Statistics of Pythagorean triples via the Romik map

July 13, 2026

1 Introduction

Pythagorian triples have been studied back to the time of antiquity, The oldest known examples come from a Babylonian clay tablet in cuneiform from about 1800 BC (discovered circa 1900 AD) and predating Pythagorus by more than a millenium. Let us consider primitive Pythagorean triples $(a, b, c) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ of natural numbers which are assumed to have the following properties:

1. $a^2 + b^2 = c^2$; and

2. $0 < a, b < c$;and a and b (and c) are coprime.

The conditions in 2. remove permutations and avoid having integer multiples of smaller triples. Let us denote the set of such triples by $\mathcal{P}$.

Example 1.1 *The choice* $(a, b, c) = (3, 4, 5)$ *is probably the best known example. The second best known example is probably* $(a, b, c) = (5, 12, 13)$.

1.1 Counting triples

Using number theoretic techniques it was shown by D. H. Lehner in 1900 that the number of primitive triples (a, b, c) with $c \leqslant N$ is asymptotic to

$$\#\{(a, b, c) \in \mathcal{P} : c \leqslant N)\} \sim \frac{1}{2\pi} N \text{ as } N \to +\infty.$$

cf. [7]. This is also discussed in the book [11]. There are also asymptotic formulae for triples with bounds on the perimeter $a + b + c \leqslant N$ (due to D. N. Lehmer [8], the son of D. H. Lehner), the area $ab/2 \leqslant N$ [6], and legs $\max\{b, c\} \leqslant N$ [2]

We want to consider the asymptotics for a slightly different ordering which arises naturally from a dynamical viewpoint.

Definition 1.2 *We can associate to each* $\underline{t} = (a, b, c) \in \mathcal{P}$ *a weight* $a + c$*.*

We would like to compare estimate the growth of the number of triples (a, b, c) increases when ordered by the weight $a + c$.

Definition 1.3 *For each* $T > 0$ *we can now consider the counting function*

$$\Pi(T) = \#\{(a, b, c) \in \mathcal{P} : a + c \leqslant T\}.$$

There is the following basic asymptotic result.

Theorem 1.4 *There exists* $C > 0$ *such that*

$$\Pi(T) \sim CT \text{ as } T \to +\infty,$$

i.e., $\lim_{T\to+\infty} \Pi(T)/T = C > 0$.

1.2 A Euclidean type algorithm

Following Romik, we can consider an associated subtractive algorithm on pairs of natural numbers $(x, y) \in \mathbb{N} \times \mathbb{N}$ for $x > y$ of the form

$$(x, y) \mapsto \begin{cases} A_1 : (x - 2y, y) & \text{if } x - 2y > y \\ A_2 : (y, x - 2y) & \text{if } y \geqslant x - 2y > 0 \\ A_3 : (y, 2y - x) & \text{if } x - 2y \leqslant 0 \end{cases}$$

This is analogous to the subtractive Euclidean algorithm in the sense that starting from (x_0, y_0) it will end in finitely many steps (in this case at $(a, 0)$ or (a, a), where a is the greatest common divisor of x_0 and y_0).

Example 1.5 *We can write* $(14, 5) \to (5, 4) \to (4, 3) \to (3, 2) \to (2, 1) \to (1, 0)$

We can "accelerate" the algorithm by replacing A_1, A_2 and A_3 by certain words formed by concatenating strings

$$w = A_{i_1}^{m_1} A_{i_2}^{m_2} \cdots A_{i_n}^{m_n} \text{ with } i_j \neq i_{j+1} \quad (1 \leqslant j \leqslant n - 1)$$

subject to some appropriate restrictions.

