MINIMIZING ENTROPY FOR TRANSLATION SURFACES

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ABSTRACT. In this note we consider the entropy [5] of unit area translation surfaces in the $SL(2,\mathbb{R})$ orbits of square tiled surfaces that are the union of squares, where the singularities occur at the vertices and the singularities have a common cone angle. We show that the entropy over such orbits is minimized at those surfaces tiled by equilateral triangles where the singularities occur precisely at the vertices.

1. INTRODUCTION

We begin by recalling for the purposes of motivation a well known classical result of Katok from 1982 for compact negatively curved surfaces. Let $\mathcal{M}_g$ denote the space of negatively curved C^∞ Riemannian metrics of unit volume on a compact orientable surface of genus $g \geqslant 2$. The entropy function $h : \mathcal{M}_g \to \mathbb{R}^+$ can be defined in terms of the growth rate of closed geodesics

$$h(\rho) = \lim_{T\to+\infty} \frac{1}{T} \log \#\{\gamma : \ell_\rho(\gamma) \leqslant T\}$$

where $\rho \in \mathcal{M}_g$ and $\ell_\rho(\gamma)$ denotes the length of a closed ρ-geodesic γ. When restricted to metrics of unit volume the entropy is minimized precisely at metrics of constant curvature [9].

In this note we want to formulate a partial analogue of this result for translation surfaces. Dankwart associated to a translation surface X with a non-empty finite singularity set Σ an analogous notion of entropy [5]. Given $k_1, \dots, k_n \geqslant 1$ we can denote by $\mathcal{H}^1(k_1, \dots, k_n)$ the space of unit area translation surfaces with n singularities in Σ with cone angles $2\pi(k_1+1), \dots, 2\pi(k_n+1)$. The entropy function $h : \mathcal{H}^1(k_1, \dots, k_n) \to \mathbb{R}^+$ can be defined in terms of the growth rate of closed geodesics containing a singular point

$$h(X) = \lim_{T\to+\infty} \frac{1}{T} \log \#\{\gamma : \gamma \cap \Sigma \neq \emptyset, \ell_X(\gamma) \leqslant T\}$$

where $\ell_X(\gamma)$ denotes the length of a closed geodesic γ on X which includes a singular point from Σ. [1] The entropy is continuous and bounded below, and can become arbitrarily large (when a closed geodesic becomes sufficiently small) even when the total area is normalized [5]. We restrict the type of translation surfaces we will consider as follows:

(1) Firstly, we fix a unit area square tiled surface X_0, which is a union of squares where the singularities occur at the vertices and the singularities have a common cone angle; and

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[1] This avoids the complication of accounting for cylinders of uncountably many parallel geodesics. Alternatively, we could account for these by counting only their free homotopy classes, but then their polynomial growth does not affect the definition of the entropy.

(2) Secondly, we consider the three dimensional orbit $SL(2,\mathbb{R})X_0$ associated to the linear action of the group $SL(2,\mathbb{R})$ [14].

The area of surfaces in the orbit $SL(2,\mathbb{R})X_0$ coincide with the area of X_0.
Our main result is the following theorem.

Theorem 1.1. *If X_0 satisfies the hypotheses in (1) then the entropy function*

$$h : SL(2,\mathbb{R})X_0 \to \mathbb{R}^+$$

is minimized at equilateral translation surfaces, by which we mean translation surfaces tiled by equilateral triangles where the singularities occur precisely at the vertices of the triangles.

We take the convention that we identify surfaces that are identical up to a rotation (i.e., the action of $SO(2)$).

Theorem 1.1 applies to the following simple example and to the examples listed in §3. Furthermore, it is known that every stratum contains an equilateral translation surface [4].

