i}| = n$ gives:

Lemma A.3 *For $n \geqslant 1$,*

$$\begin{aligned}\sum_{|\underline{i}|=n} \|\mathcal{L}_s^n\left((f - f(x_{\underline{i}}))\chi_{I_{\underline{i}}}\right)\|_\infty &= O\left(\rho^n\left(\sum_{|\underline{i}|=n}\mathrm{diam}(I_{\underline{i}})\right)\|f\|_{Lip}\right) \\ &= O\left((\rho\kappa)^n\|f\|_{Lip}\right).\end{aligned} \tag{A.1}$$

A.3 Lipschitz bounds

Let $n \geqslant 1$. If $\underline{i} = (i_1, \cdots, i_n) \in \{1, \cdots, k\}^n$ then we can write $\sigma^j\underline{i} = (i_{j+1}, \cdots, i_n)$ for each $0 \leqslant j \leqslant n-1$. We first start with the following bound for distinct $0 < x, y < 1$:

$$\begin{aligned}&\left||T_{\underline{i}}'(x)|^s - |T_{\underline{i}}'(y)|^s\right| \\ &= \left|\prod_{j=0}^{n-1}|T_{i_{j-1}}'|^s(T_{\sigma^{j-1}\underline{i}}x)\left(|T_{i_j}'|^s(T_{\sigma^j\underline{i}}x) - |T_{i_j}'|^s(T_{\sigma^j\underline{i}}y)\right)\prod_{j=0}^{n-1}|T_{i_{j-1}}'|^s(T_{\sigma^{j-1}\underline{i}}x)|\right| \\ &= O\left(\rho^{n-1}\sum_{j=0}^{n-1}|T_{\sigma^j}(x) - T_{\sigma^j}(y)|\right) = O\left(\rho^{n-1}|x-y|\right)\end{aligned} \tag{A.2}$$

using the chain rule.

For $0 \leqslant x, y \leqslant 1$ we can use the triangle inequality to bound

$$\begin{aligned}&\left|\mathcal{L}_s\left((f - f(x_{\underline{i}}))\chi_I\right)(x) - \mathcal{L}_s\left((f - f(x_{\underline{i}}))\chi_I\right)(y)\right| \\ &= \left||T_{\underline{i}}'(x)|^s(f(T_{\underline{i}}x) - f(x_{\underline{i}})) - |T_{\underline{i}}'(y)|^s(f(T_{\underline{i}}y) - f(x_{\underline{i}}))\right| \\ &\leqslant \left||T_{\underline{i}}'(x)|^s(f(T_{\underline{i}}x) - f(x_{\underline{i}})) - |T_{\underline{i}}'(y)|^s(f(T_{\underline{i}}x) - f(x_{\underline{i}}))\right| \\ &\quad + \left||T_{\underline{i}}'(y)|^s(f(T_{\underline{i}}'(x) - f(x_{\underline{i}})) - |T_{\underline{i}}'(y)|^s(f(T_{\underline{i}}y) - f(x_{\underline{i}}))\right| \\ &\leqslant \left||T_{\underline{i}}'(x)|^s - |T_{\underline{i}}'(y)|^s\right|.|f(T_{\underline{i}}x) - f(x_{\underline{i}})| + \left||T_{\underline{i}}'(y)|^s\right|.\left|f(T_{\underline{i}}x) - f(T_{\underline{i}}y)\right|.\end{aligned}$$

We can use to (A.2) to bound the first term

$$\left||T_{\underline{i}}(x)|^s - |T_{\underline{i}}(y)|^s\right|.|f(T_{\underline{i}}x) - f(x_{\underline{i}})| = O\left(\rho^n|x-y|\|f\|_{Lip}\right).$$

We can also bound

$$\left||T_{\underline{i}}'(y)|^s|f(T_{\underline{i}}x) - f(T_{\underline{i}}y)\right| = O\left(\rho^n\|f\|_{Lip}|x-y|\right).$$