Example 1.6 *In §3.3 of [10] he considers strings which end with A_2 or A_iA_1 $(i \neq 1)$ or A_jA_3 $(j \neq 3)$ and these do not previously occur in the string.*

Definition 1.7 *Given longer strings formed by further concaternation of words $\underline{w} = w_1 \cdots w_n$ we can denote $n(\underline{w}) = n$. If $\underline{w}(3,4,5) = (a,b,c)$, say, then we can associate the weight $\ell(\underline{w}) = 2\log(a+c)$*

Parallel to this, we can follow Romik and write $t = y/x$ and associate a map $T : [0,1] \to [0,1]$ on the unit interval by

$$T(t) = \begin{cases} \frac{t}{1-2t} & \text{if } 0 < t \leqslant \frac{1}{3} \\ \frac{1}{t} - 2 & \text{if } \frac{1}{3} < t < \frac{1}{2} \\ 2 - \frac{1}{t} & \text{if } \frac{1}{2} \leqslant t < 1. \end{cases}$$

An acceleration of the algorithm corresponds to choosing an interval $J \subset [0,1]$ and consider the associated jump transformation $\widehat{T} : [0,1] \to [0,1]$ defined by $\widehat{T}(x) = T^{n(x)+1}(x)$ where $n(x) = \inf\{m \geqslant 1 : T^m(x) \in J\}$. There are different choices of J possible.

Example 1.8 *Example 1.6 of Romik corresponds to a jump transformation on the interval $J = [1/5, 2/3]$.*

Associated statistical results relating $n(\underline{w})$ and $\ell(\underline{w})$ are the following.

Theorem 1.9 (Large Deviations) *There exists $\Lambda > 0$ such that for any $\epsilon > 0$:*

$$\limsup_{Y \to +\infty} \frac{1}{Y} \log \left(\frac{\#\{w_1 \cdots w_n : \ell(\underline{w}) \leqslant Y, |n/\ell(\underline{w}) - \Lambda| > \epsilon\}}{\#\{w_1 \cdots w_n : \ell(\underline{w}) < Y\}} \right) < 0.$$

Theorem 1.10 (Central Limit Theorem) *For each j there exists $\sigma_j^2 > 0$ such that for any $a \in \mathbb{R}$*

$$\frac{\#\left\{w_1 \cdots w_n : \ell(\underline{w}) < Y, n/\sqrt{Y} \leqslant a\right\}}{\#\{w_1 \cdots w_n : \ell(\underline{w}) < Y\}} \to \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{a} e^{-u^2/2\sigma_j^2} du$$

as $Y \to +\infty$.

One can induce on other intervals, but care needs to be taken that the resulting jump transformation is hyperbolic. The following standard expansions fail this test (and further inducing would be necessary).

Example 1.11 (Even Integer continued fraction) *Kim at al [5] showed that the jump transformation on $J = \{0\} \cup [1/3, 1]$ is precisely the even integer continued fraction map of Schweiger $T_{EI} : [0,1] \to [0,1]$ given by*

$$T_{EI}(x) = \begin{cases} \frac{1}{x} - 2k & \textit{if } \frac{1}{2k+1} \leqslant x \leqslant \frac{1}{2k} \\ 2k - \frac{1}{x} & \textit{if } \frac{1}{2k} \leqslant x \leqslant \frac{1}{2k-1} \\ 0 & \textit{if } x = 0 \end{cases}$$

This leads to expansions of irrationals $x \in [0,1] \setminus \mathbb{Q}$ of the form

$$x = b_0 + \cfrac{\eta_0}{b_1 + \cfrac{\eta_1}{b_2 \cfrac{\eta_2}{b_3 + \cfrac{\eta_3}{b_4 + \cdots}}}} \quad \textit{where } b_0 \in 2\mathbb{Z},\ b_n \in 2\mathbb{N}\ (n \geqslant 1) \textit{ and } \eta_n = \pm 1$$

Example 1.12 (Odd-Odd continued fraction) *Kim at al [5] show that the jump transformation on $J = \{1\} \cup [0, 1/2]$ is precisely the odd-odd continued fraction map $T_{OO} : [0,1] \to [0,1]$ given by*

$$T_{OO}(x) = \begin{cases} \frac{kx-(k-1)}{k-(k+1)x} & \textit{if } \frac{k-1}{k} \leqslant x \leqslant \frac{2k-1}{2k+1} \\ \frac{kx-(k+1)}{k-(k-1)x} & \textit{if } \frac{2k-1}{2k+1} \leqslant x \leqslant \frac{k}{k+1} \\ 1 & \textit{if } x = 1 \end{cases}$$

This leads to expansions of irrationals $x \in [0,1] \setminus \mathbb{Q}$ of the form

$$x = 1 - \cfrac{1}{a_1 + \cfrac{\epsilon_1}{2 - \cfrac{1}{a_2 + \cfrac{\epsilon_2}{2 - \cfrac{1}{\ddots}}}}} \quad \textit{where } a_n \in \mathbb{N} \textit{ and } \epsilon_n = \pm 1 \textit{ when } a_n \geqslant 2, \textit{ and } = 1 \textit{ when } a_n = 1$$

2 Generating Pythagorean triples

During the nineteenth and twentieth centuries there has been interest in different methods for generating primitive Pythagorean triples. We begin with the Berggren and Barning construction of primitive Pythagorean triples.