Example 1.2. Let X_0 be the L-shaped square tiled translation surface made up of three squares (see Figure 1). The surface X_0 has genus 2 and a single singularity of cone angle 6π. The $SL(2,\mathbb{R})$-orbit, $SL(2,\mathbb{R})X_0$, contains two non-isometric equilateral translation surfaces. These surfaces globally minimize the entropy in the orbit space.[2]

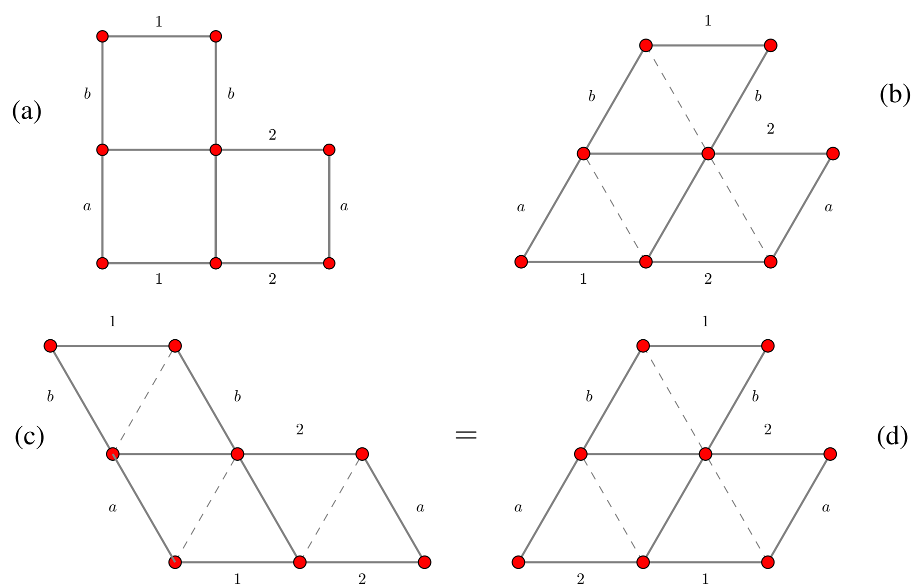

FIGURE 1. (a) The L-shaped translation surface X_0 where the horizontal and vertical sides are identified; (b) and (c) Two distinct surfaces $\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix} X_0$ and $\begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix} X_0$, respectively, in the orbit which are triangulated by equilateral triangles; (d) A surface equivalent to (c).

[2] A slight subtlety is that some authors triple count such surfaces by allowing surfaces to differ under rotation.

In Figure 2 we have plotted an approximation to the entropy of the surfaces

$$\begin{pmatrix} e^u & 0 \\ 0 & e^{-u} \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} X_0 \text{ for } -1 \leqslant s \leqslant 1 \text{ and } -\frac{1}{10} \leqslant u \leqslant \frac{1}{10}$$

and indicated the points above $(u, s) = (0, 0), (0, \pm\frac{1}{2})$. We confirm empirically that the entropy is minimized at $(u, s) = (0, \pm\frac{1}{2})$. These correspond to the surfaces in Figure 1 (b) and (c), respectively.

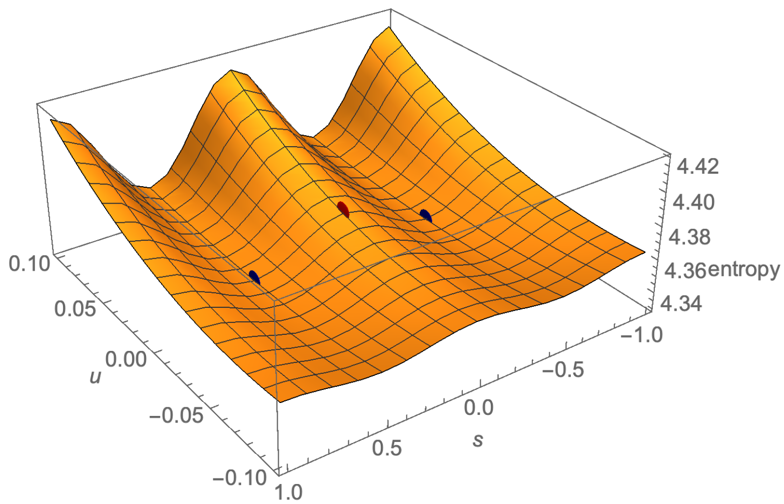

FIGURE 2. A plot of the entropy of the surface $\left(\begin{smallmatrix} e^u & 0 \\ 0 & e^{-u} \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right) X_0$, where X_0 is represented by the L-shaped region in Figure 1 (a).

