# THE HAUSDORFF DIMENSION OF THE SET WHERE THE MINKOWSKI QUESTION MARK FUNCTION HAS INFINITE DERIVATIVE

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## 1. Introduction

The famous Minkowski question mark function $Q : [0,1] \to [0,1]$ was defined by Minkowski in 1904 (at the third ICM in Heidelberg) [10]. In particular, given the continued fraction expansion of an irrational number

$$x = \cfrac{1}{a_1 + \frac{1}{a_2 + \cdots + \frac{1}{a_{n-1} + \cdots}}} \in (0,1)$$

the image of $x$ under $Q$ is given by

$$Q(x) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{a_1 + \cdots + a_n}}$$

This definition is due to Denjoy [2]. This is an example of a continuous monotone increasing function which has zero derivative almost everywhere. Furthermore, at every point the derivative cannot take any other finite value (i.e., every point has derivative zero or infinity, or the derivative will not exist).

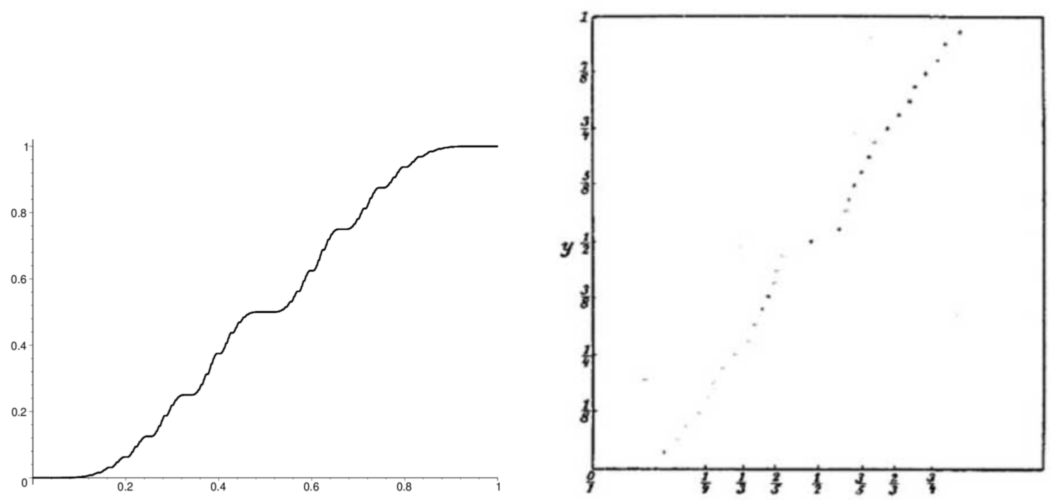


Figure 1. (i) The graph of the Minkowski function, (ii) The original plot from Minkowski's 1904 ICM talk

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We will be interested in what happens on the complement of this set.

**Definition 1.1.** We let $\Lambda_\infty \subset [0,1]$ be the set if points for which the derivative exists and has infinite derivative

In [3] there is a characterization of points in $\Lambda_\infty$ in terms of continued fraction expansions.

By the above we know that $\Lambda_\infty$ has zero Lebesgue measure. However, as was shown by Kinney [7] the set of points with infinite derivative has Hausdorff dimension $\dim_H(\Lambda_\infty) \in (0,1)$, i.e., strictly between between zero and unity. Our contribution is the following.

**Theorem 1.2.** *We can estimate the numerical value of* $\dim_H(\Lambda_\infty)$ *by:*

$$\dim_H(\Lambda_\infty) = 0.87471630510821114221515290421915975775792728975153\ldots$$

*accurate to the* 50 *decimal places given.*

*Remark* 1.3. Kesseboemer and Stratmann [6] show the Hausdorff dimension of the set of points where the derivative does not exist (nor is infinity) has the same value for its dimension as $\Lambda_\infty$.

