

Groups, drift and harmonic measures

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1 Introduction

An intriguing problem in modern geometric measure theory is the study of the harmonic measure on the unit circle which arises from a random walk on a Fuchsian group. The existing approach combines several areas of pure mathematics, such as Ergodic Theory, Probability Theory, Hyperbolic Geometry, and rigorous Numerical Analysis.

We will first recall some background. Afterwards we introduce one of the quantifiable characteristics of random walks called the *drift* and explain how it is related to properties of the harmonic measure, in particular, its Hausdorff dimension. Finally, we will draw a connection to a popular conjecture of Kaimanovich and Le Prince on the nature of the harmonic measure associated to a random walk on a Fuchsian group.

Although there is a number of partial results in special cases the general conjecture still remains open. We will offer a new perspective which covers both some known examples and some new cases. Finally, we will illustrate the question using the example of a $(4, 4, 4)$ -triangle group which can be traced back to the works of Gauss from 1805.

Wahrlich¹ es ist nicht das Wissen, sondern das Lernen, nicht das Besitzen sondern das Erwerben, nicht das Da-Seyn, sondern das Hinkommen, was den grössten Genuss gewährt.
Carl Friedrich Gauß to Wolfgang Bolyai [6].

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¹ “It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment.”

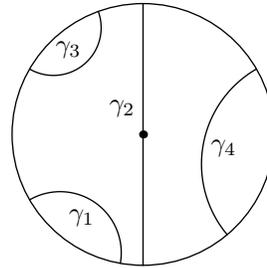
2 Preliminaries

In this section we collect together some background knowledge we need to properly formulate the problem.

2.1 Hyperbolic Geometry

We will treat Fuchsian groups as groups of isometries acting on the hyperbolic plane \mathbb{H} . For our considerations it will be convenient to consider the so-called Poincaré disk model of \mathbb{H} . This is a representation of the hyperbolic plane as an open

Fig. 1 The Poincaré disk where geodesics are either circular arcs which meet the boundary circle orthogonally or diameters.



unit disk $\mathbb{D} = \{z = x + iy : |z| < 1\}$ equipped with the Poincaré metric

$$ds^2 = 4 \frac{dx^2 + dy^2}{(1 - (x^2 + y^2))^2}$$

The geodesics are the extrema of the distance functional with respect to this metric. They are precisely the Euclidean diameters and circular arcs which are orthogonal to the boundary circle $\partial\mathbb{D}$, as shown in Figure 1.

The orientation preserving isometries of \mathbb{H} in this model are linear fractional transformations of the form

$$g(z) = \frac{az + b}{\bar{b}z + \bar{a}}, \quad \text{where } a, b \in \mathbb{C} \text{ and } |a|^2 - |b|^2 = 1.$$

As a group the orientation preserving isometries are isomorphic to the group consisting of 2×2 real matrices with determinant 1 up to multiplication by $\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Namely this is the group $\text{PSL}(2, \mathbb{R}) = \text{SL}(2, \mathbb{R}) / \{\pm I\}$.

2.2 Geometric Group Theory

In the present paper we want to consider finitely generated groups of isometries of \mathbb{H} . We call the group Γ *non-elementary* if it is not isomorphic to \mathbb{Z} . If Γ is finitely generated, then the orbit $\Gamma 0 = \{g0 : g \in \Gamma\}$ of $0 \in \mathbb{D}$ is a countable set of points in the unit disk.

Definition 1 We call a non-elementary group Γ a *Fuchsian group* if $\Gamma 0$ is a discrete set with respect to the Poincaré metric ds^2 introduced above.

In particular, if Γ is Fuchsian then all accumulation points with respect to the Poincaré metric must lie on the unit circle $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$.

Let us denote the set consisting of generators and their inverses by $\Gamma_\bullet = \{g_1^{\pm 1}, \dots, g_d^{\pm 1}\}$. We can associate to Γ_\bullet the Cayley graph of Γ . This is an infinite graph in which the vertices can be realised as the points of $\Gamma 0 = \{g0 : g \in \Gamma\}$ and two vertices $g0$ and $h0$ are connected by an edge if and only if $gh^{-1} \in \Gamma_\bullet$.

2.3 Random walks and the drift

We can now introduce our main tool. Given a set of generators and their inverses Γ_\bullet , we can consider a random walk on the Cayley graph where we allow a transition from a vertex $g0$ to a neighbouring vertex $h0$ with probability $\frac{1}{2d}$. (Here we assume that $\#\Gamma_\bullet = 2d$, as above.)