2.1 A Pythagorean tree

We begin by distinguishing two cases.

Definition 2.1 *Let $\mathcal{P}_1 = \{(a,b,c) \in \mathcal{P} : a \text{ is even and } b \text{ is odd}\}$ and $\mathcal{P}_2 = \{(a,b,c) \in \mathcal{P} : a \text{ is odd and } b \text{ is even}\}$*

From the classic 1934 paper of Berggren and the 1963 paper of Barning we have the following result.

Theorem 2.2 (Berggen, Barning) *Starting from the initial triple* $(3,4,5)$ *there is a unique presentation of all other primitive pythagorean triples* $(a,b,c) \in P$ *by either:*

$$(I) \qquad \begin{pmatrix} a \\ b \\ c \end{pmatrix} = M_{i_1}^{k_1} \cdots M_{i_n}^{k_n} \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \text{ where } (a,b,c) \in \mathcal{P}_1; \text{ or}$$

$$(II) \qquad \begin{pmatrix} a \\ b \\ c \end{pmatrix} = M_{i_1}^{k_1} \cdots M_{i_n}^{k_n} \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} \text{ where } (a,b,c) \in \mathcal{P}_2$$

where

$$M_1 = \begin{pmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{pmatrix}, M_2 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}, M_3 = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{pmatrix},$$

with $n = n(a,b,c) \in \mathbb{N}$ *and* $k_1, \cdots, k_n \in \{1,2,3\}$ *for* $n \geqslant 1$.

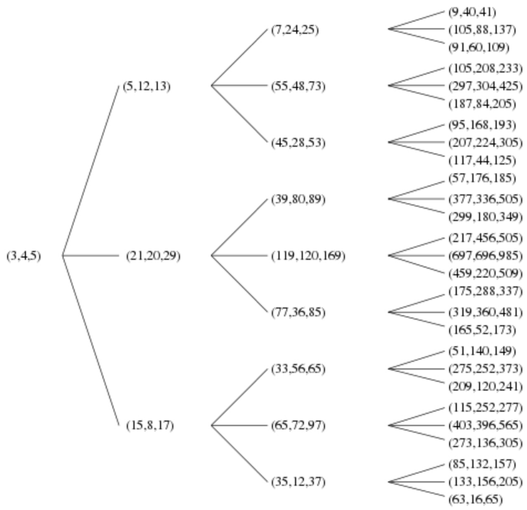

Figure 1: The first three levels in the tree in case (I)

We want to view the above method of constructing triples from a more dynamical viewpoint. We first introduce a map $S : \mathcal{P} - \{(3,4,5)\} \to \mathcal{P}$ defined by

$$S(a,b,c) = (|2c - a - 2b|, |2c - 2a - b|, 3c - 2a - 2b)$$

which moves triples down the tree, i.e., for primitive pythagorean triples $\underline{t} = (a, b, c) \in \mathcal{P} - \{(3, 4, 5)\}$ we have

$$\underline{t}^T = M_1(S(\underline{t}))^T = M_2(S(\underline{t}))^T = M_3(S(\underline{t}))^T.$$

Example 2.3 *Let* $\underline{t} = (65, 72, 97)$ *then*

$$\ldots \xrightarrow{S} (65, 72, 97) \xrightarrow{S} (15, 8, 17) \xrightarrow{S} (3, 4, 5).$$

2.2 A dynamical viewpoint

We follow Romik in first associating a map on an arc of the unit circle and then associating a map of the unit interval. It is the latter that will serve us well in the sequel.