In the literature there is a estimate accurate to 13 decimal places by Mantica [9] and an estimate accurate to 35 decimal places by Alkauskas [1]. Other early estimates include those by Lagarias (who placed the value in the range $[0.8746, 0.8749]$ [8]) and Paradis, Viader and Bibiloni [11] (who according to [1] revised their estimates to put the value in the range $[0.874716, 0.874719]$). Finally, there was an approximate estimate $\dim_H(\Lambda_\infty) \approx 0.875$ in [17].

The main purpose of this note is to illustrate a simple alternative method to get accurate and rigorous estimates.

## 2. A formula $\dim_H(\Lambda_\infty)$

In order to estimate $\dim_H(\Lambda_\infty)$ want to express this quantity in terms of dynamically defined quantities. We begin with the following classical map on the unit interval.

**Definition 2.1.** We define the *Farey map* $F : [0,1] \to [0,1]$ by

$$F(x) = \begin{cases} \frac{1}{1-x} & \text{if } 0 \leq x \leq \frac{1}{2} \\ \frac{1-x}{x} & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

We let $\mu$ denote the measure of maximal entropy (or Parry measure) for $F$. (Since Farey map is topologically conjugate to the usual tent map on the unit interval and then $\mu$ is the image of the Lebesgue under this conjugating map, which is the question mark function). Associated to this measure is its entropy which is precisely $\log 2$.

A second useful numerical quantity associated to this measure is the following:

**Definition 2.2.** We define the *Lyapunov exponent* $\lambda(\mu)$ for the measure $\mu$ by

$$\lambda(\mu) = \int \log |F'(x)| d\mu(x) \left(= 2\int_0^1 \log(1+x) d\mu(x)\right)$$

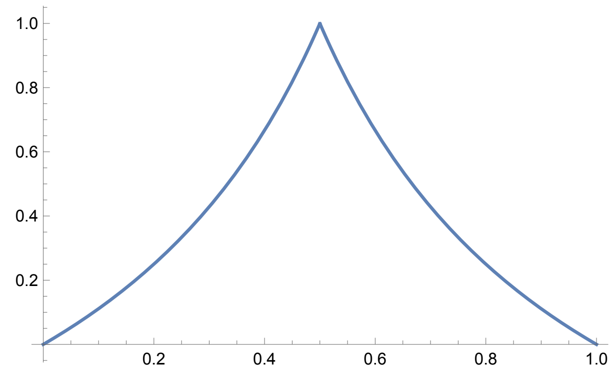


FIGURE 2. The graph of the Farey map

Often in the dynamical approach to the Hausdorff dimension of appropriate sets the value of the Hausdorff dimension turns out to be the ratio of the entropy to the Lyapunov exponent. Kinney [7] and Kesseboemer-Stratmann [6] showed that this was indeed the case for $\dim_H(\Lambda_\infty)$.

**Proposition 2.3** ([7], [6])**.** *One can write*

$$\dim_H(\Lambda_\infty) = \frac{\log 2}{\lambda(\mu)}.$$

In particular, this result translates the problem of estimating $\dim_H(\Lambda_\infty)$ into the problem of estimating $\lambda(\mu)$.

## 3. Approach to estimating $\lambda(\mu)$

We can now introduce a family of bounded linear operators on the Banach space $C([0,1])$ of continuous functions on the unit interval with the usual supremum norm

$$\|w\|_\infty = \sup_{0 \le x \le 1} |w(x)|.$$

**Definition 3.1.** Let $t \in \mathbb{R}$. We define $\mathcal{L}_t : C([0,1]) \to C([0,1])$ by

$$\mathcal{L}_t w(x) = \frac{1}{2(1+x)^{2t}} \left( w\left(\frac{x}{1+x}\right) + w\left(\frac{1}{1+x}\right)\right)$$

where $w \in C([0,1])$.

In particular, the functions $T_1 : x \mapsto \frac{x}{1+x}$ and $T_2 : x \mapsto \frac{1}{1+x}$ arise as inverse branches to the usual Farey map. We can denote the *spectral radius* of $\mathcal{L}_t$ by using the standard spectral radius formula: [1]

$$\rho(t) := \lim_{n \to +\infty} \|\mathcal{L}_t^n\|_\infty^{1/n}.$$

**Example 3.2.** *For example, when* $t = 0$ *then* $\mathcal{L}_0$ *preserves the constant functions are eigenfunctions for the maximal eigenvalue* 1*, which is also the spectral radius, i.e.,* $\rho(0) = 1$.

