Estimating characteristic values in Dynamics, Diophantine Approximation and Geometry

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Abstract

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1 introduction

There are certain mathematical problems where accurately attributing a numerical value to some quantity can be important in establishing rigorous results. Examples of such quantities might be:

(a) The size of sets (e.g., a Cantor set) in the unit interval can be quantified by their (Hausdorff) dimension $d \in (0,1)$; and

(b) The growth of random products of matrices, e.g., Fix the pair of matrices $A_1 = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and $A_2 = \left(\begin{smallmatrix} 3 & 1 \\ 2 & 1 \end{smallmatrix}\right)$. Then we can consider the products of all pairs of matrices
$A_1A_1 = \left(\begin{smallmatrix} 5 & 3 \\ 3 & 2 \end{smallmatrix}\right)$, $A_1A_2 = \left(\begin{smallmatrix} 8 & 3 \\ 5 & 2 \end{smallmatrix}\right)$, $A_2A_1 = \left(\begin{smallmatrix} 7 & 4 \\ 5 & 3 \end{smallmatrix}\right)$, $A_2A_2 = \left(\begin{smallmatrix} 11 & 4 \\ 8 & 3 \end{smallmatrix}\right)$,
and triples of matrices
$A_1A_1A_1 = \left(\begin{smallmatrix} 13 & 8 \\ 8 & 5 \end{smallmatrix}\right)$, $A_1A_1A_2 = \left(\begin{smallmatrix} 21 & 8 \\ 13 & 5 \end{smallmatrix}\right)$, $A_1A_2A_1 = \left(\begin{smallmatrix} 19 & 11 \\ 12 & 7 \end{smallmatrix}\right)$, $A_1A_2A_2 = \left(\begin{smallmatrix} 30 & 11 \\ 19 & 7 \end{smallmatrix}\right)$,
$A_2A_1A_1 = \left(\begin{smallmatrix} 18 & 11 \\ 13 & 8 \end{smallmatrix}\right)$, $A_2A_1A_2 = \left(\begin{smallmatrix} 29 & 11 \\ 21 & 8 \end{smallmatrix}\right)$, $A_2A_2A_1 = \left(\begin{smallmatrix} 26 & 15 \\ 19 & 11 \end{smallmatrix}\right)$, $A_2A_2A_2 = \left(\begin{smallmatrix} 41 & 15 \\ 30 & 11 \end{smallmatrix}\right)$,
etc.,
and we continue with products of longer strings matrices. The growth rate of the norms (or the entries) of typical products

$$A_{i_1}A_{i_2}\cdots A_{i_n} \text{ with } i_1, i_2, \ldots, i_n \in \{1,2\}$$

as $n \to +\infty$ is given by the Lyapunov exponent $\lambda > 0$.

We would like to estimate these quantities as efficiently as possible. We would, of course, like useful estimates as quickly as possible. But more importantly, we want to have complete confidence in our estimates, i.e., they really are accurate to the number of decimal places given. [1]

[1] To quote the legendary western marshal Wyatt Earp (1848 – 1929) "Fast is fine, but accuracy is everything"

A natural question at this stage might be: *Who cares?*. Sometimes precise bounds on these values has (surprising) applications to other problems in mathematics (e.g., in number theory and geometry).

2 Dimension of Cantor sets

We begin with the first problem of estimating the dimension of certain special Cantor [2] sets in the interval.

2.1 Infinite continued fractions

In order to define the sets we are interested in we first recall some basic facts about continued fractions. For any irrational $0 < x < 1$ we can write

$$x = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}} := \lim_{n \to +\infty} \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots + \frac{1}{N}}}}$$

where $a_1, a_2, a_3, \ldots \mathbb{N}$, which we denote by $x = [a_1, a_2, a_3, \ldots]$. Indeed, from the classical 1938 book *An Introduction to the Theory of Numbers* by Hardy and Wright [8] [3] we recall the following lemma.

Lemma 2.1 ([8], Theorem 170)**.** *For any irrational number $0 < x < 1$ there exists $a_1, a_2, a_3, \ldots \in \mathbb{N}$ such that $x = [a_1, a_2, a_3, \ldots]$.*

We can consider those continued fractions where we use fewer digits.