Definition 2 Given a specific set of generators Γ_\bullet , we can associate the drift (or the rate of escape) defined by

$$\ell = \ell(\Gamma_\bullet) = \lim_{n \rightarrow +\infty} \frac{1}{(2d)^n} \sum_{g_{j_1}, \dots, g_{j_n} \in \Gamma_\bullet} \frac{d(g_{j_1} \cdots g_{j_n} 0, 0)}{n}.$$

The limit always exists by a standard subadditivity argument and quantifies the rate at which typical points $g_{j_1} \cdots g_{j_n} 0$ escape towards the boundary circle $\partial\mathbb{D}$.

3 Some examples

Let us now turn to specific examples of Fuchsian groups. For basic results on hyperbolic polygons we refer the reader to an excellent book by Beardon [3].

3.1 Regular octagon tilings

Let $P \subset \mathbb{H}$ be a regular octagon with angles $\frac{\pi}{4}$ and sides of equal length which are geodesics as shown in Figure 2. We can consider groups of isometries generated by four transformations which identify the sides of the regular octagon. In this case the images of the octagon under the group action tile the hyperbolic plane, so that $\Gamma P = \mathbb{H}$. There are four different identifications which yield a surface of genus 2 as factor space \mathbb{H}/Γ [15]. This property implies, in particular, that the group Γ generated by these identifications is discrete. We will consider only two identifications which lead to well-known surfaces: the Bolza surface and the Gutzwiller surface.

The *Gutzwiller group* [11] $\Gamma_G = \langle g_1, g_2, g_3, g_4 \rangle$ is generated by four isometries which identify the opposite sides of the regular octagon P . They satisfy the identity $g_1 g_2^{-1} g_3 g_4^{-1} g_1 g_2^{-1} g_3 g_4^{-1} = I$ (see Figure 2, Left).

The *Bolza group* [5] $\Gamma_B = \langle g_1, g_2, g_3, g_4 \rangle$ is generated by four isometries which identify the opposite sides of the regular octagon P . They satisfy the identity $g_1 g_2^{-1} g_3 g_4^{-1} g_1^{-1} g_2 g_3^{-1} g_4 = I$ (see Figure 2, Right).

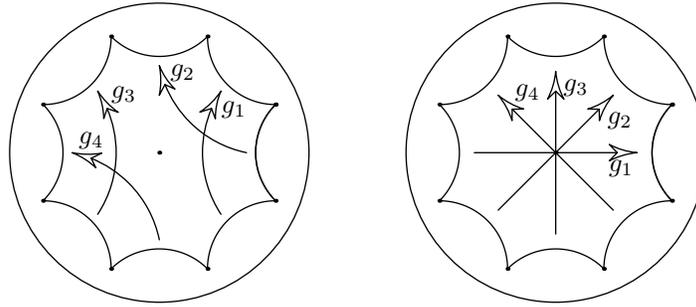


Fig. 2 Left: Generators of the Gutzwiller group identify alternating sides of the regular octagon; Right: Bolza group generated by isometries identifying opposite sides of the regular octagon.

It turns out the drift doesn't depend on the identification chosen, in particular, these two examples share the same value for the drift ℓ .

Theorem 1 *With the choice of generators specified above, the drift ℓ for the Bolza group Γ_B and for the Gutzwiller group Γ_G is the same and satisfies*

$$1.690771 < \ell < 1.691313.$$

The method we use for estimating ℓ involves looking at the action of Γ on $\partial\mathbb{D}$ and computing the maximal Lyapunov exponent. This is achieved by obtaining estimates on the spectral radius of transfer operators acting on the space of α -Hölder continuous functions $\mathcal{L}_t : C^\alpha(\partial\mathbb{D}) \rightarrow C^\alpha(\partial\mathbb{D})$ defined by $[\mathcal{L}_t f](z) = \frac{1}{2n} \sum_{g \in \Gamma} |g'(z)|^t f(gz)$ for t close to 1 and suitably small $\alpha > 0$. A more detailed exposition of the technical computer-assisted argument will appear elsewhere.

