Meromorphic extension of the dimension series for hyperbolic Möbius actions on circles

Mark Pollicott

1 Introduction

Let $\mathcal{A} = \{A_1, \cdots, A_d\} \subset SL(2, \mathbb{R})$ ($d \geq 2$) be a finite family of matrices. We call this family *hyperbolic* if there exists $c > 0$ and $\lambda > 1$ such that

$$\|A_{i_1} \cdots A_{i_n}\| \geq c\lambda^n \text{ for all } n \geq 1 \text{ and } 1 \leq i_1, \cdots, i_n \leq d.$$

Such families have been studied by Avila, Bochi and Yoccoz [4].

Definition 1.1. *Associated to this family is the function*

$$\eta_{\mathcal{A}}(s) = \sum_{n=1}^{\infty} \sum_{1 \leq i_1, \cdots, i_n \leq d} \|A_{i_1} \cdots A_{i_d}\|^{-2s}, s \in \mathbb{C}$$

which converges for $Re(s)$ sufficiently large (e.g., $d\lambda^{-2Re(s)} < 1$).

Several authors have studied the case that $s = \sigma > 0$ is real valued and associated the real number

$$s(\mathcal{A}) = \inf\{\sigma > 0 : \eta_{\mathcal{A}}(\sigma) < +\infty\}.$$

cf. work of de Leo [3] and Solomyak-Takahasi [2].

In this note we will be interested in the meromorphic domain of $\eta_{\mathcal{A}}(s)$ as a complex function and the dependence of $s(\mathcal{A})$ on $\mathcal{A}$

Remark 1.2. Let $\hat{A}_i : K \to K$ be the projective action on the unit circle K associated to the matrix A_i. For a hyperbolic family there is a (unique) non-empty closed set $\Lambda \subset K$ such that $\Lambda = \cup_{i=1}^{d} \hat{A}_i \Lambda$. In the paper of Solomyak-Takahasi they further assume that the family $\mathcal{A}$ satisfies a diophantine condition which is implied, for example when the entries of the matrices are algebraic [2]. Under these additional hypotheses they can identify $\dim_H(\Lambda) = \min\{1, s(\mathcal{A})\}$.

2 Results

In this note we will consider the complex analytic nature of the function $s(\mathcal{A})$ and its implications.

Theorem 2.1. *Let $\mathcal{A}$ be a uniformly hyperbolic family.*

1. *The function $\eta_{\mathcal{A}}(s)$ has a meromorphic extension to $\mathbb{C}$;*
2. *The value $s = s(\mathcal{A})$ is a simple pole of residue $C > 0$, say; and*
3. *The function $\eta_{\mathcal{A}}(s)$ has no poles at $s(\mathcal{A}) + it$ $(t \neq 0)$.*

Part of the motivation for Part 1 of this Theorem is the perspective of Lapidus by which the poles of a function such as $\eta(s)$ might has an interpretation as "complex dimensions". The proof of the theorem will be given in the next section.

The following Corollaries are standard.

Corollary 2.2. *Let $\mathcal{A}$ be a uniformly hyperbolic family. The value $s = s(\mathcal{A})$ depends real analytically on $\mathcal{A}$.*

Proof. The follows from the implicit function theorem applied to

$$(\sigma, \mathcal{A}) \mapsto 1/\eta_{\mathcal{A}}(\sigma) = (1/C)(\sigma - s(\mathcal{A})) + \psi(\sigma)$$

where $\psi(s)$ is an analytic function in a neighbourhood of $s(\mathcal{A})$. □

Corollary 2.3. *Let $\mathcal{A}$ be a uniformly hyperbolic family. We have the asymptotic estimate*

$$\#\{(i_1, \cdots, i_n) : \|A_{i_1} \cdots A_{i_d}\| \leq T\} \sim CT^{s(\mathcal{A})T} \text{ as } T \to +\infty$$