Definition 2.2. *Given $m \geq 2$ the finite digit set E_m denotes the set of points in the unit interval whose continued fraction expansion contains only digits from $\{1, \ldots, m\}$., i.e.,*

$$E_m = \{[a_1, a_2, a_3, \ldots,] : a_1, a_2, a_3, \ldots \{1, \ldots, m\}\}.$$

It is an easy exercise to see that:

1. E_m is a Cantor set (i.e., homeomorphic to the middle third Cantor set, say); and
2. E_m has zero Lebesgue measure.

Example 2.3. *The simplest case $m = 2$ gives:*

$$E_2 = \{[a_1, a_2, a_3, \ldots,] : a_1, a_2, a_3, \ldots \{1, 2\}\}.$$

i.e., the set of points in the unit interval whose continued fraction expansion contains only digits from $\{1, 2\}$.

[2] Georg Cantor (1845–1818) was one of the many characters in this story who had an unfortunate end.

[3] G. H. Hardy (1877–1947) was the preeminent English number theorist of his time. Sir Edward Wright (1906-2005) went on to be chancellor of Aberdeen University

2.2 Dimension of sets

A convenient way to describe the "size" of (Cantor) sets $X \subset [0,1]$ of zero Lebesgue measure is to define their dimension [4]. The more subtle definition is that of the Hausdorff dimension.[4] On the other hand the simplest definition is the Box dimension (also called the Minkowski dimension). Fortunately, for sets like E_m the values of these two dimensions coincide. Therefore we can use this as an excuse to take the easy option of just recalling the definition of the Box dimension.

For any $\epsilon > 0$ we let $N(\epsilon)$ be the smallest number of ϵ-intervals needed to cover X .

Definition 2.4. *We define the (Box) dimension by*

$$\dim(X) := \limsup_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log(1/\epsilon)} \in [0,1].$$

In this definition the $\limsup$ is actually a genuine limit.

Remark 2.5. *For the middle third Cantor set C there is a well known explicit value* $\dim(C) = \log 2 / \log 3$ *.*

Unfortunately, there is no explicit expression for $\dim(E_m)$ for any $m \geq 2$. Therefore, one needs to resort to (rigorous) numerical estimates.

Example 2.6 (E_2 revisited)**.** *Jack Good showed* $0.5306 < \dim(E_2) < 0.5320$ *[7] (which was definitely* good *for 1941.) In 2016 Falk and Nussbaum computed* $\dim(E_2) = 0.53128050627720$ *to* 16 *decimal places. In 2018 Jenkinson and the author. used a "zeta function approach" to compute* $\dim(E_2)$ *to 100 decimal places:*[5]

$$\begin{aligned}\dim(E_2) = 0.&531280506277205141624468647368471785493059109018 39\\ &8779888397803927529535643831345918109570181185239 8\\ &8042805724307518763342238933948082230901786959653 2\\ &8712235464299794896637840337287630454110150804519 1\\ &39697680713\ldots\end{aligned}$$

[6]

Remark 2.7 (An aside: A Good story)**.** *Jack Good's estimate was from his PhD thesis supervised at Cambridge by Hardy and Besicovitch. During the Second World War he worked at Bletchley Park in England, helping to break the german enigma communication codes. This wartime work features in the 2014 film* The Imitation Game *about Alan Turing in which Good appeared as a character. In fact, he had a more direct personal connection with the film industry when he worked with the film director Stanley Kubrick as an advisor on the movie* 2001: A space odyssey*.*

[4] Unfortunately, Felix Hausdorff (1868-1942) came to a sad end too.

[5] Now this now known to over 200 decimal places (by Vytnova and the author) using a method I will explain later.

[6] One might wonder if people who compute values to lots of decimal places might not have better things to do? We recall a quote of Isaac Newton (from 1666) on computing 15 digits for π "I am ashamed to tell you to how many figures I carried these computations, having no other business"'

Example 2.8 (E_5)**.** *In 2018, Jenkinson and the author estimated* $\dim(E_5) = 0.83682944\ldots$ *accurate to 8 decimal places [10]. In 2020, Vytnova and the author improved this to 29 decimal places:* $\dim(E_5) = 0.83682944368120882244159438727\ldots$ *[14]*

2.3 Application to Number Theory: Finite continued fractions

We return to continued fractions, but this time *finite* continued fractions. These were described in the 1202 manuscript Liber Abacci by Leonardo of Pisa (Fibonacci, c.1170-c.1250), which was eventually published in 1857 (in Italian). Given a finite sequence $a_1, a_2, \ldots, a_n \in \mathbb{N}$ we can write

$$\frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots + \frac{1}{a_n}}}} = \frac{p}{q} \in \mathbb{Q},$$

which we denote by $p/q = [a_1, a_2, \ldots, a_n]$. We recall the following classical result on finite continued fractions..