Combining these bounds we have that

$$\begin{aligned}\sum_{|\underline{i}|=n}\|\mathcal{L}_s^n\left((f - f(x_{\underline{i}}))\chi_{I_{\underline{i}}}\right)\|_{Lip} &= O\left(\|f\|_{Lip}\rho^n\sum_{|\underline{i}|=n}\mathrm{diam}(I_{\underline{i}})\right) \\ &= O\left(\|f\|_{Lip}(\kappa\rho)^n\right).\end{aligned}$$

A.4 Quasi-compactness and essential spectral radius

We now want to to convert the bounds above in the previous two subsections into estimates on the essential spectral radius $\rho_{ess}(\mathcal{L}_s)$. For $n \geqslant 1$, we can define $K_n : C^{Lip}([0,1]) \to C^{Lip}([0,1])$ by

$$K_n f(x) = \sum_{|\underline{i}|=n} |T'_{\underline{i}}(x)|^s f(x_{\underline{i}}) \chi_{I_{\underline{i}}}(x).$$

This image is finite dimensional and thus the operator K_n is compact. We can write

$$(\mathcal{L}_s^n - K_n) f = \sum_{|\underline{i}|=n} \mathcal{L}_s^n \left((f - f(x_{\underline{i}})) \chi_{I_{\underline{i}}} \right)$$

Thus we can use (A.1) and (A.2) to bound

$$\begin{aligned}
\|(\mathcal{L}_s^n - K_n) f\| &= \|(\mathcal{L}_s^n - K_n) f\|_\infty + \|(\mathcal{L}_s^n - K_n) f\|_{Lip} \\
&= \sum_{|\underline{i}|=n} \|\mathcal{L}_s^n \left((f - f(x_{\underline{i}})) \chi_{I_{\underline{i}}} \right)\|_{Lip} + \sum_{|\underline{i}|=n} \|\mathcal{L}_s^n \left((f - f(x_{\underline{i}})) \chi_{I_{\underline{i}}} \right)\|_\infty \\
&= O\left(\|f\|_{Lip} (\kappa\rho)^n \right) + O\left(\|f\|_\infty (\kappa\rho)^n \right) \\
&= O\left(\|f\| (\kappa\rho)^n \right).
\end{aligned}$$

By the Nussbaum essential spectral radius theorem [13].

$$\rho_{ess}(\mathcal{L}_s) \leqslant \limsup_{n\to+\infty} \|(\mathcal{L}_s^n - K_n) f\|^{1/n} \leqslant \kappa\rho < 1.$$

B A heuristic description of patterns

We want to present a heuristic description for the location of complex dimensions for self-affine Cantor sets (at least those close to $Re(s) = \delta$).

Assume that $\alpha = |\log r_1|, \beta = |\log r_2| > 0$ then we want to understand why in some cases the complex zeros s, which occur as the solutions $s = \sigma + it$ to the equation

$$r_1^s + r_2^s = e^{-s\alpha} + e^{-s\beta} = 1 \qquad (B.1)$$

appear to lie on curves. This is expected to be related to the approximability of the ratio $\beta/\alpha > 0$. Let us denote $\theta = \frac{\beta}{\alpha}$ and write $s = \sigma + it$ then by rescaling (C.1) (by $s \mapsto s/\alpha$) the equation becomes

$$e^{-(\sigma+it)} + e^{-(\sigma+it)\theta} = 1. \qquad (B.2)$$

To begin we would like to look for solutions such that $(t, \theta t)$ (mod 1) is close to $(0,0)$ which should be a neighbourhood in which to look for solutions to (B.2).

A more general equation. Fix a small value $\epsilon > 0$. [14] More generally, we can consider solutions $(t_1, t_2) \in (-\epsilon, \epsilon) \times (-\epsilon, \epsilon)$ to the more general equation

$$e^{-\sigma} e^{it_1} + e^{-\sigma\theta} e^{it_2} = 1. \qquad (B.3)$$

This is shown geometrically in Figure 9.