Proof. The function $\pi(T) = \#\{(i_1, \cdots, i_n) : \|A_{i_1} \cdots A_{i_d}\| \leq T\}$ is monotone increasing and we can write

$$\eta_{\mathcal{A}}(s) = \int_1^\infty T^{-2s} d\pi(T)$$

By the Ikehara-Wiener Theorem applied to $\eta_{\mathcal{A}}(s)$ the result follows. □

3 Transfer operators

We begin with the following useful lemma (cf. [4])

Lemma 3.1. *The family $\mathcal{A} = (A_1, \cdots, A_d) \in \underline{SL(2, \mathbb{C})^d}$ is* hyperbolic *if there exists a finite disjoint union $\mathcal{U} = \cup_{i=1}^k U_i$ of arcs such that $\overline{\cup_{j=1}^d \hat{A}_j \mathcal{U}} \subset \mathcal{U}$.*

Given $A \in SL(2, \mathbb{C})$ we can assume that the action $\widehat{A} : K \to K$ has two fixed points $x_A^+, x_A^- \in K$. A simple calculation gives

Lemma 3.2. *We can write $\|A_i\|^2 = |A_i'(x_A^+)|$*

We can associate to $\mathcal{U}$ a neighbourhood $\widetilde{\mathcal{U}} \subset C$ in the complexification and we still denote by $\widehat{A}_j : \widetilde{\mathcal{U}} \to \widetilde{\mathcal{U}}$ the extension of the actions on $\widehat{A}_j : \mathcal{U} \to \mathcal{U}$ $(j = 1, \cdots, d)$.

We let $\mathcal{B}$ denote the Banach space of bounded analytic functions $f : \widetilde{\mathcal{U}} \to \mathbb{C}$ with the supremum norm.

Definition 3.3. *We define the family of linear operators* $\mathcal{L}_s : \mathcal{B} \to \mathcal{B}$ $(s \in \mathbb{C})$ *by*

$$\mathcal{L}_s f(z) = \sum_{i=1}^{d} |\widehat{A}_i'(z)|^s f(\widehat{A}_i z).$$

The following result is classical.

Lemma 3.4. *Each operator* $\mathcal{L}_s$ *is a nuclear operator and thus trace class. More precisely there exists*

1. $(v_n)_{n=1}^{\infty} \subset \mathcal{B}$;
2. $(\ell_n)_{n=1}^{\infty} \subset \mathcal{B}^*$ *with* $\ell_n(v_n) = 1$; *and*
3. $(\alpha_n)_{n=1}^{\infty} \subset \mathbb{C}$ *where* $|\alpha_n| = O(\rho^n)$ *for some* $0 < \rho < 1$,

such that $\mathcal{L}_s(\cdot) = \sum_{n=1}^{\infty} \alpha_n v_n \ell_n(\cdot)$.

Nuclear operators are trace class, i.e., there are countably many eigenvalues $(\lambda_n(s))_{n=1}^{\infty}$ and $\mathrm{tr}(\mathcal{L}_s) = \sum_{n=1}^{\infty} \lambda_n(s)$ is finite.

Given an n-tuple $\underline{i} = (i_1, \cdots, i_n)$ we can write $A_{\underline{i}} = A_{i_1} \cdots A_{i_n}$ we can write $|\underline{i}| = n$ and $A_{\underline{i}} = A_{i_1} \cdots A_{i_n}$. Let $\widehat{A}_{\underline{i}} : K \to K$ be the associated projective action and let $x_{\underline{i}}^+ = \widehat{A}_{\underline{i}}(x_{\underline{i}}^+)$ with $|\widehat{A}_{\underline{i}}'(x_{\underline{i}}^+)| < 1$ be the attracting fixed point. In particular, we can explicitly compute the traces.

Lemma 3.5. *For* $n \geq 1$, *we can write*

$$\mathrm{tr}(\mathcal{L}_s^n) = \sum_{|i|=n} \frac{|A_{\underline{i}}'(x_{\underline{i}}^+)|^s}{1 - A_{\underline{i}}'(x_{\underline{i}}^+)} = \sum_{|i|=n} \frac{\|A_{\underline{i}}\|^{-2s}}{1 - \|A_{\underline{i}}\|^{-2}}.$$

We can now follow Grothendieck-Ruelle viewpoint in writing the following for $z, s \in \mathbb{C}$

$$\det(I - z\mathcal{L}_s) = \exp\left(\sum_{n=1}^{\infty} \mathrm{tr}(\mathcal{L}_s^n)\right)$$

which converges for $Re(s)$ sufficiently large and $|z|$ sufficiently small.