Lemma 2.9 (Hardy and Wright, Theorem 161)**.** *It is well known that any rational* $p/q \in \mathbb{Q} \cap (0,1)$ *can be written as a finite continued fraction, i.e., there exist* $a_1, \ldots, a_n \in \mathbb{N}$ *such that* $p/q = [a_1, \ldots, a_n]$.

Application I: Zaremba Conjecture. The Zaremba conjecture asks if we can still get any $q \in \mathbb{N}$ as a denominator for some numerator p if we uniformly bound the digits a_i. More precisely:

Conjecture (*Zaremba*, [17]) [7] For any natural number $q \in \mathbb{N}$ there exists p (coprime to q) and $a_1, \ldots, , a_n \in \{1, 2, 3, 4, 5\}$ such that $\frac{p}{q} = [a_1, \ldots, a_n]$.

Unfortunately, this conjecture is still open. However, there is the following important result.

Theorem 2.10 (Bourgain-Kontorovich-Huang Theorem [2], [9])**.** *The Zaremba conjecture is true for most denominators q, i.e., a density one result of the form*

$$\lim_{Q \to +\infty} \frac{1}{Q} \# \left\{ 1 \leq q \leq Q \, : \, \frac{p}{q} = [a_1, \ldots a_n] \text{ with } a_1, \ldots, a_n \in \{1, 2, 3, 4, 5\} \right\} = 1.$$

However, their proof is conditional on the fact $dim(E_5) = 0.8368\ldots > \frac{5}{6} = 0.833\ldots$ [8] which was finally rigorously established in Remark 2.8

2.4 Method of estimating $\dim(E_m)$ in two steps

The ideas behind the estimates on $\dim(E_m)$ are actually very simple to explain and we will attempt to sketch them in this subsection.

[7]Interestingly, S. Zaremba (1903- 1990) died in Wales, UK
[8]This estimated is used in the "circle method" component of the proof

Step 1: Averaging operators. For simplicity of exposition, we concentrate on E_2 the other examples being similar. Let $C([0,1],\mathbb{R})$ denote the Banach space of continuous functions $f : [0,1] \to \mathbb{R}$.

Definition 2.11. *For each $0 < t < 1$ define a linear operator $L_t : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ by*

$$Ltf(x) = \frac{1}{(1+x)^{2t}} f\left(\frac{1}{(1+x)}\right) + \frac{1}{(2+x)^{2t}} f\left(\frac{1}{(2+x)}\right)$$

for $f \in C([0,1],\mathbb{R})$, $x \in [0,1]$.

We need two standard results.

Lemma 2.12 (Ruelle)**.** *There exists a C^∞ decreasing function $P : \mathbb{R} \to \mathbb{R}$ such that for any positive $f > 0$ we have*

$$e^{P(t)} := \lim_{n \to +\infty} \|L_t^n f\|_\infty^{1/n}.$$

The connection to dimension is the following:

Lemma 2.13 (Ruelle [16], Bowen[9])**.** *The value $t = \dim(E_2)$ is the solution to $e^{P(t)} = 1$.*

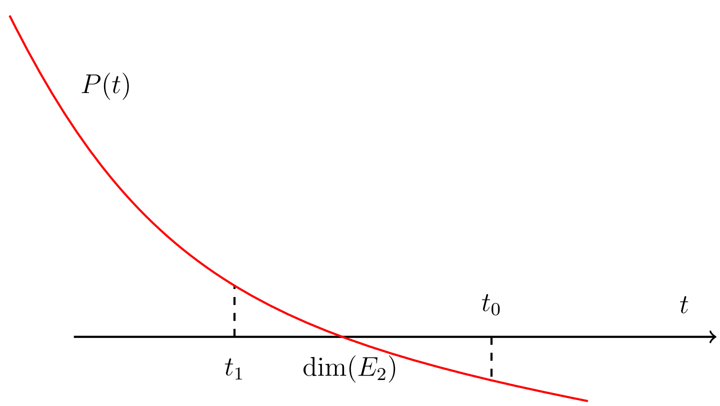


Figure 1: The graph of $P(t)$

Combining these two lemmas gives the conclusion that we need to find t so that $e^{P(t)} = 1$ then $t = \dim(E_2)$.