[1] For simplicity, we formulate this for $\mathcal{L}_t$ acting on continuous functions. However, in the proofs one uses the operator acting on $C^1$ functions and the fact that the spectral radii are the same for both Banach spaces.

We can relate the spectral radius $\rho(t)$ of $\mathcal{L}_t$ to the value $\lambda(\mu)$ by the following bounds.

**Proposition 3.3.** *For $\epsilon > 0$ we can bound*

$$\frac{|\log \rho(\epsilon)|}{\epsilon} \leq \lambda(\mu) \leq \frac{\log \rho(-\epsilon)}{\epsilon} \tag{1}$$

In practice, the values of $\rho(\pm\epsilon)$ maybe difficult to estimate directly from the definition. However, we can implement more tractable bounds using the following simple lemma.

**Proposition 3.4.** *For any $\epsilon > 0$ assume we have*

*(1) a pair of real numbers $r_- > 1 > r_+$; and*
*(2) a pair of functions $w_-, w_+ \in C([0,1])$ with $w_-, w_+ > 0$*

*satisfying*

$$r_+ \leq \inf_{0\leq x\leq 1} \frac{\mathcal{L}_\epsilon w_+(x)}{w_+(x)} \text{ and } \sup_{0\leq x\leq 1} \frac{\mathcal{L}_{-\epsilon} w_-(x)}{w_-(x)} \leq r_-$$

*then $r_+ \leq \rho(+\epsilon)$ and $\rho(-\epsilon) \leq r_-$.*

Comparing Proposition 3.3 and Proposition 3.4 we can deduce that

$$\frac{|\log r_+|}{\epsilon} \leq \lambda(\mu) \leq \frac{|\log r_-|}{\epsilon}. \tag{2}$$

## 4. Proof of Propositions 3.3 and 3.4

We need to formulate some properties of the transfer operators $\mathcal{L}_{\pm\epsilon}$ used in the proof. It is useful to first consider it acting on the (smaller) space $C^1([0,1]) \subset C([0,1])$ of $C^1$ functions which is a Banach space with the new norm

$$\|w\| := \|w\|_\infty + \|w'\|_\infty.$$

We recall that the essential spectral radius $\rho_{ess}(t)$ is given by

$$\rho_{ess}(t) = \lim_{n\to+\infty} \inf \left\{ \|\mathcal{L}_t - K\| : K \text{ is a compact operator} \right\}.$$

The following lemma describes the spectrum of $\mathcal{L}_t$ on $C^1([0,1])$.

**Lemma 4.1** (Spectrum of $\mathcal{L}_t$)**.** *Let $\epsilon < 1$. For $-\epsilon \leq t \leq \epsilon$:*

1. *The operator $\mathcal{L}_t : C^1([0,1[) \to C^1([0,1[)$ has an essential spectral radius $\rho_{ess}(t) < \rho(t)$;*
2. *The operator $\mathcal{L}_t : C^1([0,1[) \to C^1([0,1[)$ has a simple maximal positive eigenvalue $\rho(t)$ with eigenfunction $h_t$ (i.e., $\mathcal{L}_t h_t = \rho(t) h_t$);*
3. *The dual operator $\mathcal{L}_t^* : C^1([0,1[)^* \to C^1([0,1[)^*$ has a simple maximal positive eigenvalue $\rho(t)$ with eigenmeasure $\mu_t$ (i.e., $\mathcal{L}_t^* \mu_t = \rho(t)\mu_t$); and*
4. *The rest of the spectrum of $\mathcal{L}_t : C^1([0,1[) \to C^1([0,1[)$ lies inside a disk of radius strictly smaller than $\rho(t)$.*

This result is standard and well known in the case that $T_1$ and $T_2$ are strict contractions. (In this case, the value $\log \rho(t)$ is identified with the topological pressure of the function $-\log|T'|$.) For the current case of $\mathcal{L}_t$ we outline the modified proof in the appendix. [2]