A simple symmetry argument confirms that $(u, s) = (0, 0)$ is also a critical point.

If we homothetically scale any translation surface by a factor $c > 0$ then the entropy scales by $1/c$, but the area scales by c^2. Therefore, it is appropriate to consider translation surfaces scaled to have unit area, say. Let $\mathcal{H}^1[k, n]$ denote a stratum of unit area surfaces with n singularities, each with the same cone angle $2\pi(k + 1)$, where $n, k \geqslant 1$. In light of Theorem 1.1 we conjecture the following:

Conjecture 1.3. *The restriction* $h : \mathcal{H}^1[k, n] \to \mathbb{R}^+$ *has global minima at equilateral translation surfaces.*

In section 2 we present some preliminary results on entropy. In section 3 we will present some more examples of surfaces satisfying the hypotheses in (1). In sections 4 and 5 we introduce the main technical ingredients in the proof: Montgomary's and Bernstein's Theorems, respectively. In the penultimate section we complete the proof of Theorem 1.1. In the final section we collect together some final comments and questions.

2. TRANSLATION SURFACES AND ENTROPY

Fix a translation surface X. A saddle connection s is a straight line on X between two singularities (which does not contain a singularity in its interior).

The entropy can be defined in terms of the growth of saddle connection paths, which are geodesics joining singularities. Given a translation surface X, let $\underline{s} = s_1 \ldots s_n$ denote an oriented saddle connection path of length $\ell(\underline{s})$ where consecutive saddle connections s_i and s_{i+1} form a locally distance minimizing geodesic. In particular the angle between s_i and s_{i+1}, for $1 \leqslant i \leqslant n-1$, should be greater than or equal to π on both sides. We write $\ell(\underline{s}) = \sum_{i=1}^n \ell(s_i)$ where $\ell(s_i)$ denotes the length of s_i. The following definition is equivalent to that from the introduction.

Definition 2.1. The entropy $h(X)$ of a translation surface X is given by the growth rate of saddle connection paths on X

$$h(X) = \lim_{T\to+\infty} \frac{1}{T} \log \# \{\underline{s} : \ell(\underline{s}) \leqslant T\}.$$

Whenever $\Sigma \neq \emptyset$ we have that $h(X) > 0$.

There is a useful alternative formulation which we now present in the next lemma.

Lemma 2.2. *We can write*

$$h(X) = \inf \left\{ t > 0 : \sum_{\underline{s}} \exp\left(-t\ell(\underline{s})\right) < +\infty \right\}$$

where the summation is over all saddle connection paths on X.

Lemma 2.2 follows from the Definition 2.1.

Fix X_0 to be a unit area square tiled surface satisfying the conditions in (1) in the introduction. In order to study surfaces $A(X_0)$, where $A \in SL(2,\mathbb{R})$, we can denote by

$$\Lambda = \left\{ \left(\frac{a}{\sqrt{n(k+1)}}, \frac{b}{\sqrt{n(k+1)}} \right) : (a,b) \in \mathbb{Z}^2 \setminus \{(0,0)\} \right\}$$

the scaled standard lattice (minus the origin) and $A(\Lambda) = \{(A \left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right))^T : (p,q) \in \Lambda\}$ its image under the linear action of A. We can then associate functions $f_t : SL(2,\mathbb{R}) \to \mathbb{R}^+$ for each $t > 0$ by

$$f_t(A) = \sum_{\underline{v} \in A(\Lambda)} \exp\left(-t\|\underline{v}\|\right)$$

where $\|\underline{v}\|$ is the usual Euclidean length of $\underline{v} \in \mathbb{R}^2$.

The functions $f_t(\cdot)$ have the following properties.