Step 2: Applying the Bowen and Ruelle results. Given $0 < t_0 < t_1 < 1$ we want to check if $t_0 < \dim_H(E_2) < t_1$.

[9] Sadly, Bowen died at the age of 31. His ideas on Hausdorff dimension and quasi-circles was subsequently put into a more general context by Ruelle

For the upper bound. Assume there exists a (positive) function $g : [0,1] \to \mathbb{R}^+$ such that

$$\sup_x \frac{L_{t_1} g(x)}{g(x)} < 1 \text{ then } 1 \leq e^{P(t_1)}.$$

In particular, the hypothesis implies $L_{t_1}^n g \leq g$ and then it is an easy application of Lemma 2.12.

For the lower bound. Assume there exists a (positive) function $g : [0,1] \to \mathbb{R}^+$ such that

$$\sup_x \frac{L_{t_0} f(x)}{f(x)} > 1 \text{ then } e^{P(t_0)} \geq 1$$

In particular, the hypothesis implies $L_{t_1}^n f \geq f$ and then it is an easy application of Lemma 2.12.

Combining these bounds gives $e^{P(t_0)} \geq 1 \geq e^{P(t_1)}$ and since $e^{P(t)}$ is continuous and monotone we get: $t_0 \leq \dim(E_2) \leq t_1$.

Remark 2.14. *The practical issue of finding such f and g uses numerical analysis (and collocation). This is usually the part which is more time consuming.*

2.5 Application to Diophantine approximation.

Our starting point is the following classical theorem.

Theorem 2.15 (Dirichlet's Theorem [3]). *Let $0 < \alpha < 1$ be irrational. There are infinitely many rationals $\frac{p}{q} \in \mathbb{Q}$ such that $|\alpha - \frac{p}{q}| < \frac{1}{q^2}$.*

The inequality can be made slightly sharper.

Theorem 2.16 (Hurwitz's Theorem [5]). *Let $0 < \alpha < 1$ be irrational. There exist infinitely many rationals $\frac{p}{q} \in \mathbb{Q}$ such that $|\alpha - \frac{p}{q}| < \frac{1}{\sqrt{5}q^2}$.*

We can hope to get better approximations for individual irrational numbers $0 < \alpha < 1$

Definition 2.17. *For each individual irrational number $\alpha \in (0,1)$ we can choose*

$$c(\alpha) := \sup \left\{ c > 0 \,:\, \left| \alpha - \frac{p}{q} \right| < \frac{1}{cq^2} \text{ for infinitely many } \frac{p}{q} \in \mathbb{Q} \right\}$$

Example 2.18. $c(\frac{2}{1+\sqrt{5}}) = \sqrt{5}$

This leads us to the subset of $[\sqrt{5}, +\infty)$ which we would like to study.

Definition 2.19. *We can define the Lagrange spectrum* [10] $\mathcal{L} \subset [\sqrt{5}, +\infty)]$ *by*

$$L = \{c(\alpha) \,:\, \alpha \in \mathbb{R} \setminus \mathbb{Q})\}$$

[10] The Lagrange spectrum was not studied by Lagrange, but by Markoff (1856 - 1922) who came to a sad end.

Some parts of the spectra are well understood.

1. To the left of 3: $L \cap (\sqrt{5}, 3) = \{\sqrt{5} < \sqrt{8} < \sqrt{221}/5 < \dots \sqrt{5} < \sqrt{8} < \sqrt{221}/5 < \dots\}$ is an (explicit) countable set.
2. To the right of $\sqrt{21}$ every point lies in L, i.e., $[\sqrt{21}, +\infty) \subset L$. Hall (1947), Schecker-Freiman (1963)

To understand the (more) complicated nature of the set $L \cap (3, \sqrt{21})$ we can consider the Hausdorff dimension function $t \mapsto f(t) := \dim_H(L \cap (3, t))$. We begin with the following interesting result.