Clearly, equation (B.3) is both 2π-periodic in both t_1 and t_2. We can use the sine and cosine rules [15] applied to the figure to write:

$$\begin{aligned}
\sin(t_1) &= e^{\sigma(1-\theta)} \sin(t_2) \\
1 + e^{-2\sigma} - e^{-2\sigma\theta} &= 2e^{-\sigma} \cos(t_1).
\end{aligned} \qquad (B.4)$$

The function $\sigma(t_1, t_2)$. We can use the first equality in (B.4) to write $\sigma : (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \to \mathbb{R}^+$ explicitly in the form

$$e^{\sigma(t_1,t_2)} = \left(\frac{\sin(t_1)}{\sin(t_2)} \right)^{\frac{1}{1-\theta}}. \qquad (B.5)$$

[14] This is just for safety. Probably $\epsilon > 0$ can be quite large

[15] For a triangle with sides of lengths $a, b, c > 0$ and angles α, β, γ then $\sin(\alpha)/a = \sin(\beta)/b$ and $b^2 + c^2 - a^2 = 2bc\cos(\alpha)$.

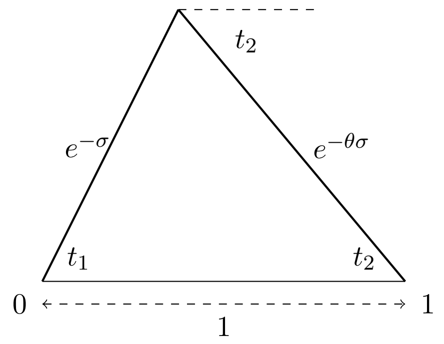

Figure 9: A geometric representation of equation (B.2) in the complex plane

For $(0,0) \in \mathcal{C}$ (corresponding to the solution $\sigma = \delta$ and $t = 0$ to (B.2)) we can consider the value $\delta = \sigma(0,0)$ at that point, i.e., the real value $\delta > 0$ such that $e^{-\delta} + e^{-\delta\theta} = 1$. A plot of $\sigma(\cdot,\cdot)$ for Example B.2 below appears in Figure 10 (a).

Using (B.5) to eliminate σ in the equations (B.4) leaves an implicit equation relating t_1 and t_2 and describing a one dimensional curve: $\mathcal{C} \subset (-\epsilon,\epsilon) \times (-\epsilon,\epsilon)$, i.e.,

$$\mathcal{C} = \left\{ (t_1,t_2) : 1 + \left(\frac{\sin(t_1)}{\sin(t_2)}\right)^{\frac{-2}{1-\theta}} - \left(\frac{\sin(t_1)}{\sin(t_2)}\right)^{\frac{-2\theta}{1-\theta}} - 2\left(\frac{\sin(t_1)}{\sin(t_2)}\right)^{\frac{-1}{1-\theta}} \cos(t_1) = 0 \right\}.$$

The (blue) curve $\mathcal{C}$ for Example B.2 below appears in Figure 10 (b).

To look for true solutions to (B.2) we now need to (re)-impose the condition that $t_2 = \theta t_1$. We can consider the line segment on the torus $\mathbb{T}^2 = [0,1) \times [0,1)$ given by

$$\mathcal{L}_T = \{(t, \theta t) \pmod 1 : 0 \leqslant t \leqslant T\}$$

for $T > 0$, i.e., the projection onto the torus of the line segment in $\mathbb{R}^2_+$ passing through $(0,0)$ and of slope θ. The (red) line in Example B.2 (c) corresponds to the choice $T = 60$

We can summarize this discussion as follows: *The solutions to (B.2) occur for imaginary values t such that $(t,\theta t) \in \mathcal{L}_T \cap \mathcal{C} \pmod 1$. The corresponding real value is then* $\sigma = \sigma(t,\theta t)$ i.e., we want to consider solutions $s_n = \sigma_n + it_n$ to (B.2) where:

- for large T the values $0 \leqslant t_n \leqslant T$ with $(t_n, \theta t_n) \in \mathcal{L}_N \cap \mathcal{C} \pmod 1$ are clustered close to $(0,0)$; and
- there are associated values $\sigma_n = \sigma(t_n, \theta t_n)$.