Lemma 2.3. *Let $X_0 \in \mathcal{H}^1[k,n]$ satisfy the hypotheses in (1) . Fix $A \in SL(2,\mathbb{R})$.*

(1) *The function $f_t(A)$ is well defined and monotone decreasing for all $t > 0$.*
(2) *The entropy $h = h(A(X_0)) > 0$ is the unique solution to $f_h(A) = \frac{1}{k}$*

Proof. Part 1 follows from the definition of $f_t(A)$

For part 2, first note that the square tiled surface X_0 factors over the homothetically scaled version $\frac{1}{\sqrt{n(k+1)}}\mathbb{R}^2/\mathbb{Z}^2$ of the standard torus. If X_0 has singularities with cone angles $2\pi(k+1)$ then the oriented saddle connections s based at some $x \in \Sigma$ are in bijection with $(k+1)$ copies of lines in $\mathbb{R}^2$ of the form $(0,0)$ to (p,q), where p and q are coprime.

There is a correspondence between oriented saddle connection paths $\underline{s}$ and

$$(x; j_1, (p_1,q_1); j_2, (p_2,q_2); \cdots ; j_m, (p_m,q_m)) \in$$
$$\Sigma \times \left(\mathbb{Z}_{k+1} \times \mathbb{Z}^2 \setminus \{(0,0)\}\right) \times \left(\mathbb{Z}_k \times \mathbb{Z}^2 \setminus \{(0,0)\}\right) \times \cdots \times \left(\mathbb{Z}_k \times \mathbb{Z}^2 \setminus \{(0,0)\}\right)$$

for $m \geqslant 1$, depending on which singularity the path starts at. Moreover, we can write

$$\ell(\underline{s}) = \frac{1}{\sqrt{n(k+1)}} \sum_{i=1}^{m} \|(p_j, q_j)\| \text{ where } \|(p_j, q_j)\| = \sqrt{p_j^2 + q_j^2}.$$

Using the above correspondence we can write

$$\begin{aligned}
\sum_{\underline{s}} \exp\left(-t\ell(\underline{s})\right) &= n(k+1) \sum_{\underline{v}\in A(\Lambda)} \exp\left(-t\ell(\underline{v})\right) \sum_{m=0}^{\infty} \left(k \sum_{\underline{v}\in A(\Lambda)} \exp\left(-t\ell(\underline{v})\right)\right)^m \\
&= n(k+1) f_t(A) \sum_{m=0}^{\infty} (k f_t(A))^m \\
&= n \frac{(k+1) f_t(A)}{1 - k f_t(A)}
\end{aligned}$$

provided $f_t(A) < \frac{1}{k}$. Moreover, since $f_t(A)$ is monotone in t, by part 1 of Lemma 2.3, then Lemma 2.2 and the above identity complete the proof. □

Lemma 2.3 gives a particularly useful characterization of the entropy. As a first application we have the following.

Corollary 2.4. *Let $X_0 \in \mathcal{H}^1[k, n]$ satisfy the hypotheses in (1). Then the entropy function $h : SL(2, \mathbb{R})X_0 \to \mathbb{R}^+$ is real analytic when restricted to the orbit $SL(2, \mathbb{R})X_0$.*

Proof. In the definition of $f_t(A)$ the dependence of the saddle connections on A is real analytic. The function also has an analytic dependence on t. By Lemma 2.3 we see that $h(AX_0)$ satisfies $f_{h(AX_0)}(AX_0) = \frac{1}{k}$ and then applying the Implicit Function Theorem gives the result. □

3. Examples

We are interested in square tiled surfaces where the vertex of each square is a singularity with a common cone angle. We recall that for translation surfaces $X \in \mathcal{H}^1(k_1, \ldots, k_n)$ the genus g satisfies $2g - 2 = \sum_{i=1}^{n} k_i$ [14]. We can consider a few simple examples.

Example 3.1 (O_k, $k \geqslant 2$)**.** This is a translation surface of genus k with two singularities each with cone angle $2\pi k$ (see [12], Definition 5.3, p.53).