Theorem 2.20 (C. Moreira, [13])**.** *The function $f(t)$ is continuous.*

Furthermore, we can estimate the value $t = t_1$ when the dimension function $f(t)$ first reaches 1.

Theorem 2.21 (Matheus, Moreira, M.P., Vytnova [12])**.** *We can estimate*

$$t_1 := \inf\{t : f(t) = 1\} = 3.334384\dots$$

The basic idea of the proof is to approximate L using suitable Cantor sets which are generalizations of the sets E_m. Previously, Hall (1947) and Moreira (2018) gave rigorous bounds $\sqrt{10} = 3.1622\dots < t1 < \sqrt{12} = 3.4641\dots$ and Bumby (1982) had given non-rigorous bounds: $3.33437 < t_1 < 3.33440$.

3 Lyapunov exponents for random matrix products

We now turn from estimating dimension (the first problem) to estimating Lyapunov exponents (the second problem).

3.1 Random products of matrices

We begin by presenting some basic notation. Fix a finite set of $k \times k$ real matrices $k \geq 2)$: $A_1, \dots, , A_d$ with $d \geq 2$. For each $n \geq 1$ we can consider the d^n products of matrices $A_{i_1} A_{i_2} \cdots A_{i_n}$ where strings $i_1, i_2, \dots, i_n \in \{1, 2, \dots, d\}$ which are chosen with equal probability $1/d$.

Definition 3.1. *We can define the (top) Lyapunov exponent to be the "asymptotic average"*

$$\lambda = \lim_{n \to +\infty} \frac{1}{d^n} \sum_{i_1, \cdots, i_n \in \{1, \cdots, d\}} \frac{\log \|A_{i_1} A_{i_2} \cdots A_{i_n}\|}{n}$$

where $\|A\| = \max_{1 \leq, i, j \leq d} |A_{ij}|$, say.

The existence of the limit comes from a simple subadditivity argument.

Remark 3.2. *Of course, for a single matrix A the value $\lim_{n \to +\infty} \frac{1}{n} \log \|A^n\|$ is merely the logarithm of its spectral radius (i.e., $\max\{|\rho| : \rho =$ eigenvalue for A $\}$)*

Example 3.3. *We recall the example from the introduction. Let $d = 2$ and*

$$A_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } A_1 = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}.$$

But one can ask: *How easy it is to estimate λ.?* [11]

Theorem 3.4 (M.P.-Vytnova, [15]). *In Example 3.3 one can estrimate*

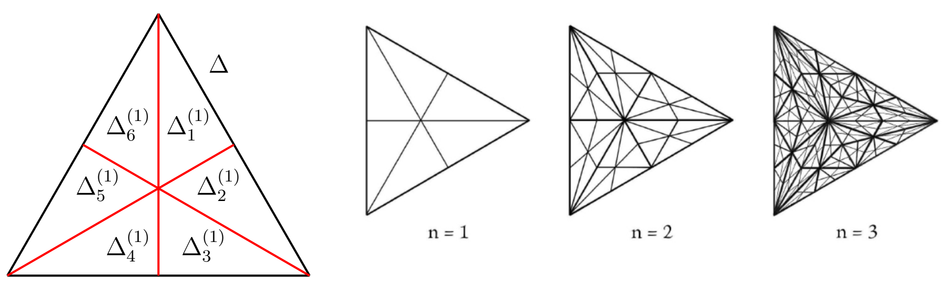
$\lambda = 1.14331103510294924584325185365558829940254614248358355161600419871806898559\ldots$

accurate to the number of decimal places presented.

Previous estimates were only accurate to a third of these decimal places. However, we should ask: *Why should we care about the value of Lyapunov exponents?*

3.2 Application to planar geometry

We will describe a simple geometric construction in Euclidean space. Begin with an (equilateral) triangle $\Delta \subset \mathbb{R}^2$. Draw the three median lines (from each vertex to the midpoint of the other side). The triangle Δ is subdivided into 6 smaller triangles $\Delta_1(1), \cdots, \Delta_6(1)$.