A precise knowledge of $\mathcal{C}$ could be used to reconstruct the locations of the solutions to (B.2). However, it also gives an heuristic explanation of the patterns of the complex zeros where β/α can be well approximated by rationals. This is illustrated by the following examples.

Example B.1 *Consider again $\beta = 1$ and $\alpha = \pi$. A "diophantine property" for β/α suggests the reason for the apparent bunching of parts of the red line in Figure 10 (c). In particular, this bunching of lines in $\mathcal{L}_T$ means that the intersections $\mathcal{L}_T \cap \mathcal{C}$ with the (transverse) one dimensional curve $\mathcal{C}$ should be close (and also have a monotone ordering on the curve). Since the restriction $\sigma : \mathcal{C} \to \mathbb{R}$ of the smooth function $\sigma(\cdot,\cdot)$ is again a smooth function whereby the values $\sigma_n = \sigma(t_n, \theta t_n)$ should be appropriately close (and suitably ordered), i.e., the real parts σ_n of complex zeros will change slowly as their imaginary parts increase.*

Example B.2 *Consider again $\beta = \log((\sqrt{53} - 7)/2)$ and $\alpha = \log((\sqrt{85} - 9)/2)$ A "diophantine property" for β/α suggests the reason for the apparent bunching of parts of the red line in Figure 11 (c). In particular, this bunching of lines in $\mathcal{L}_T$ means that the intersections $\mathcal{L}_T \cap \mathcal{C}$ with the (transverse) one dimensional curve $\mathcal{C}$ should be close (and also have a monotone ordering on the curve). Since the restriction $\sigma : \mathcal{C} \to \mathbb{R}$ of the smooth function $\sigma(\cdot,\cdot)$ is again a smooth function known the values $\sigma_n = \sigma(t_n, \theta t_n)$ should be appropriately close (and suitably ordered), i.e., the real parts σ_n of complex zeros will change slowly as their imaginary parts increase.*