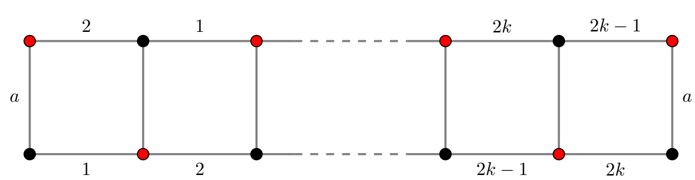

FIGURE 3. A translation surface of genus k and two singularities.

We can next consider two different types of stair examples.

Example 3.2 ($St_k = E_{2k-1}$, $k \geqslant 2$)**.** This is a translation surface of genus k with one singularity with cone angle $2\pi(2k-1)$ (see [12], Definition 5.10, p.61). The special case $k = 2$ corresponds to Example 1.2.

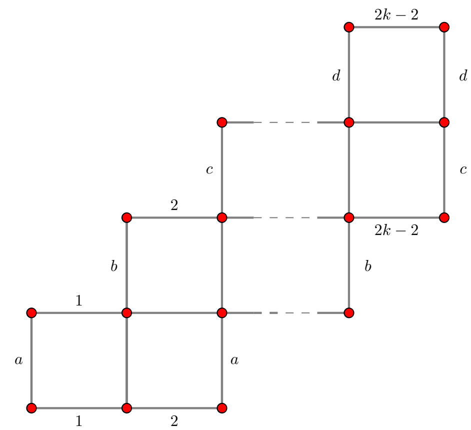

FIGURE 4. A translation surface of genus k and one singularity.

Example 3.3 ($G_k = E_{2k}$, $k \geqslant 2$)**.** This is a translation surface of genus k with two singularities each with cone angle $2\pi k$ (see [12], Definition 5.8, p.59).

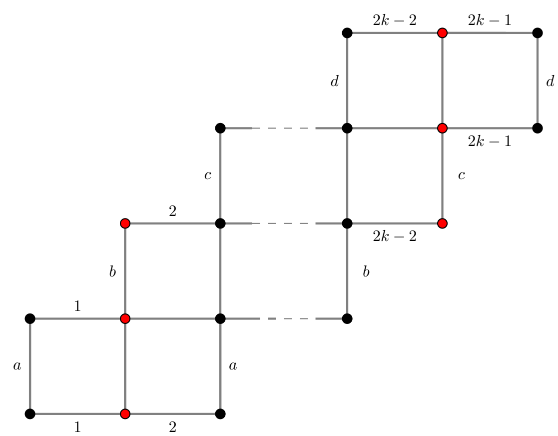

FIGURE 5. A translation surface of genus k and two cone singularities.

Finally, we can consider a well known example of Forni [6] and Herrlich-Schmithüsen [7].

Example 3.4 (Eierlegende Wollmilchsau[3])**.** This is a translation surface of genus 3 with four singularities each with cone angle 4π.

[3]This literally translates as “egg-laying wool-milk-sow” and is a reference to the many different useful properties this example has.

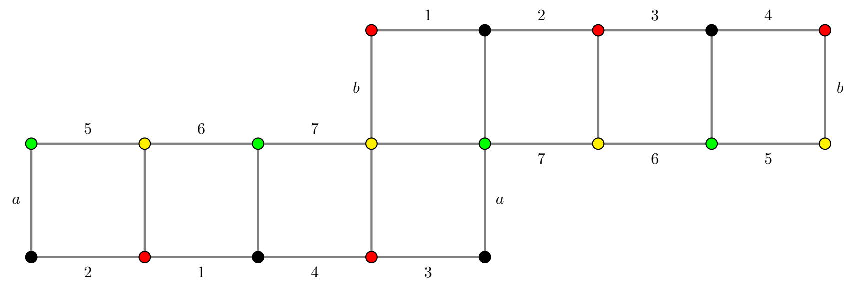

FIGURE 6. A translation surface of genus 3 and 4 singularities.