We can repeat the process with each of the six subtriangles $\Delta_i(1)$. $(i = 1, \cdots, 6)$ to get a total of 36 triangles $\Delta_j(2)$ $(j = 1, \cdots, 36)$. We can carry on iteratively to get smaller and smaller triangles. For $n \geq 1$ we get 6^n triangles $\Delta_i(n)$ $(i = 1, \ldots, 6^n)$. We are interested in the shape of a "typical" triangle $\Delta_i(n)$ $(i = 1, \ldots, 6^n)$ as $n \to +\infty$. The qualitative observation is that the triangles become "degenerate", i.e., for a (Lebesgue) typical point x the shape of the sub-triangle containing it at the nth stage gets "flattened".

There is a more quantitive description of how fast this happens. For a (Lebegue) typical point $x \in \Delta$ we let $\Delta(n)(x)$ denote the n-th level triangle containing x. To measure quantitative degeneracy for triangles we let $\frac{\pi}{3} \leq \theta(n)(x) \leq \pi$ be the largest internal angle of $\Delta(n)(x)$. Typically triangles will degenerate (in the sense that the largest angle $\Delta(n)(x)$ tends to π). This can be quantified as follows.

[11] But who exactly is asking? We recall the following quotations: "Pride of place among the unsolved problems of subadditive ergodic theory must go to calculation of the value λ ... and indeed this usually seems to be a problem of some depth." - Sir John Kingman (1973); and "We turn now to the excruciating problem of the subject: Devise reasonably general and effective algorithms for explicit calculation (or at least approximation) of Lyapunov exponents." -Yuval Peres (1992).

Theorem 3.5. *(Bárány, Beardon and Carne) There exists $\lambda > 0$ such that for almost every (Lebesgue) $x \in \Delta$*

$$\lim_{n \to +\infty} \frac{1}{n} \log\left(\pi - \theta(n)(x)\right) = -2\lambda$$

The proof of Bárány, Beardon and Carne is based on random products using six 2×2 matrices and their Lyapunov exponent λ. The 6 matrices are essentially:

$$A_1 = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ 0 & \frac{3}{\sqrt{6}} \end{pmatrix}, A_2 = \begin{pmatrix} \frac{4}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ \frac{3}{\sqrt{6}} & -\frac{3}{\sqrt{6}} \end{pmatrix}, A_3 = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{3}{\sqrt{6}} & 0 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} -\frac{2}{\sqrt{6}} & \frac{4}{\sqrt{6}} \\ 0 & \frac{3}{\sqrt{6}} \end{pmatrix}, A_5 = \begin{pmatrix} -\frac{2}{\sqrt{6}} & \frac{4}{\sqrt{6}} \\ -\frac{3}{\sqrt{6}} & \frac{3}{\sqrt{6}} \end{pmatrix}, A_6 = \begin{pmatrix} -\frac{4}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ -\frac{3}{\sqrt{6}} & 0 \end{pmatrix}$$

These are used in the definition of the affine maps which map Δ to $\Delta_i(1)$ ($i = 1, \cdots, 6$).

Remark 3.6. *Diaconis and McMullen rediscovered this result (McMullen has a very readable [as always] unpublished note on this topic from 2011). Hough (2009) rigorously showed that $0.0585 < \lambda < 0.0946$ and guessed $\lambda \approx 0.071$. In 2016, Wilkinson popularized this construction in a survey in the Notices of the AMS (although her estimate on λ seems to be different).*

Theorem 3.7 (M.P.+ Vytnova)**.** *The value λ has the rigorous bounds $0.077316 < \lambda < 0.077331$.*

The method of proof is similar in nature to that of the dimension estimate, but with a few more steps. For this reason we consign it to Appendix 3.2

Appendix I: Method of estimating λ in five steps

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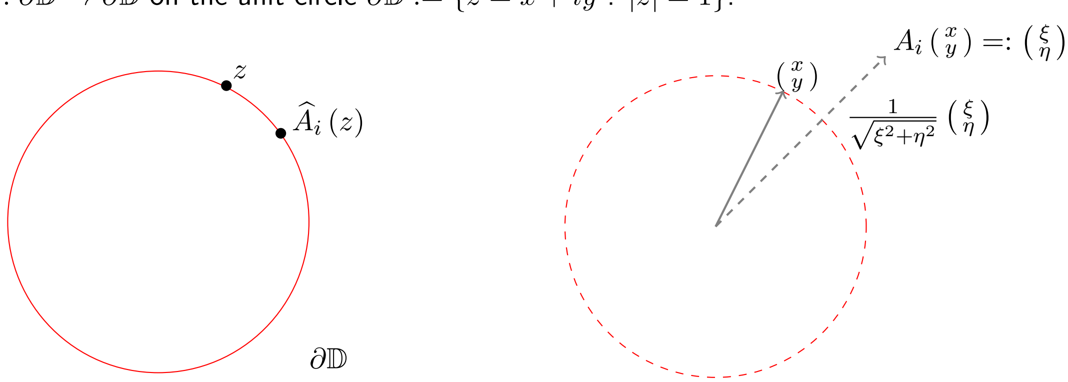
Step 1. Recall we associate to any matrices $A_1, \cdots, A_d$ their "projectivized" actions $\widehat{A}_i : \partial\mathbb{D} \to \partial\mathbb{D}$ on the unit circle $\partial\mathbb{D} := \{z = x + iy : |z| = 1\}$:



1. Given $z = x + iy$ we associate the vector $\binom{x}{y}$.
2. Multiply by the matrix: $A_i \binom{x}{y} = \binom{\xi}{\eta}$.

3. Scale back to $\partial\mathbb{D}$: $\widehat{A}_i(z) = \left(\frac{\xi}{\sqrt{\xi^2+\eta^2}}\right) + i\left(\frac{\eta}{\sqrt{\xi^2+\eta^2}}\right)$.

Remark 3.8. *For some z we have $|\widehat{A}_i'(z)| < 1$ but for some z we have $|\widehat{A}_i'(z)| > 1$.*

Step 2. The transfer operator: Hölder functions. Fix $\alpha > 0$ and let $C^\alpha(\mathbb{S})$ be the Banach space of α-Hölder continuous functions $f : \mathbb{S} \to \mathbb{C}$ with the norm

$$\|f\|_\alpha := \sup_{x\in\mathbb{S}} |f(x)| + \sup_{x\neq y\in\mathbb{S}} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

Provided α is sufficiently small, for each $t \in \mathbb{R}$ we can define a linear transfer operator $\mathcal{L}_t : C^\alpha(\mathbb{D}) \to C^\alpha(\mathbb{D})$ by

$$\mathcal{L}_t f(x) = \frac{1}{d}\sum_{i=1}^{d} |\widehat{A}_i'(z)|^t f(\widehat{A}_i(z)).$$

(a) When $t = 0$ then $\mathcal{L}_0\mathbf{1} = \mathbf{1}$ where $\mathbf{1}$ is the constant function taking value 1. In particular, 1 is an isolated eigenvalue for $\mathcal{L}_0$.

(b) If $A_1, \cdots, A_d$ *do not have a common eigenvector* (i.e., "strong irreducibility") then 1 is an isolated eigenvalue for $\mathcal{L}_0$.

(c) By perturbation theory for $|t|$ sufficiently small $\mathcal{L}_t$ (still) has an isolated eigenvalue $e^{Q(t)}$, say, with $Q(0) = 0$.

Step 3. Maximal eigenvalue and the Lyapunov exponent. Let $e^{P(t)}$ be maximal eigenvalue for $\mathcal{L}_t$ and $|t|$ sufficiently small.

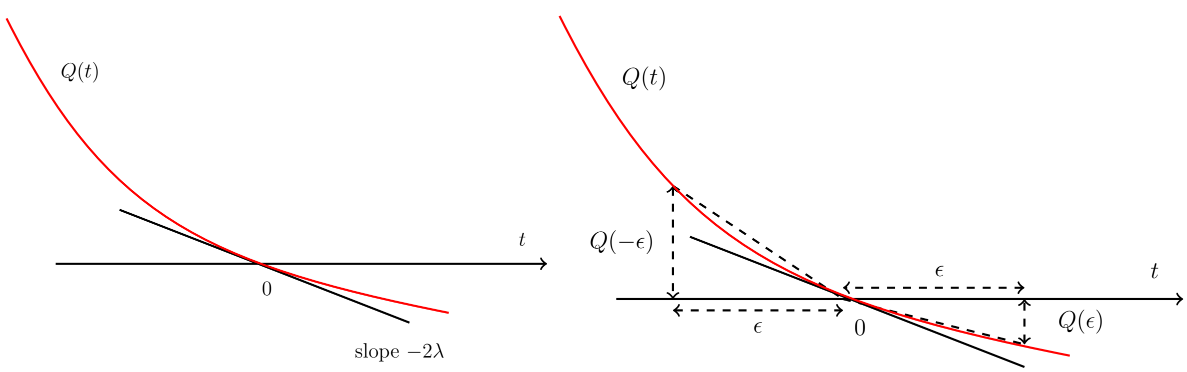


Figure 2: (i) The graph of $Q(t)$; (ii) Using $Q(\epsilon)$ and $Q(-\epsilon)$ to bounds λ.