**Lemma 4.2** (Properties of $\rho(t)$)**.** *The spectral radius $\rho(t)$ has the following properties:*

*(1)* $\mathbb{R} \ni t \mapsto \log \rho(t)$ *is smooth and monotone decreasing;*
*(2)* $\frac{d \log \rho(t)}{dt}|_{t=0} = -\lambda(\mu)$; *and*
*(3)* $\mathbb{R} \ni t \mapsto \log \rho(t)$ *is convex.*

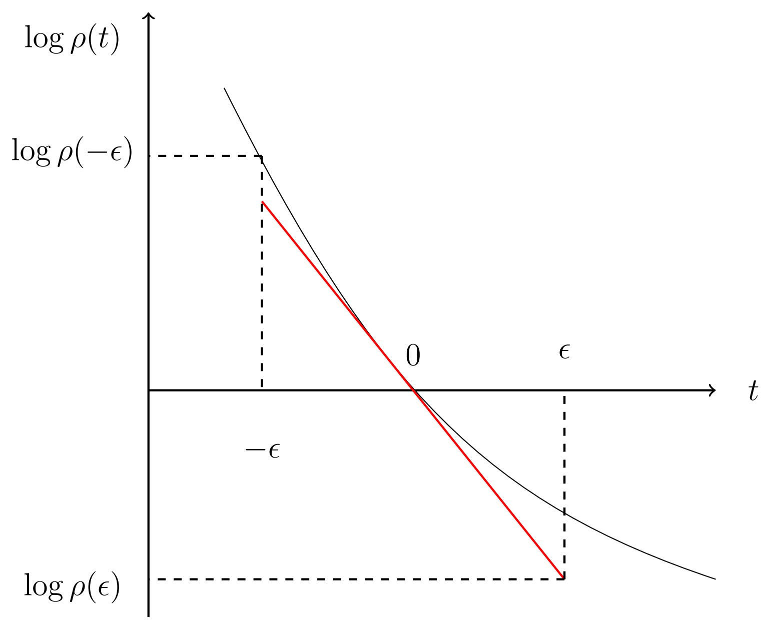


FIGURE 3. The proof of Proposition 3.3

This result is again standard in the case that $T_1$ and $T_2$ are strict contractions. For the current case of $\mathcal{L}_t$ we outline the modified proof in the appendix.

*Proof of Proposition 3.3.* By the previous lemma we can write

$$\frac{d \log \rho(t)}{dt} \begin{cases} \leq -\lambda(\mu) & \text{if } -\epsilon < t \leq 0 \\ \geq -\lambda(\mu) & \text{if } 0 \leq t < \epsilon \end{cases}$$

and integrating this derivative over the intervals $[-\epsilon, 0]$ and $[0, \epsilon]$, respectively, gives the required bounds. This is illustrated in Figure 3. □

This brings us to the proof of the propositions.

[2]For the Farey map it is well known that when $t = 1$ the transfer operator does not have good spectral properties. In particular, $\rho(1) = \rho_{ess}(1)$ and the function $h_1$ is not integrable, as is reflected in the absolutely continuous invariant probability being $\sigma$-finite, but not finite finite. However when, for example, $t = 0$ the situation is much better and the ambient measure $\mu_0$ is the measure of maximal entropy and $h_0 = 1$. So for $t = 0$, and nearby values, we can expect the operator to have good spectral properties

*Proof of Proposition 3.4.* This proof follows the lines argument [14]. Using the positivity of the operators and iterating the inequalities in the hypotheses gives that

$$r_-^n w_-(x) \geq r_-^{n-1}(\mathcal{L}_{-\epsilon} w_-)(x) \geq r_-^{n-2}(\mathcal{L}_{-\epsilon}^2 w_-)(x) \geq \ldots \geq (\mathcal{L}_{-\epsilon}^n w_-)(x) \tag{3a}$$

and

$$r_+^n w_+(x) \leq r_+^{n-1}(\mathcal{L}_{\epsilon} w_+)(x) \leq r_+^{n-2}(\mathcal{L}_{\epsilon}^2 w_+)(x)(\mathcal{L}_{\epsilon}^n w_+)(x) \tag{3b}$$

for all $0 \leq x \leq 1$ and all $n \geq 1$.