3.2 Hyperbolic triangle groups

Another class of interesting examples is perhaps the class of Coxeter groups generated by reflections in the sides of a hyperbolic triangle, the so-called triangle groups. It is easy to see that the group is discrete if and only if all angles of the triangle are rational multipliers of π . We will restrict our considerations to the case when triangle has angles $\frac{\pi}{k}$, $\frac{\pi}{l}$ and $\frac{\pi}{m}$ where k, l and m are integers. Recall that sum of the angles of the hyperbolic triangle is strictly less than π , and therefore k, l, m should satisfy the inequality $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} < 1$.

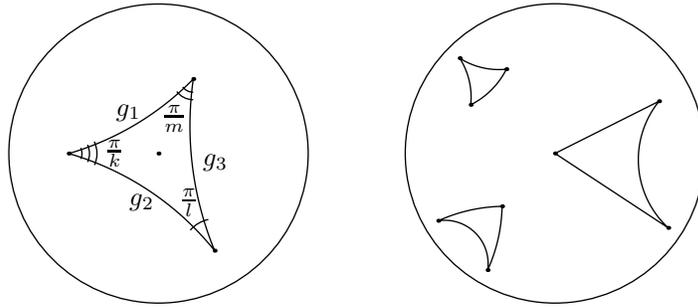


Fig. 3 Left: a triangle in \mathbb{D} with boundaries which are geodesics with respect to the Poincaré metric and internal angles $\frac{\pi}{k}$, $\frac{\pi}{l}$ and $\frac{\pi}{m}$, containing the centre of the disk in its interior. Right: hyperbolic triangles with different shape.

Definition 3 The (k, l, m) -triangle group is a group generated by reflections in the three sides of a hyperbolic triangle with angles $\frac{\pi}{k}$, $\frac{\pi}{l}$ and $\frac{\pi}{m}$.

It follows from the properties of reflections, that the generators g_1, g_2, g_3 of the (k, l, m) -triangle group, shown in Figure 3 satisfy the following relations

$$g_1^2 = g_2^2 = g_3^2 = I \quad \text{and} \quad (g_1 g_2)^k = (g_2 g_3)^l = (g_3 g_1)^m = I.$$

These are the defining relations in the sense that any group with three generations which satisfy these condition is the (k, l, m) -triangle group. Evidently, this group is also cocompact. Furthermore, similarly to the case of the regular octagon, the images of the original triangle with respect to the group form a tessellation of the hyperbolic plane.

The study of the groups generated by reflections with respect to the sides of curvilinear triangles can be traced back to the works of Gauss. Bolyai, commenting on the Gauss' work, suggests that in a drawing from "Cereri Palladi Junoni Sacrum" dated February 1805 Gauss introduced the idea of reflection with respect to the circle. A copy of the drawing, taken from [9, p. 104] is shown in Figure 4 on the left. On the right we see a tessellation of the hyperbolic plane generated by $(4, 4, 4)$ -triangle group. The difference between the two drawings is due to the choice of the

location of the original triangle. In the Gauss' drawing the centre of the disk is one of the vertices. In our drawing, the centre of the disk is the barycentre of the triangle. Despite the appearance of a tessellation of the Poincaré disk in the Gauss' drawing it was written 49 years before the birth of Poincaré! Lobachevsky laid the foundations of the hyperbolic geometry in 1823.

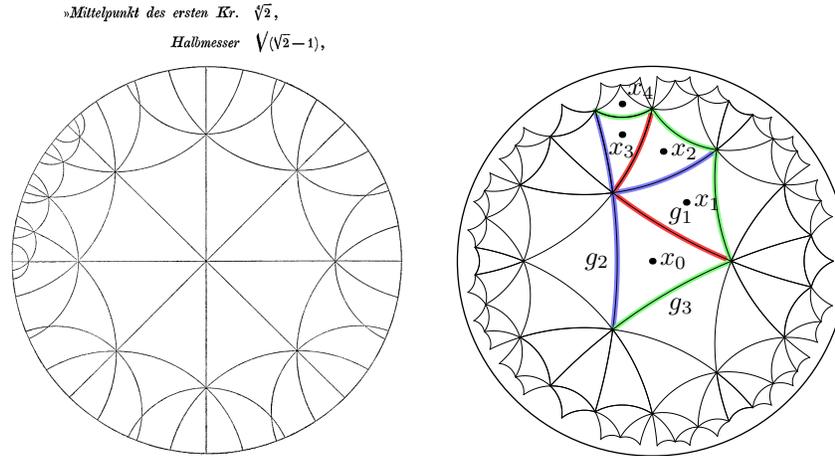


Fig. 4 Left: An original drawing by Gauss of a part of the $(4, 4, 4)$ -triangle group tessellation; Right: An orbit of a random walk corresponding to the $(4, 4, 4)$ -triangle group: $x_1 = g_1 x_0$, $x_2 = g_1 g_2 x_0$, $x_3 = g_1 g_2 g_1 x_0$, and finally $x_4 = g_1 g_2 g_1 g_3 x_0$.