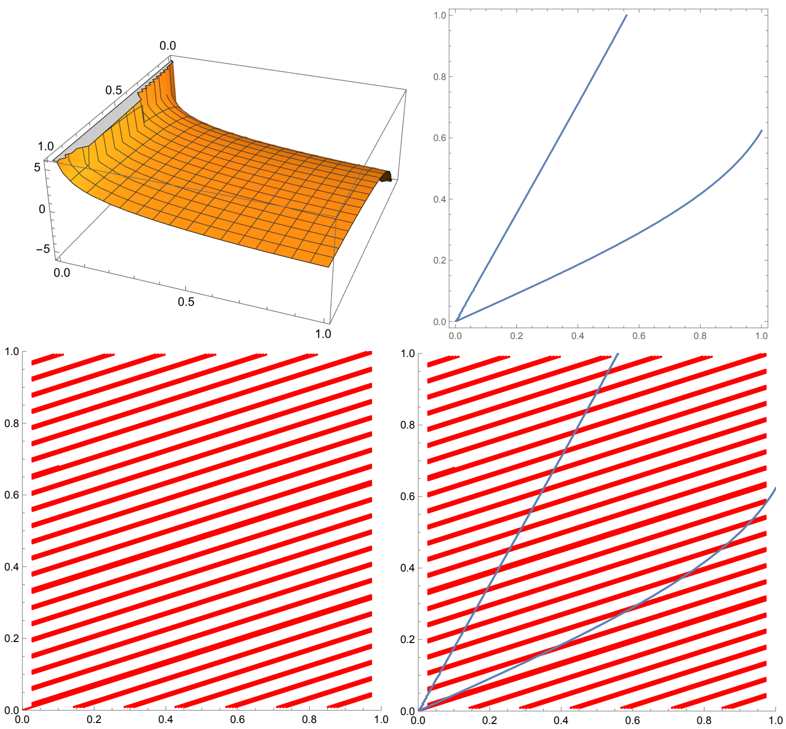

Figure 10: For example B.2 (a) the (orange) plot of $\sigma(\cdot,\cdot)$; (b) the (blue) curve $\mathcal{C}$, (c) The red line $\mathcal{L}_T$ for $T = 90$ wrapping around the torus; (d) The intersections or the red and blue curves give values $(t_n, \theta t_n)$

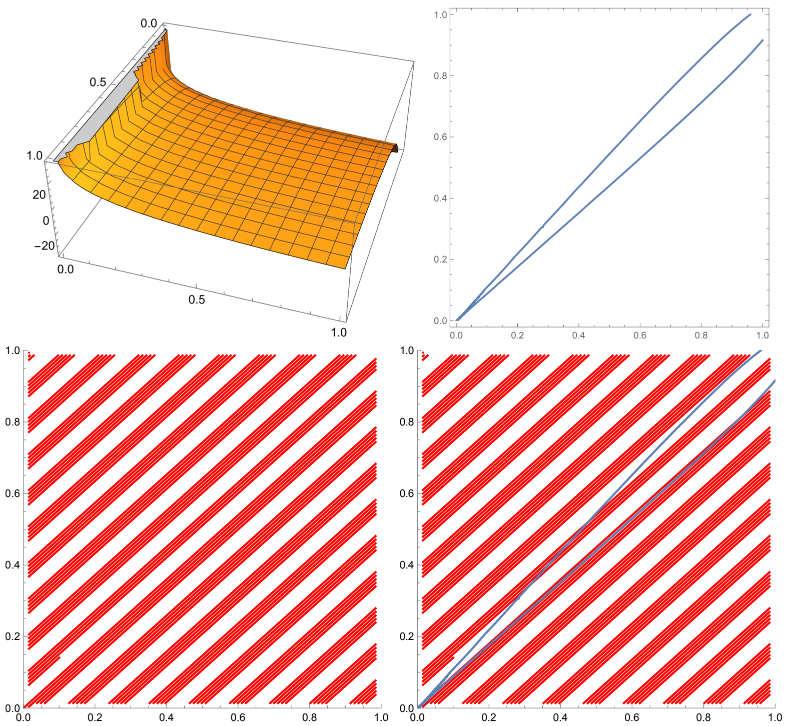

Figure 11: For example B.2 (a) the (orange) plot of $\sigma(\cdot,\cdot)$; (b) the (blue) curve $\mathcal{C}$, (c) The red line $\mathcal{L}_T$ for $T = 40$ wrapping around the torus; (d) The intersections or the red and blue curves give values $(t_n, \theta t_n)$

C Appendix: Speculation on other applications

The properties on $\zeta_{\mathcal{L}}(s)$ can be related to various other properties. In this appendix we collect together various miscellaneous applications.

C.1 Counting functions for length spectra

We want to consider the following counting function for lengths of strings.

Definition C.1 *We can denote by $N_{\mathcal{L}}(x) = \#\{l_j \geqslant 1/x\}$ for $x > 0$ the counting function for lengths greater than $1/x$.*

In particular, in our setting it is easy to that there exists $C > 0$ such that $N_{\mathcal{L}}(x) = Cx + o(x)$ as $x \to +\infty$ as a consequence of δ being the only pole of $\zeta_{\mathcal{L}}(s)$ on $Re(s) = \delta$ and that this is a simple pole (with residue C)