Since by Lemma 4.1 the operators $\mathcal{L}_{\pm\epsilon}$ have simple maximal eigenvalues $\rho(\pm\epsilon)$ (with eigenvectors $h_{\pm\epsilon}$ and eigendistributions $\mu_{\pm\epsilon}$ with $\mu_{\pm\epsilon}(h_{\pm\epsilon}) = 1$, without loss of generality) with no other eigenvalues on the unit circle we have that

$$\lim_{n\to+\infty} \|\rho(\pm\epsilon)^{-n}\mathcal{L}_{\pm\epsilon}^n w_{\pm\epsilon} - \underbrace{\mu_{\pm\epsilon}(w_{\pm\epsilon})}_{=1}\|_\infty = 0$$

and, in particular,

$$\lim_{n\to+\infty} \sup_{0\leq x\leq 1} |\mathcal{L}_\epsilon^n w_+(x)|^{1/n} = \rho(\epsilon) \text{ and } \lim_{n\to+\infty} \inf_{0\leq x\leq 1} \left|\mathcal{L}_{-\epsilon}^n w_-(x)\right|^{1/n} = \rho(-\epsilon) \tag{4}$$

Moreover, we trivially have

$$\lim_{n\to+\infty} \inf_{0\leq x\leq 1} |w_+(x)|^{1/n} = 1 = \lim_{n\to+\infty} \sup_{0\leq x\leq 1} |w_-(x)|^{1/n}. \tag{5}$$

Thus, by (3a), (3b), (4) and (5) we have

$$\frac{\rho(\epsilon)}{r_+} \geq 1 \text{ and } \frac{\rho(-\epsilon)}{r_-} \leq 1,$$

as required. □

## 5. Implementing the estimates

We can find candidate functions for $w_-$ and $w_+$ using standard ideas from numerical analysis. More precisely, as in [14] let us fix $N \geq 2$ and choose

(1) Chebychev nodes $\{x_n\}_{n=1}^N \subset [0,1]$ (where $x_n = \frac{1}{2}\left(1 + \cos\left(\frac{n-\frac{1}{2}}{N}\right)\right)$); and
(2) Lagrange polynomials $\{\ell\}_{n=1}^N$ (where $\ell_n(x) = \prod_{k=1}^N (x - x_n)/\prod_{k\neq n}(x_n - x_k)$).

We can then associate the $N \times N$ matrices $M_{\pm\epsilon}$ with entries

$$M_{\pm\epsilon}(i,j) = (\mathcal{L}_{\pm\epsilon}\ell_i)(x_j) \text{ for } 1 \leq i,j \leq N$$

For sufficiently large $N$ the matrix will have a maximal positive eigenvalue with eigenvector $v_\pm = (v_{\pm,1}, \cdots, v_{\pm,N})$. Finally one can let

$$w_\pm(x) = \sum_{n=1}^N v_{\pm,n}\ell_n(x).$$

We first give a simple implementation of these estimates to illustrate the method.

**Example 5.1** (Simple bound). *Let $\epsilon = \frac{1}{10^3}$ and $N = 6$. With this choice of $N$ the functions $w_-, w_+ : [0, 1] \to \mathbb{R}^+$ coming from the above constructions are polynomials of degree* 5

$$w_+(x) = (0.408614\ldots + (0.001189\ldots)x + (0.001228\ldots)x^2 - (0.001309\ldots)x^3 + (0.000869\ldots)x^4 - (0.000246\ldots)x^5$$

*and*

$$w_-(x) = (0.407882\ldots) + (0.001184\ldots)x - (0.001218\ldots)x^2 + (0.001296\ldots)x^3 - (0.000860\ldots)x^4 + (0.000243\ldots)x^5$$

*These are illustrated in Figures 3 and Figure 4. The plots of $w_+$ and $w_-$ are sufficiently close as to appear indistinguishable.*