4. MONTGOMERY'S THEOREM

In order to analyze $f_t(\cdot)$, and thus use Lemma 2.3 to study the entropy, it is convenient to first study a related function $F_t(L)$, where L is a unimodular lattice, i.e., of the form $A(\Lambda)$ for some $A \in SL(2, \mathbb{R})$. Throughout we consider the lattices up to rotation. In particular, this will allow us to use a result of Montgomery.

Definition 4.1. We can associate to a unimodular lattice L and $t > 0$ the function

$$F_t(L) = \sum_{p \in L} \exp\left(-t\|p\|^2\right)$$

where $\|p\|$ denotes the Euclidean norm.

We see that $F_t(L)$ is finite provided $t > 0$. Moreover, on this domain the function $F_t(L)$ has a smooth dependence on t and L. The next result describes lattices which minimize $F_t(L)$ [11] (see also [1], Appendix A). Let L_Δ denote the equilateral triangular lattice.

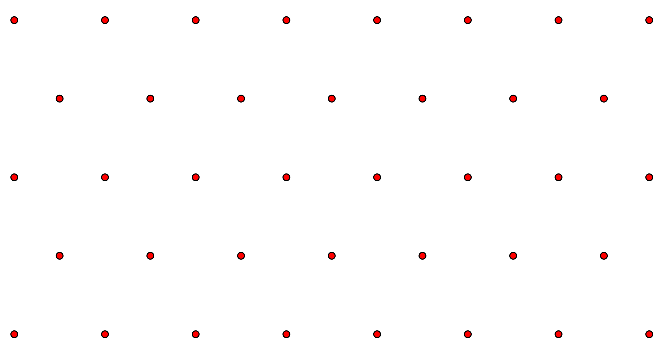

FIGURE 7. Part of the equilateral triangular lattice L_Δ.

Proposition 4.2 (Montgomery's Theorem)**.** *For each $t > 0$ and all unimodular lattices L we have that $F_t(L) \geqslant F_t(L_\Delta)$, with equality iff $L = L_\Delta$.*

Remark 4.3 (Comment on the proof of Proposition 4.2)**.** There is a standard correspondence between unimodular lattices L in $\mathbb{R}^2$ and the standard Modular domain, i.e., $z = x + iy$ with $-\frac{1}{2} \leqslant x < \frac{1}{2}$ and $|z| \geqslant 1$, with suitable identifications on the boundary. Let us denote $L = L_z$. Let $L_\Delta = \mathbb{Z} + z\mathbb{Z}$ be the equilateral triangular lattice with $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. The work of Montgomery established, in particular, the following properties:

(1) If $0 < x < \frac{1}{2}$ and $y \geqslant \frac{1}{2}$ then $\frac{\partial F_t(L_z)}{\partial x} < 0$ for all $t > 0$; and
(2) If $0 < x \leqslant \frac{1}{2}$ and $x^2 + y^2 \geqslant 1$ then $\frac{\partial F_t(L_z)}{\partial y} \geqslant 0$ for all $t > 0$ with equality iff $(x, y) = (0, 1)$ or $(\frac{1}{2}, \frac{\sqrt{3}}{2})$, i.e., the ramification points on the modular surface.

By the definitions we have $F_t(L_{x+iy}) = F_t(L_{-x+iy})$ and so we can assume without loss of generality that $0 \leqslant x \leqslant \frac{1}{2}$. Thus given a lattice L_z we consider a path consisting of a straight line path from $z = x + iy$ to $\frac{1}{2} + iy$ and then a straight line path from $\frac{1}{2} + iy$ to $\frac{1}{2} + i\frac{\sqrt{3}}{2}$ along which $F_t(L_{x+iy})$ decreases for any $t > 0$ (by (1) and (2), respectively).

5. Bernstein's theorem

To proceed we need to relate $F_t(\cdot)$ and $f_t(\cdot)$. This requires a result of Bernstein on completely monotone functions.

Definition 5.1. We call a smooth function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *completely monotone* if for all $x > 0$:

$$(-1)^n \frac{d^n \psi}{dx^n}(x) < 0.$$

We will be particularly interested in the following example.