2.2.1 A map on the positive arc of a circle

Since the work of Diophantus, it has been known that the set of pythgorean triples is related to the standard rational parameterization of the unit circle

$$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

More precisely, the pythagorean triples $\underline{t} = (a, b, c) \in \mathcal{P}$ correspond to (rational) points in the unit circle S^1 intersected with the positive quadrant, which we denote by

$$\mathcal{Q} = \{(x, y) \in S^1 : x \geqslant 0, y \geqslant 0\},$$

by a map $\Phi : \mathcal{P} \to S^1 \cap \mathcal{Q}$ given by

$$\Phi : \underline{t} = (a, b, c) \mapsto \left(\frac{a}{c}, \frac{b}{c}\right) \in S^1 \cap \mathcal{Q}.$$

Example 2.4 *In particular, the triple* $(3, 4, 5)$ *gives rise to the point* $\Phi(3, 4, 5) = (\frac{3}{5}, \frac{4}{5})$ *on the unit circle and the triple* $(4, 3, 5)$ *gives the point* $(\frac{4}{5}, \frac{3}{5})$ *on the unit circle.*

Following Romik [10] we can define a map R on the arc $\mathcal{Q}$ using the projectivized action by the matrices on the positive quadrant of the unit circle as follows.

Definition 2.5 *We define $R : \mathcal{Q} \to \mathcal{Q}$ by*

$$R(x,y) = \begin{cases} \left(\frac{2-x-2y}{3-2x-2y}, -\frac{2-2x-y}{3-2x-2y}\right) & \textit{if } \frac{4}{3} < \frac{x}{y} \\ \left(-\frac{2-x-2y}{3-2x-2y}, -\frac{2-2x-y}{3-2x-2y}\right) & \textit{if } \frac{3}{4} < \frac{x}{y} < \frac{4}{3} \\ \left(-\frac{2-x-2y}{3-2x-2y}, \frac{2-2x-y}{3-2x-2y}\right) & \textit{if } \frac{4}{3} < \frac{x}{y} \end{cases}$$

In particular, R preserves rational points on the unit circle, cf. ([10], Theorem 2) and ([1], Lemma 16). Moreover, by construction we have that $\Phi \circ S = R \circ \Phi$.

2.3 A map on the unit interval

Finally, we want to replace the map $R : \mathcal{Q} \to \mathcal{Q}$ by a map $T : [0,1] \to [0,1]$ on the unit interval. Following Romik [10] we achieve this by conjugating R by the homeomorphism $D : [0,1] \to \mathcal{Q}$ defined by

$$D :\mapsto \left(\frac{t^2-1}{t^2+1}, \frac{2t}{t^2+1}\right)$$

cf. ([1], §3.6), i.e., $T = D^{-1} \circ R \circ D$. We can then write the inverse $D^{-1} : \mathcal{Q} \to [0,1]$ as

$$D^{-1}(x,y) = \begin{cases} \frac{y}{1+x} = \frac{1-x}{y} & \text{if } y > 0 \\ 0 & \text{if } (x,y) = (1,0). \end{cases}$$

Remark 2.6 *Observe $\frac{y}{1+x} = \frac{1-x}{y}$ is equivalent to $y^2 = 1 - x^2$ which holds for $(x,y) \in \mathcal{Q}$. A triple (a,b,c) gives rise to $x = \frac{a}{c}$ and $y = \frac{b}{c}$ in which case*

$$D^{-1}\left(\frac{a}{c}, \frac{b}{c}\right) = \frac{b/c}{1 + a/c} = \frac{b}{a+c} \in \mathbb{Q}.$$

We recall a basic property of the map D (cf. [1], Lemma 17).

Lemma 2.7 *The map D gives a bijection between the rational points $[0,1] \cap \mathbb{Q}$ and the rational points $\mathcal{Q} \cap \mathbb{Q}^2$.*

The two special points $(\frac{3}{5}, \frac{4}{5}), (\frac{4}{5}, \frac{3}{5}) \in \mathcal{Q}$ can be identified with particular rationals in the interval.