In many cases one can get effective error terms on the asymptotic estimates for $N_{\mathcal{L}}(x)$. This is based on the fact that the geometric zeta function $\zeta_{\mathcal{L}}(s)$ can be expressed in terms of function $N_{\mathcal{L}}(x)$ by the natural identity

$$\zeta_{\mathcal{L}}(s) = \int_0^\infty x^{-s} dN_{\mathcal{L}}(x).$$

A very classical approach based on analytic number theory leads to effective asymptotic estimates. For example, under the hypotheses of Theorem 4.3 one can deduce that there exists $0 < \eta < 1$ such that

$$N_{\mathcal{L}}(x) = Cx + O(x^\eta) \text{ as } T \to +\infty.$$

(The argument would follow the lines of in [15]).

Similarly, when the limit set is $E_{n,m}$ $(n \neq m)$ one can deduce that there exists $\epsilon > 0$ such that

$$N_{\mathcal{L}}(x) = Cx + O(e^{-\epsilon x}) \text{ as } T \to +\infty.$$

(The argument would follow the lines of in [16]).

C.2 Analogues of the Berry conjecture

We can associate to each interval of length ℓ, say, a sequence of frequencies k/ℓ $(k = 1, 2, 3, \cdots)$ which one would associate to an oscillating string. [16] We can denote thet otality of such frequencies by $\mathcal{F} = \{k/l_j : k, j \geqslant 1\}$.

Definition C.2 *We can denote by $N_{\mathcal{F}}(x) = \#\{k/l_j \geqslant 1/x\}$ for $x > 0$ the counting function for lengths greater than $1/x$.*

Giving the values k/l_j an interpretation as the spectrum of the Cantor set. Sir Michael Berry conjectured analogues of the Weyl Theorem for certain domains with fractal domains.[17]

Definition C.3 *We can associate to $\mathcal{F}$ the zeta function*

$$\zeta_{\mathcal{F}}(s) = \sum_{j,k \geqslant 1} \left(\frac{k}{l_j}\right)^{-s}.$$

which converges for $Re(s) > 1$.

Lapidus showed that the zeta functions are related by

$$\zeta_\nu(s)/\zeta_{\mathcal{L}}(s) = \zeta(s)$$

and $\zeta(s) = \sum_{n=1}^\infty n^{-s}$ is the usual Riemann zeta function.

Moreover, we can write

$$\zeta_{\mathcal{F}}(s) = \int_0^\infty x^{-s} dN_{\mathcal{F}}(x).$$

The asymptotics of $N_{\mathcal{F}}(x)$ are discussed in greater detail in ([8], p.149),

[16] The frequencies for the Laplacian on an interval of length ℓ take the form k/ℓ, for $k \in \mathbb{N}_0$

[17] The original conjecture had Hausdorff Dimension rather than Box Dimension, although there are examples (cf. p.370) which show D is more appropriate

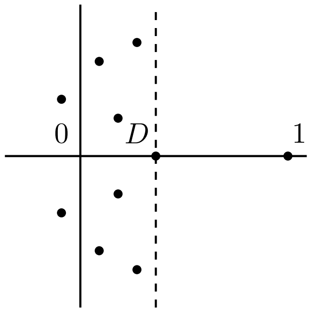

Figure 12: The poles for $\zeta_{\mathcal{L}}(s)$

C.3 Tubular neighbourhoods

We can consider the following "fattening up" of the Cantor set $X \subset [0,1]$.