(i) $Q(t)$ is (locally) C^∞ monotone decreasing and convex (in a neighbourhood of $t = 0$).

(ii) $Q(0) = 0$.

(iii) $\frac{d}{dt}Q(t)|_{t=0} = -\lambda$

Thus we can estimate the Lyapunov exponent λ if we can estimate the derivative of (the logarithm of) the maximal eigenvalue $e^{P(t)}$ of the operator $\mathcal{L}_t$ (at $t = 0$).

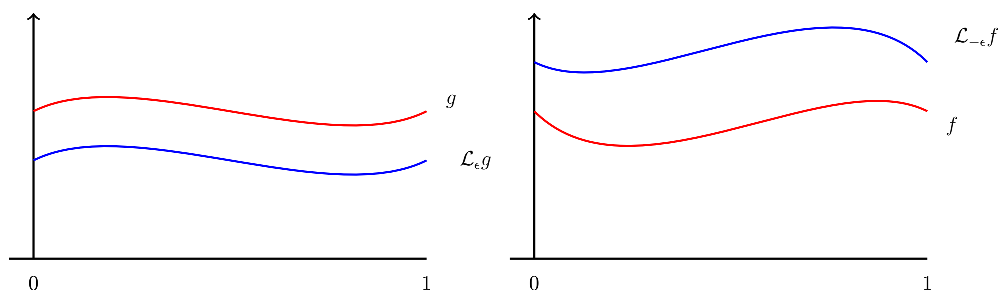
Step 4. Estimating λ using $Q(-\epsilon)$ and $Q(\epsilon)$. Let $\epsilon > 0$ then to estimate $\frac{d}{dt}Q(t)|_{\lambda=0} = -\lambda$ it is sufficient to estimate $Q(\epsilon)$ and $Q(-\epsilon)$ since by convexity we can (rigorously) bound

$$\frac{Q(-\epsilon)}{\epsilon} \geq \lambda \geq -\frac{|Q(\epsilon)|}{\epsilon}.$$

Thus it only remains to estimate $Q(-\epsilon)$ and $Q(\epsilon)$ using whatever is our favourite method, e.g. the following min-max type result.

Remark 3.9. *In practice we need to take care to choose ϵ small enough to be in the domain of $Q(t)$ in (i).*

Step 5. Finally, bounds on λ. We can use a sort of "min-max" estimate on operators:



Lemma 3.10 (Lower bound)**.** *If there exists (positive) function $g : \mathbb{S} \to \mathbb{R}^+$ and $\alpha > 0$ then*

$$\sup_x \frac{\mathcal{L}_\epsilon g(x)}{g(x)} < e^{-\alpha} \implies |P(\epsilon)| \geq \alpha.$$

Lemma 3.11 (Upper bound)**.** *If there exists (positive) function $f : \mathbb{S} \to \mathbb{R}^+$ and $\beta > 1$ then*

$$\sup_x \frac{\mathcal{L}_{-\epsilon} f(x)}{f(x)} < e^{\beta} \implies P(-\epsilon) \leq \beta$$

This brings us to the conclusion.

Corollary 3.12 (Bounds on the Lyapunov exponent λ)**.**

$$\frac{\beta}{\epsilon} \geq \lambda \geq \frac{\alpha}{\epsilon}.$$

Appendix II: Good's formula

in addition to his estimates on $\dim(E_2)$ we want to recall another mathematical contribution of Jack Good (published 49 years later in 1990) which applies more widely.

"A very rough guide to the maximum length that a paper should have is given by the formula $10^{9px/2}$ *words where*

- $0 \leq x \leq 1$ *is the importance of the topic,*
- *a partly-baked idea has a "bakedness" of* $0 \leq p \leq 1$*"*

For calibration: We call that the expression "half-baked idea" $(p = \frac{1}{2}))$ is defined to mean *poorly developed; foolish; unlikely to work.*

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