Example 2.8 *In particular,*

$$D^{-1}\left(\frac{3}{5}, \frac{4}{5}\right) = \frac{1-\frac{3}{5}}{\frac{4}{5}} = \frac{1}{2} \textit{ and } D^{-1}\left(\frac{4}{5}, \frac{3}{5}\right) = \frac{1-\frac{4}{5}}{\frac{3}{5}} = \frac{1}{3}$$

The piecewise smooth interval map $T = D^{-1} \circ R \circ D$ on the unit interval takes the following explicit form.

Definition 2.9 *We define a map $T : [0,1] \to [0,1]$ on the unit interval by*

$$T(t) = \begin{cases} \frac{t}{1-2t} & \text{if } 0 < t < \frac{1}{3} \\ 1 & \text{if } t = \frac{1}{3} \\ \frac{1}{t} - 2 & \text{if } \frac{1}{3} < t < \frac{1}{2} \\ 0 & \text{if } t = \frac{1}{2} \\ 2 - \frac{1}{t} & \text{if } \frac{1}{2} < t < 1 \end{cases}$$

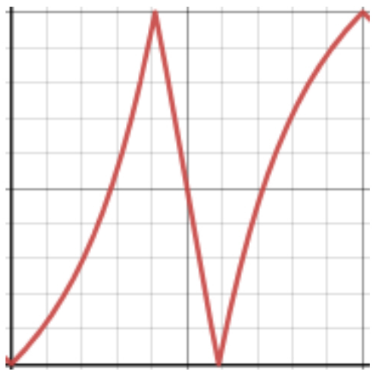

Figure 2: The graph of T

The properties of the map were studied in [10] and [1]. We are particularly interested in the following.

Lemma 2.10 *The map $T : [0,1] \to [0,1]$ has the following properties.*

1. *The images of the three branches are full, i.e.,*

$$T\left[0, \frac{1}{3}\right] = T\left[\frac{1}{3}, \frac{1}{2}\right] = T\left[\frac{1}{3}, 1\right] = [0,1].$$

2. *The map is (non-uniformly) expanding, i.e., $|T'(t)| > 1$ for $0 < t < 1$ and $|T'(0)| = |T'(1)| = 1$.*

3. *$T(0) = 0$ and $T(1) = 1$ with asymptotic expansions*

$$T(t) = t + 2t^2 + o(t^2) \text{ and } T(t) = (1-t) - 2(1-t)^2 + o\left((1-t)^2\right)$$

A consequence of Lemma 2.10 is that there exists a sigma finite (but not finite) T-invariant probability measure.

In order to apply a dynamical approach to the statistics we need to introduce a dynamical weighting to orbits which is a multiplicative cocycle along the orbit. The following is natural.

Definition 2.11 *Let us assume that $x \in T^{-n}(x_0)$ (for some $n \geqslant 0$) then we can associate the dynamical weight $|(T^n)'(x)|$.*

We have the following asymptotic result (by applying a result from [9]).

Proposition 2.12 *For any $x \in [0,1]$ we have*

$$\#\{y : T^n y = x \text{ and } |(T^n)'(y)| \leqslant Y\} \sim Y \text{ as } Y \to +\infty$$

Remark 2.13 *Unfortunately, the indifferent fixed points make a central limit theorem in the present context impossible (without inducing). In this context other (stable) limit laws would be appropriate.*

The connection back to triples in $\mathcal{P}$ is the following. Each triple $\underline{t} = (a,b,c)$ can be written as a product of matrices $A_{i_1}^{k_1} A_{i_2}^{k_2} \cdots A_{i_n}^{k_n}$ as in (I) or (II). The corresponding point on $[0,1]$ is the image under a sequence of contractions $T_{i_n}^{k_n} \circ \cdots \circ T_{i_2}^{k_2} \circ T_{i_1}^{k_1}$ applied to $\frac{1}{2}$ or $\frac{1}{3}$, respectively. However, we want to interpret Proposition 2.12 in terms of triples $\underline{t} = (a,b,c)$ and the weight $w(\cdot)$ which we will address in the next subsection.

2.4 The derivative weighting of preimages

In particular, if $p/q \in \mathbb{Q} \cap [0,1]$ satisfies $T^n(p/q) \in \{1/3, 1/2\}$ (with n chosen least with this property) then we can write this in terms of inverse branches by choosing $i_1, \cdots, i_n \in \{1,2,3\}$ and $k_1, \cdots, k_n \in \mathbb{N}$ with $T_{i_n}^{k_n} \circ \cdots \circ T_{i_2}^{k_2} \circ T_{i_1}^{k_1}(x) = p/q$.