Definition C.4 *For each $\epsilon > 0$ we can associate an ϵ-neighbourhood $B(X, \epsilon)$ and denote its volume by $V(\epsilon) = \mathrm{Vol}\,(B(X, \epsilon))$.*

For $0 < \epsilon < \frac{1}{2}$ we can then explicitly write

$$V(\epsilon) = (2\epsilon) \sum_{l_j \geqslant 2\epsilon} 1 + \sum_{l_j \geqslant 2\epsilon} l_j .$$

(cf. [8], p. 13-14])

Motivated by the classical 1840 theorem of Steiner (for regions in $\mathbb{R}^2$ and $\mathbb{R}^3$), one can ask about the asymptotics of $V(\epsilon)$ as $\epsilon \to 0$. This is related to $\zeta_{\mathcal{L}}(s)$ by

$$V(\epsilon) = \sum_{w \in \mathcal{D}} c_w(\epsilon) + \frac{1}{2\pi i} \int_S (2\epsilon)^{1-s} \zeta(s) \frac{ds}{s(1-s)}$$

where $c_w = \mathrm{res}_{s=w} \left(\frac{\zeta(s)(2\epsilon)^{1-s}}{s(1-s)} \right)$ (see [8], Theorem 8.1, p. 235])

Example C.5 *As observed in (8.70)-(8.71) on p.263 of [8] if we have $C > 0$ and $p > 0$ such that $S(t) = D - C/(1+t)^p$ then there exists $M > 0$ such that for all $\delta > 0$ we have* [18]

$$V(\epsilon) = M\epsilon^{1-D}(1 + O(|\log \epsilon|^{\delta - 1/p}))$$

In particular, this will hold under the diophantine hypothesis of Theorem 4.3

C.4 Spectral Triples

Following Lapidus and Connes one can associate a spectral triple to the Cantor set (cf. [7],p. 157). For each of the intervals (a_j, b_j) let $\mathcal{H}_j = \ell^2(\{a_j, b_j\}) = \mathbb{C}^2$ be the two dimensional space (with basis $e_{-,j} = (1,0)$ and $e_{+,j} = (0,1)$). One can then define the linear operator $\mathcal{D}_j : \mathcal{H}_j \to \mathcal{H}_j$ using the basis as

$$\mathcal{D}_j = \frac{1}{\ell_j} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which clearly has eigenvalues $\pm 1/\ell_j$. We may then define the finite dimensional Hilbert space $\mathcal{H} = \oplus_j \mathcal{H}_j$ and the (self-adjoint) linear operator $\mathcal{D} = \oplus_j \mathcal{D}_j : \mathcal{H} \to \mathcal{H}$.

Any continuous function $f : K \to \mathbb{R}$ also gives a second linear operator $M_f : \mathcal{H} \to \mathcal{H}$ by multiplication, i.e.,

$$M_f(\Phi)(x) = f(x)\Phi(x), \text{ where } x \in \cup_j \{a_j, b_j\}, \Phi \in H.$$

We can then consider the commutator operator $[\mathcal{D}, M_f] = \mathcal{D}M_f - M_f\mathcal{D} : \mathcal{H} \to \mathcal{H}$ then we can explicitly write

$$[\mathcal{D}, M_f]e_{\pm j} = \frac{1}{\ell_j} \left(f(a_j) - f(b_j) \right) e_{\pm,j}$$

[18] There is probably an additional bound on $\zeta(s)$ required in W

Moreover, if we assume that f lies in the space of Lipschitz functions $\mathcal{A} = C^{Lip}(K)$ then $[\mathcal{D}, M_f]$ is a bounded linear operator.

Definition C.6 *The triple* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *is called a* spectral triple *(in the sense of Connes).*

The eigenvalues of $|\mathcal{D}|^{-1}$ are $\{\ell_1, \ell_1, \ell_2, \ell_2, \ell_3, \ell_3, \ell_4, \ell_4 \cdots\}$ i.e., the lengths of the intervals (with multiplicity two). It then follows that the Diximier trace takes the form

$$\mathrm{tr}(|\mathcal{D}|^{-D}) = 2^D(1-D)\mathcal{M}(d, K)$$

where $\mathcal{M}(d, K)$ is the Minkowski content of K. (The Minkowski measurability follows from the asymptotic $\ell_j \sim cj^{-1/D}$ as $j \to +\infty$, for some $c > 0$ which, in turn frollows from the position of the poles for $\zeta(s)$).

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