*One can easily check that the associated values are $r_- = 0.99920825\ldots$ and $r_+ = 1.00079246\ldots$.*

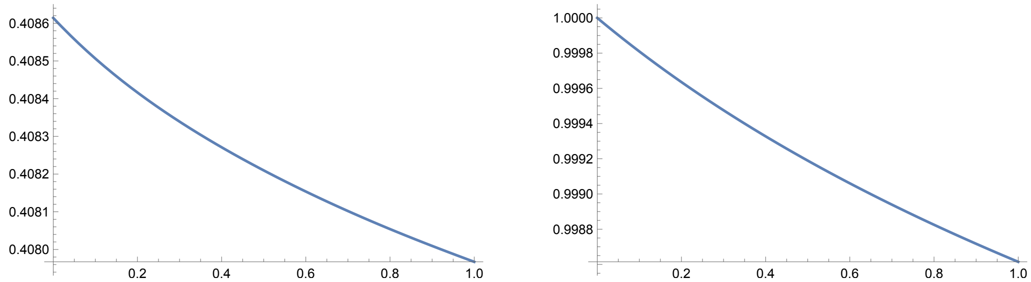


FIGURE 4. (i) The graph of the polynomial $w_-$ (ii) The graph of the ratio $(\mathcal{L}_{-\epsilon}w_-)/w_-$

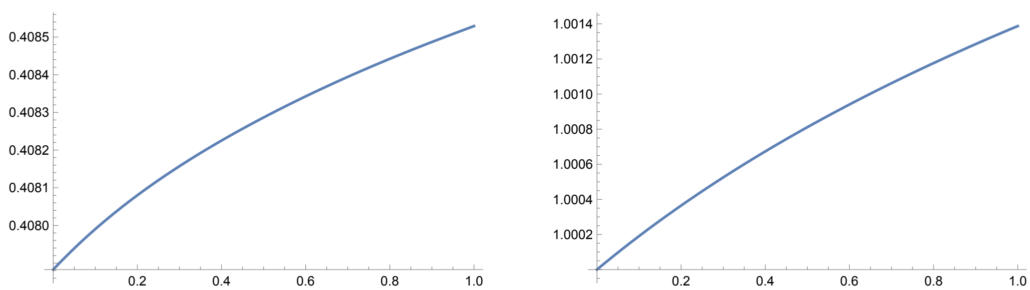


FIGURE 5. (i) The graph of the polynomial $w_+$ (ii) The graph of the ratio $(\mathcal{L}_{-\epsilon}w_+)/w_+$

*Working to* 50 *decimal places one obtains from (2) an estimate for the Lyapunov exponent of*

$$\mathbf{0.792}4371\ldots \leq \lambda(\mu) \leq \mathbf{0.792}5251\ldots$$

*This leads to basic bounds on the dimension of*

$$\mathbf{0.87}4169\ldots \leq \dim_H(\Lambda_\infty) \leq \mathbf{0.87}5016\ldots$$

*with the difference between the upper and lower bounds being* $0.000847\ldots$.

**Example 5.2** (Better bounds)**.** *Let $\epsilon = \frac{1}{120}$ and $N = 300$. Working to 200 decimal places one obtains an estimate for the Lyapunov exponent of*

$$\mathbf{0.79242512859548911819121151529989139888941278204380}0623726386297092 91$$
$$\leq \lambda(\mu) \leq$$
$$\mathbf{0.79242512859548911819121151529989139888941278204380}0933137291218528 86$$

*This leads to bounds on theHausdorff dimension of $\Lambda_\infty$*

$$\mathbf{0.87471630510821114221515290421915975777579272897515 3}223751895331660433$$
$$\leq \dim_H(\Lambda_\infty) \leq$$
$$\mathbf{0.87471630510821114221515290421915975777579272897515 3}189597657442616397$$

*with the difference between the upper and lower bounds being* $3.416 \times 10^{-52}$

## Appendix A. Proof of [Proofs of Lemmas 4.1 and 4.2]

We present the proofs of the two lemmas which were postponed from earlier.