We can consider the inverse branches $T_1, T_2, T_3 : [0,1] \to [0,1]$ of $T : [0,1] \to [0,1]$ which are explicitly given by the maps

$$\begin{aligned} T_1(x) &= \frac{t}{1+2t} \in [0,1/3] \\ T_2(x) &= \frac{1}{2+t} \in [1/3,1/2] \\ T_3(x) &= \frac{1}{2-t} \in [1/2,1] \end{aligned}$$

(see [10], equation (1)). We can then write the (absolute value of the) derivatives:

$$|T_1'(x)| = \frac{1}{(1+2t)^2}$$
$$|T_2'(x)| = \frac{1}{(2+t)^2}$$
$$|T_3'(x)| = \frac{1}{(2-t)^2}$$

We will want to consider the specific case of a point $x \in \{1/2, 1/3\}$.

Lemma 2.14 *Let $i_1, \ldots, i_n \in \{1,2,3\}$ and $k_1, \cdots, k_n \in \mathbb{N}$ (with $n \geqslant 1$) such that $S_{i_n}^{k_n} \circ \cdots \circ S_{i_1}^{k_1}(x) = p/q$.* [1]

1. *When $x = x_1 := \frac{1}{3}$. then $|(T_{i_1}^{k_1} \circ \cdots \circ T_{i_k}^{k_n})'(x_0)| = \frac{3^2}{q^2}$ and*
2. *When $x = x_2 := \frac{1}{2}$ then $|(T_{i_1}^{k_1} \circ \cdots \circ T_{i_n}^{k_n})'(x_1)| = \frac{2^2}{q^2}$*

This is a proof by induction.

Initial case: ($n = 1$ and $k_1 = 1$). Starting from $x_1 = 1/3$ we can observe that

$$T_1(x_1) = \frac{1/3}{1+2/3} = \frac{1}{5} \text{ and } |T_1'(x_1)| = \frac{1}{(1+2/3)^2} = \frac{3^2}{5^2}$$
$$T_2(x_1) = \frac{1}{2+1/3} = \frac{3}{7} \text{ and } |T_2'(x_1)| = \frac{1}{(2+1/3)^2} = \frac{3^2}{7^2}$$
$$T_3(x_1) = \frac{1}{2-1/3} = \frac{3}{5} \text{ and } |T_3'(x_1)| = \frac{1}{(2-1/3)^2} = \frac{3^2}{5^2}$$

and starting from $x_2 = 1/2$ we can consider

$$T_1(x_2) = \frac{1/2}{1+2/2} = \frac{1}{2^2} \text{ and } |T_1'(x_2)| = \frac{1}{(1+2/2)^2} = \frac{2^2}{4^2}$$
$$T_2(x_2) = \frac{1}{2+1/2} = \frac{2}{5} \text{ and } |T_2'(x_2)| = \frac{1}{(2+1/2)^2} = \frac{2^2}{5^2}$$
$$T_3(x_2) = \frac{1}{2-1/2} = \frac{2}{3} \text{ and } |T_3'(x_2)| = \frac{1}{(2-1/3)^2} = \frac{2^2}{3^2}$$

Inductive step. The two cases are similar. We will assume that

$$(T_{i_1}^{k_1} \circ \cdots \circ T_{i_n}^{k_n})(x_i) = \frac{p}{q} \text{ and } |(T_{i_1}^{k_1} \circ \cdots \circ T_{i_k}^{k_n})'(x_i)| = \frac{\ell_i^2}{q^2},$$

[1] Some care is needed that these fractions are in reduced form

for $i = 1, 2$, where

$$\ell_j = \begin{cases} 3^2 & \text{if } j = 0 \\ 2^2 & \text{if } j = 1. \end{cases}$$

Case I (Compose with T_1). When we apply T_1 to $\frac{p}{q}$ then

$$S_1(p/q) = \frac{p/q}{1 + 2p/q} = \frac{p}{q + 2p} =: \frac{P}{Q}.$$

If p, q are coprime then P, Q are also coprime (since if $Q = kP$ with $k \geqslant 2$ then $q + 2p = kp$ and so $q = (k-2)p$ implying that either $q = 0$ (which is impossible), or q divides p, which is a contradiction).