It is convenient to introduce another bi-complex function.

Definition 3.6. *We formally define a* zeta function *by*

$$\zeta(z, s) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{|\underline{i}|=n} \|A_{\underline{i}}\|^{-2s}\right)$$

This converges to a non-zero analytic function provided $Re(s)$ *is sufficiently large and* $|z|$ *is sufficiently small.*

In particular, we have the following

Lemma 3.7. *We can write*

$$\zeta(z, s) = \frac{\det(I - z\mathcal{L}_{s+1})}{\det(I - z\mathcal{L}_s)}$$

and deduce that $\zeta(z, s)$ *is bi-meromorphic.*

3.1 Proof of Theorem 2.1

We can now relate $\zeta(z,s)$ to $\eta(s)$ as follows. We can write

$$\frac{\partial}{\partial z}\log\zeta(z,s)|_{z=1} = \frac{\partial}{\partial z}\left(\sum_{n=1}^{\infty}\frac{z^n}{n}\sum_{1\le i_1,\cdots,i_n\le d}\|A_{i_1}\cdots A_{i_d}\|^{-2s}\right)|_{z=1}$$
$$= \left(\sum_{n=1}^{\infty} z^{n-1}\sum_{1\le i_1,\cdots,i_n\le d}\|A_{i_1}\cdots A_{i_d}\|^{-2s}\right)|_{z=1} = \eta(s)$$

By the previous lemma we can write

$$\frac{\partial}{\partial z}\log\zeta(z,s)|_{z=1} = \frac{\frac{\partial}{\partial z}\log\det(I - z\mathcal{L}_{s+1})|_{z=1}}{\det(I - z\mathcal{L}_{s+1})} - \frac{\frac{\partial}{\partial z}\log\det(I - z\mathcal{L}_{s})|_{z=1}}{\det(I - z\mathcal{L}_{s})}$$

Since $\det(I - z\mathcal{L}_s)$ is bi-analytic we can deduce from these two identities that $\eta(s)$ is a meromorphic function.

Moreover, the pole at $s = s(\mathcal{L})$ arises as a zero for $\det(I - z\mathcal{L}_s)$, which in turn means 1 is an eigenvalue for $\mathcal{L}_s$). The simplicity of the pole at $s = s(\mathcal{A})$ comes from the simplicity of the eigenvalue $\lambda(s)$ for which $\lambda(s(\mathcal{A})) = 1$ and that $\frac{\partial}{\partial s}\lambda(s(\mathcal{A}))|_{s=s(\mathcal{A})} \neq 0$.

One can show that if $s = s(\mathcal{A} + it)$ is a pole for $\eta(s)$ then $t = 0$ by observing that this condition implies that 1 is an eigenvalue for $\mathcal{L}_s$. But if we assume for a contradiction that there is a solution to the eigenvalue equation $\mathcal{L}_s h = h$ then we deduce by taking absolute values that $\mathcal{L}_{s(\gamma)}|h| \geq |h|$ which leads to the conclusion that

References

[1] A. Avila, J. Bochi, J.-C. Yoccoz, Uniformly Hyperbolic Finite-Valued SL(2,R)-Cocycles

[2] Boris Solomyak and Yuki Takahashi Diophantine property of matrices and attractors of projective iterated function systems in $\mathcal{R}P^1$

[3] R. De Leo, On the exponential growth of norms in semigroups of linear endomorphisms and the Hausdorff dimension of attractors of projective iterated function schemes, J. Geom. Anal., 25 (2015) 1798-1827

[4] A. Avila, J. Bochi, J.-C. Yoccoz, Uniformly Hyperbolic Finite-Valued SL(2,R)-Cocycles, Commentarii Mathematici Helvetici 85 (2010), 813-884