*Proof of Lemma 4.1.* For part (1): Unfortunately, since the maps $T_1$ and $T_2$ are not strictly contracting we cannot directly apply the traditional results (e.g, from [15]). It should be possible to apply some version of the general results for systems that contract on average [13]. However, we will sketch how the proofs in the hyperbolic case can be adapted to the present context. We can write

$$\begin{aligned}\mathcal{L}_t^2 w(x) &= |(T_1 \circ T_1)'(x)|^t w(T_1 \circ T_2(x)) + |(T_1 \circ T_2)'(x)|^t w(T_1 \circ T_2(x)) \\ &\quad + |(T_2 \circ T_1)'(x)|^t w(T_1 \circ T_2(x)) + |(T_2 \circ T_2)'(x)|^t w(T_1 \circ T_2(x))\end{aligned}$$

where we recall that $T_1(x) = \frac{1}{1-x}$. and $T_1(x) = \frac{1-x}{x}$. Observe that

$$\begin{aligned} T_1 \circ T_1(x) &= \frac{1+x}{2+x} \qquad & T_1 \circ T_2(x) &= \frac{1+x}{1+2x} \\ T_2 \circ T_2(x) &= \frac{x}{2+x} \qquad & T_2 \circ T_1(x) &= \frac{1}{2+x} \end{aligned}$$

and. therefore

$$\begin{aligned} \frac{1}{9} &\leq |(T_1 \circ T_1)'(x)| = |(T_2 \circ T_1)'(x)| = \frac{1}{(2+x)^2} \leq \frac{1}{4} \\ \frac{1}{9} &\leq |(T_2 \circ T_2)'(x)| = |(T_1 \circ T_2)'(x)| = \frac{1}{(1+2x)^2} \leq 1. \end{aligned}$$

Furthermore,

$$\begin{aligned} |(T_1 \circ T_1)''(x)| = |(T_2 \circ T_1)''(x)| &= \frac{2}{(2+x)^3} \leq 1 \\ |(T_2 \circ T_2)''(x)| = |(T_1 \circ T_2)''(x)| &= \frac{4}{(1+2x)^2} \leq 4 \end{aligned}$$

We can now bound

$$\left| \left(\mathcal{L}_t^2 w\right)'(x) \right| \leq \frac{1}{2} \sum_{i,j=1}^{2} \frac{|t|.|(T_i \circ T_j)''(x)|}{|(T_i \circ T_j)'(x)|^{1-t}} |w(T_i \circ T_j(x))| + \frac{1}{2} \sum_{i,j=1}^{2} |(T_i \circ T_j)'(x)|^{t+1} |w'(T_i \circ T_j(x))|$$

and deduce that

$$\|(\mathcal{L}_t^2 w)'\|_\infty \le \underbrace{\left(\frac{5|t|}{9^{1-t}}\right)}_{=C} \|w\|_\infty + \underbrace{\frac{1}{2}\left(\frac{1}{4^{t+1}}+1\right)}_{=:\theta<1} \|w'\|_\infty .$$

By iteration one can show that for $n \ge 2$,

$$\|(\mathcal{L}_t^{2n} w)'\|_\infty \le \frac{C}{1-\theta}\|w\|_\infty + \theta^n \|w'\|_\infty$$

(i.e., a Doeblin-Fortet-Ionescu-Tulcea-Lasota-Yorke inequality). From here the proof of quasi-compactness follows by the standard approach involving compact operators coming from the anti-derivative (cf. [4]).