By the chain rule.

$$\begin{aligned} |(T_1 \circ T_{i_n}^{k_n} \circ \cdots \circ T_{i_{k_1}}^{k_1})'(x_j)| &= |T_1'(p/q)|.|(T_{i_n}^{k_n} \circ \cdots \circ T_{i_1}^{k_1})'(x_j)| \\ &= \frac{1}{(1 + 2p/q)^2} \frac{\ell_j^2}{q^2} \\ &= \frac{1}{(q + 2p)^2} = \frac{\ell_j^2}{Q^2}. \end{aligned}$$

Case II (Compose with T_2). If we apply T_2 then

$$T_2(p/q) = \frac{p/q}{2 + p/q} = \frac{p}{2q + p} =: \frac{P}{Q}$$

If p, q are coprime then P, Q are also coprime (since if $Q = kP$ with $k \geqslant 2$ then $2q + p = kp$ and so $2q = (k-1)p$ implying that either $q = p$ when $k = 2$ (which is impossible), or when $k \geqslant 3$ then $P/Q \leqslant \frac{1}{3}$ (contradicting $T_2[0, 1] \subset [1/3, 1/2]$)).

By the chain rule.

$$\begin{aligned} |(T_2 \circ T_{i_n}^{k_n} \circ \cdots \circ T_{i_{k_1}}^{k_1})'(x_j)| &= |T_2'(p/q)|.|(T_{i_n}^{k_n} \circ \cdots \circ T_{i_1}^{k_1})'(x_j)| \\ &= \frac{1}{(2 + p/q)^2} \frac{\ell_j^2}{q^2} \\ &= \frac{1}{(2q + 2)^2} = \frac{\ell_j^2}{Q^2}. \end{aligned}$$

Case III (Compose with T_3). If we assume that we apply T_3 then

$$T_3(p/q) = \frac{p/q}{2 - p/q} = \frac{p}{2q - p} =: \frac{P}{Q}$$

If p, q are coprime then P, Q are also coprime (since if $Q = kP$ with $k \geqslant 2$ then $2q - p = kp$ and so $2q = (k+1)p$ implying that $P/Q \leqslant \frac{1}{2}$ (contradicting $S_3[0,1] \subset [1/2, 1]$)).

By the chain rule.

$$\begin{aligned}
|(T_3 \circ T_{i_n}^{k_n} \circ \cdots \circ T_{i_{k_n}}^{k_n})'(x_j)| &= |(T_3'(p/q)|.|(T_{i_1}^{k_1} \circ \cdots \circ T_{i_k}^{k_n})'(x_j)| \\
&= \frac{1}{(2 - p/q)^2} \frac{\ell_j^2}{q^2} \\
&= \frac{3^2}{(2q - p)^2} = \frac{\ell_j^2}{Q^2}.
\end{aligned}$$

Thus we would associate to $\underline{t} = (a, b, c)$ the value $p/q = \frac{b}{a+c}$. The associated weight is $Q = a + c$. If $\underline{t} \in \mathcal{P}_j$ ($j = 1, 2$) then, in particular, substituting into Proposition 2.12 gives the asymptotic

$$\#\{\underline{t} : \ell_j^2/Q^2 \leqslant T\} \sim T.$$

Example 2.15 *We can consider the level three of the tree corresponding to* $M(i_1).M(i_2).M(i_3).\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$