Using the same machinery as in the case of the regular octagon we can estimate the drift. In Table 1 we list different examples of triangle groups and give upper and lower bounds on the associated drift. In a special case of the $(4, 4, 4)$ -triangle group, we have the following result.

Theorem 2 *The drift of the random walk on the $(4, 4, 4)$ -triangle group with the choice of generators specified above satisfies*

$$0.1282273 < \ell < 0.1282264.$$

4 Two problems

After a brief discussion of the groups we are concerned with we continue by introducing one of the central objects of the theory — *the harmonic measure*.

Table 1 Upper and lower bounds on the drift for some (k, l, m) -hyperbolic triangle groups.

k	l	m	lower bound on ℓ	upper bound on ℓ
3	7	2	0.009936413804542	0.009974294432083
3	8	2	0.016242376981342	0.016295700460901
3	9	2	0.020422904820936	0.020508218335138
4	5	2	0.024263195172778	0.024341830945392
4	6	2	0.037765501277040	0.037870175386186
4	8	2	0.050724918174930	0.050934249274956
5	5	2	0.046019792084900	0.046155635941842
5	6	2	0.058159239428682	0.058334985605960
5	7	2	0.065329026703739	0.065563197936118
6	6	2	0.069559814745121	0.069846131636394
4	3	3	0.046694831446660	0.046816105401585
5	3	3	0.069435926662536	0.069689191304812
6	3	3	0.081515978567027	0.081925767935374
7	3	3	0.088431558608918	0.089059709051931
3	4	4	0.088752444507380	0.088919437571219
3	6	6	0.148515148139248	0.149179933451390
4	4	4	0.128086862380309	0.128344145942091
5	5	5	0.182618423778876	0.183286144055414
6	6	6	0.209779208475952	0.2111031605163552
7	7	7	0.224864828238411	0.228908301867331
8	8	8	0.232248419011566	0.238574707256068
9	9	9	0.236782098913020	0.247054233672500
10	10	10	0.240409132283172	0.252180931190328

4.1 The harmonic measure on the unit circle

As we have seen already, a typical orbit of the random walk associated to a Fuchsian group converges to a point on the boundary circle with respect to the Euclidean and the hyperbolic metric. The distribution ν of the limit points on the boundary $\partial\mathbb{D}$ defines a probability measure. More precisely, given a generating set Γ_\bullet of a Fuchsian group we can define a family of probability measures on \mathbb{D} by

$$\nu_n = \left(\frac{1}{2d}\right)^n \sum_{g_{j_1}, \dots, g_{j_n} \in \Gamma_\bullet} \delta_{g_{j_1} \cdots g_{j_n} 0}$$

where $\delta_{g_{j_1} \cdots g_{j_n} 0}$ is the Dirac measure supported at $g_{j_1} \cdots g_{j_n} 0$. The measures ν_n converge in the weak star topology (on the closed unit disk) to a probability measure ν on $\partial\mathbb{D}$.

Definition 4 The measure ν is called the *harmonic measure* or the *hitting measure*.

We can denote by $\Lambda \subset \partial\mathbb{D}$ the support of this measure (i.e., the smallest closed set of the full measure). It is known that either $\Lambda = \partial\mathbb{D}$ or $\Lambda \subsetneq \partial\mathbb{D}$ is a Cantor set.

4.2 Singularity of the harmonic measure

The following natural question was posed by Kaimanovich and Le Prince [12]:

Question 1

Can we characterise Fuchsian groups for which the associate harmonic measure is absolutely continuous with respect to the Lebesgue measure?

Of course, if the support of the harmonic measure is a Cantor set then the measure is singular with respect to Lebesgue measure. Therefore, we will only consider the case that $\Lambda = \partial\mathbb{D}$. In the special case when one of the generators in Γ_\bullet is parabolic it was shown by Gadre, Maher, and Tiozzo that the harmonic measure is always singular [8].