For parts (2) and (3): The existence of a simple maximal eigenvalue follows by first considering the map on the convex weak-star compact space $\mathcal{M}$ of probability measures defined by $\mu \mapsto \mathcal{L}_t^*\mu/(\mathcal{L}_t^*\mu(1))$. Then by the Schauder - Tychonoff. fixed point theorem there exists the fixed point $\mu_t \in \mathcal{M}$. This can then be used to define the cones

$$\mathcal{C}_\lambda = \left\{ f \in C([0,1],\mathbb{R}^+) : f(x) \le e^{\lambda|x-y|} f(y), \forall x,y \in [0,1] \text{ and } \mu_t(f)=1 \right\}$$

By Doeblin-Fortet-Ionescu-Tulcea-Lasota-Yorke inequality we see that $\mathcal{L}_t^n \mathcal{C}_\lambda \subset \operatorname{int}(\mathcal{C}_\lambda)$ for $n$ sufficiently large. It then follows from the theory of Birkhoff metric on cones and the contraction mapping principle that there exists a Lipschitz function $h_\lambda \in C([0,1),\mathbb{R})$ which is an eigenfunction of the maximal eigenvalue. However, by quasi-compactness one can see that actually $h_\lambda \in C^1([0,1])$.

For part (4): We can assume for a contradiction that there exists an eigenfunction $k_t$ for an eigenvalue $\rho(t)r^{i\theta}$ with $0 < \theta < 2\pi$. However, then

$$\frac{1}{\rho(t)}|\mathcal{L}_t k_t(x)| \le \frac{1}{2\rho(t)}\left|\sum_i |T_i'(x)|^t k_t(T_i x)\right| \le \frac{1}{2\rho(t)}\sum_i |T_i'(x)|^t |k_t(T_i x)|$$

and by comparing this with $\frac{1}{\rho(t)}\mathcal{L}_t h_t(x) = h_t(x) > 0$ and using a simple convexity argument we can deduce that $h_t = k_t$ and therefore $e^{i\theta}=1$, giving a contradiction. □

*Proof of Lemma 4.2.* There is a routine thermodynamic proof for the usual transfer operators which we adapt to the present setting [15], [16]. Since $\rho(t)$ is an isolated eigenvalue it follows by analytic perturbation theory that $\rho(t)$ depends analytically on $t$ [5].

To show (1) we can follow ([12], p.60) and differentiate (in $t$) the eigenvalue equation $\mathcal{L}_t h_t = \rho(t) h_t$ to get

$$\mathcal{L}_t\left(\frac{dh_t}{dt}\right) - \mathcal{L}_t\left(\log|F'|h_t\right) = \frac{d\rho(t)}{dt}h_t + \rho(t)\frac{dh_t}{dt}$$

(where we use that $|T_i'| = (1/|T'|)\circ T_i$ (for $i=1,2$) by the chain rule applies to $T\circ T_i(x) = x$). We can then apply the dual eigenmeasure identity $\mathcal{L}_t^*\mu_t = \rho(t)\mu_t$ to write

$$\underbrace{\mu_t\left(\mathcal{L}_t\left(\frac{dh_t}{dt}\right)\right)}_{=\rho(t)\mu_t\left(\frac{dh_t}{dt}\right)} - \underbrace{\mu_t\left(\mathcal{L}_t\left(\log|F'|h_t\right)\right)}_{=\rho(t)\mu_t(\log|F'|h_t)} = \frac{d\lambda_t}{dt}\underbrace{\mu_t(h_t)}_{=1} + \rho(t)\mu_t\left(\frac{dh_t}{dt}\right).$$

which after canceling (the first term on the left hand side with the last term on the right hand side) gives gives $\frac{d\rho(t)}{dt} = -\mu_t(\log|F'|h_t) < 0$ and thus $\frac{d(\log\rho)(t)}{dt} = \frac{d\rho(t)}{dt}\frac{1}{\rho(t)} < 0$, i.e., the function $\log\rho(t)$ is monotone decreasing. In particular, when $t = 0$ then $h_0 = 1$ and $\rho(0) = 1$ and so this expression reduces to $\frac{d\log\rho(t)}{dt}|_{t=0} = -\mu_0(\log|F'|) = -\lambda(\mu)$, as required.

The estimate on the second derivative follows the argument in ([12], p.60-61) and gives

$$\frac{d^2\log\rho(t)}{dt^2} = \lim_{n\to+\infty}\frac{1}{n}\int\left(-\log|(T^n)'(x)| + \int\log|(T^n)'|d\mu_t\right)^2 d\mu_t \geq 0$$

This shows the function is convex. □

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