Example 5.2. The function $\psi(x) = \exp(-\sqrt{x})$ is completely monotone (see the corollary on p. 391 of [10]). More generally, given $\psi_1(x)$ and $\psi_2(x)$ with ψ_1 and ψ_2' completely monotone one has that the composition $\psi_1 \circ \psi_2$ is completely monotone (see Theorem 1 in [10]). We can apply this result with $\psi_1(x) = \exp(-x)$ and $\psi_2(x) = x^{\frac{1}{2}}$.

The interest in completely monotone functions is that they are the Laplace transform of positive functions, as is shown in the following classical theorem [3].

Proposition 5.3 (Bernstein's Theorem)**.** *If ψ is completely monotone, then there exists a positive function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ (i.e., $\rho(t) \geqslant 0$) such that*

$$\psi(r) = \int_0^\infty \exp(-ru)\, \rho(u) du.$$

An account appears, for example, in the book of Widder (see Chapter IV, §12 [13]).

6. Proof of Theorem 1.1

We want to use use Proposition 5.3 to convert Proposition 4.2 for $F_t(\cdot)$ into the corresponding result for $f_t(\cdot)$, in Proposition 6.1 below.

Proposition 6.1 (Bétermin)**.** *For each $t > 0$ and all lattices L we have that $f_t(L) \geqslant f_t(L_\Delta)$ with equality iff $L = L_\Delta$.*

For completeness we recall the elegant short proof of Bétermin.

Proof. First one uses Theorem 6.1 to write

$$\exp(-t\|p\|) = \int_0^\infty \exp\left(-ut\|p\|^2\right) \rho(u) du$$

for $t > 0$ and $p \in L$. Summing gives

$$f_t(L) - f_t(L_\Delta) = \int_0^\infty \left(F_{ut}(L) - F_{ut}(L_\Delta)\right) \rho(u) du \geqslant 0$$

with $F_{ut}(L) - F_{ut}(L_\Delta) \geqslant 0$ and with equality iff $L = L_\Delta$, using Proposition 4.2. □

We can now complete the proof of Theorem 1.1.

Let $A \in SL(2,\mathbb{R})$ be chosen so that AX_0 has a triangulation by equilateral triangles and let $B \in SL(2,\mathbb{R})$ be an element in the group which does *not* correspond to triangulation by equilateral triangles.

We can use the functions $f_t(A)$ and $f_t(B)$ to compare the entropies $h(A)$ and $h(B)$ of AX_0 and BX_0, respectively. By Proposition 6.1 we know that that $f_t(A) < f_t(B)$ for all $t > 0$. By part 2 of Lemma 2.3 the entropy $h(A)$ for the surface AX_0 is the unique value such that $f_{h(A)}(A) = \frac{1}{k}$. However, by part 1 of Lemma 2.3 the function $t \mapsto f_t(B)$ is monotone decreasing so the solution $f_{h(B)}(B) = \frac{1}{k}$ implies that $h(B) > h(A)$.

7. Final comments and questions

(1) If X is an equilateral translation surface of genus $g \geqslant 2$ and with n singularities then the number of equilateral triangles used is $2(2g-2+n)$.

(2) Let X_0 be a square-tiled surface satisfying the hypotheses in (1). Then the number of equilateral surfaces in $SL(2,\mathbb{R})X_0$ corresponds to the index of the Veech group of X_0 in $SL(2,\mathbb{Z})$. The Veech groups for our examples in §3 are computed in [12].

(3) The characterization of the entropy in Part 2 of Lemma 2.3 has the advantage of giving very good numerical approximations. For example, the entropy of the equilateral surfaces in Example 1.2, scaled to have unit area, is

$$h = 4.349345046141502903031389021 37\ldots.$$

Moreover, for this example easy explicit estimates show additionally that the Hessian is non-degenerate.

(4) We can also consider the entropy function on general strata, without the additional restriction that the singularities share the same cone angle. In this broader context it is not clear what the correct candidates for the global minima are, let alone how to prove they minimize the entropy. It is also natural to ask if the entropy function is smooth, by the analogy with Riemannian manifolds with negative sectional curvature [8].

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