Furthermore, there are examples of non-discrete groups due to Bourgain for which ν is absolutely continuous [7] (see also [2]). In the more general setting when the weights in the random walk differ the measure ν may be singular [12].

In the setting of the surface groups, this question has been intensively studied [13], [14]. One set of examples is Fuchsian groups generated by isometries identifying the sides of hyperbolic polygons. However, many of the examples with more than four sides have harmonic measures that are singular (see [13, Theorem 1]).

4.3 Dimension of the harmonic measure

An important quantitative characteristic of the measure is its Hausdorff dimension.

Definition 5 The Hausdorff dimension of the measure is the infimum of Hausdorff dimensions of sets of the full measure:

$$\dim_H(\nu) = \inf\{\dim_H(X) \mid X \subset \partial\mathbb{D} \text{ Borel and } \nu(X) = 1\}.$$

There is a useful result due to Tanaka which relates the question of absolute continuity of the harmonic measure to the numerical value of its Hausdorff dimension [17].

Proposition 1 (Tanaka) *The harmonic measure ν is absolutely continuous if and only if $\dim_H(\nu) = 1$.*

This leads to the following stronger version of Question 1.

Question 2

Assuming the harmonic measure is not absolutely continuous with respect to Lebesgue measure, can we estimate its Hausdorff dimension?

We now return to our examples.

It follows from the result of Kosenko [13, Theorem 1.2] (see also [14]) that the harmonic measure ν_B associated to the Bolza group Γ_B is singular. We can improve on this result.

Theorem 3 *The dimension of the harmonic measure ν_B for the Bolza group satisfies*

$$\dim_H(\nu_B) \leq 0.86116.$$

The dimension of the harmonic measure ν_G for the Gutzwiller group Γ_G satisfies

$$\dim_H(\nu_G) \leq 0.86317.$$

In particular, the harmonic measure ν_G is also singular.

In order to explain the proof, we need one extra ingredient.

4.4 Relation to the Avez entropy

We introduce another numerical characteristic of the random walk which is commonly used to estimate the dimension of a measure.

Definition 6 We can associate to a harmonic measure ν the Avez random walk entropy defined by

$$h_A(\nu) = \lim_{n \rightarrow +\infty} \frac{1}{n} H(\nu^{*n})$$

where $H(\cdot)$ is the usual Shannon entropy function and ν^{*n} denotes the n -fold convolution [1].

The limit always exists by subadditivity. The dimension, entropy and drift are related using the following identity [4], [10], [17].

Proposition 2 *For the harmonic measure ν we have that $\dim_H(\nu) = h_A(\nu)/\ell(\nu)$.*

Now we are ready to prove Theorem 3.

Proof of Theorem 3. Combining Proposition 1 and Proposition 2 we see that in order to establish that the harmonic measure is singular it is sufficient to show that $h(\nu) \geq \ell(\nu)$. In particular, we need to establish an upper bound on the entropy and a lower bound on the drift. For the Gutzwiller group an estimate on the entropy is given in a beautiful paper [10]. Even the most basic bound they give in Example 2.3 $h(\nu_B) \leq 1.46$ in combination with the estimate on the drift $\ell(\nu_B)$ from Theorem 1 allows us to deduce that the measure is singular. In the case of the Bolza group, we can use the estimate on the drift $\ell(\nu_G)$ from Theorem 1 and the upper bound on the entropy of $h(\nu_G) \leq \frac{3}{4} \log 7$ coming from the free group on four generators.

4.5 Final remarks

Should one wish to apply the same approach to show that the harmonic measure for the triangle groups are singular it will be necessary to obtain an effective upper bound on the Avez entropy. Unfortunately, the naive bound of $\frac{1}{3}\log 2 \approx 0.231049\dots$ corresponding to the Avez entropy of the random walk on the free product $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ isn't quite low enough to show that the measure is singular for most of triangle groups listed in Table 1. Nevertheless in the case of $(8, 8, 8)$ -, $(9, 9, 9)$ -, and $(10, 10, 10)$ -triangle groups we may conclude that the measure is singular. It is reasonable to suggest that the drift for (k, k, k) -triangle group is monotone increasing as $k \rightarrow \infty$. This would imply that the harmonic measure is singular for $k \geq 8$.

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