(i_1, i_2, i_3)	(a, b, c)	$\frac{b}{a+c}$	$\lvert (T^3)'(\frac{b}{a+c})\rvert$
(1, 1, 1)	(9, 40, 41)	4/5	4/25
(1, 1, 2)	(57, 176, 185)	8/11	4/121
(1, 1, 3)	(51, 140, 149)	7/10	4/100
(1, 2, 1)	(105, 208, 233)	8/13	4/169
(1, 2, 2)	(217, 456, 505)	12/19	4/ 361
(1, 2, 3)	(115, 252, 277)	9/14	4/196
(1, 3, 1)	(95, 168, 193)	7/12	4/144
(1, 3, 2)	(175, 288, 337)	9/16	4/256
(1, 3, 3)	(85, 132, 157)	6/11	4/ 121
(2, 1, 1)	(105, 88, 137)	4/11	4/121
(2, 1, 2)	(377, 336, 505)	8/21	4/ 441
(2, 1, 3)	(275, 252, 373)	7/18	4/324
(2, 2, 1)	(297, 304, 425)	8/19	4/361
(2, 2, 2)	(697, 696, 985)	12/29	4/ 841
(2, 2, 3)	(403, 396, 565)	9/22	4/484
(2, 3, 1)	(207, 224, 305)	7/16	4/256
(2, 3, 2)	(319, 360, 481)	9/20	4/400
(2, 3, 3)	(133, 156, 205)	6/13	4/ 169
(3, 1, 1)	(91, 60, 109)	3/10	4/100
(3, 1, 2)	(299, 180, 349)	5/18	4/324
(3, 1, 3)	(209, 120, 241)	4/15	4/ 225
(3, 2, 1)	(187, 84, 205)	3/14	4/196
(3, 2, 2)	(459, 220, 509)	5/22	4//484
(3, 2, 3)	(273, 136, 305)	4/17	4/ 289
(3, 3, 1)	(117, 44, 125)	2/11	4/ 121
(3, 3, 2)	(165, 52, 173)	2/13	4/ 169
(3, 3, 3)	(63, 16, 65)	1/8	4/64

Furthermore, we can break this down to the two counting functions

$\Pi_1(T) = \#\{(a, b, c) \in \mathcal{P}_1 : a + c \leqslant T\}$ and $\Pi_2(T) = \#\{(a, b, c) \in \mathcal{P}_2 : a + c \leqslant T\}$.

In particular, $\Pi(T) = \Pi_1(T) + \Pi_2(T)$. We then have the following basic asymptotic result.

Proposition 2.16 *There exist $C_1, C_2 > 0$ such that*

$$\Pi_1(T) \sim C_1 T \text{ and } \Pi_2(T) \sim C_2 T \text{ as } T \to +\infty.$$

Finally, we have that

$$\Pi(N) = \Pi_1(N) + \Pi_2(N) \sim (C_1 + C_2)T$$

3 Associated continued fraction maps

It is relatively difficult to prove results directly about the transformation $T : [0,1] \to [0,1]$

3.1 Jump transformations

A general approach to studying the properties of maps with indifferent fixed points of the above form is to accelerate the map using an interval $E \subset [0,1]$.

Definition 3.1 *Let $E \subset [0,1]$. We can consider the* jump transformation $\widehat{T} : [0,1] \to [0,1]$ *defined by $\widehat{T}(x) = T^{n_E(x)+1}$ where*

$$n_E(x) = \min\{j \geqslant 0 \,:\, T^j(x) \in E\}$$

is the return time to E.

1. When $E = \{0\} \cup [1/3, 1]$ then according to [5] the jump transformation $\widehat{T}$ is the even integer continued fraction of Schweiger. Romik considers instead the induced map associated to $(1/5, 2/3)$.

2. When $E = \{1\} \cup [0, 1/2]$ then $\widehat{T}$ is the odd-odd integer continued fraction as described by [5].

3.2 Statistics of the jump maps

Let $\widehat{T} : [0.1] \to [0,1]$ be a transitive countable branch C^1 expanding map for which:

1. the inverse branches $h \in \mathcal{H}_n$ for $\widehat{T}^n$ are uniformly contracting, i.e., there exists $0 < \rho < 1$ such that for all $h \in \mathcal{H}_n$ and $n \geqslant 1$

$$|h(x) - h(y)| \leqslant \rho^n |x - y| \text{ for all } x, y \in [0,1]$$

2. the inverse branches have bounded distortion, i.e., there exists $M > 0$ such that for all $h \in \mathcal{H}_n$ then

$$\left| \frac{h'(x)}{h'(y)} - 1 \right| \leqslant M|x - y|.$$

3. there doesn't exist $a > 0$ such that $\{\log |T'(y)| \,:\, h(x) = x\} \subset a\mathbb{Z}$..

We need the following additional notation. Let $[0,1] = \cup_j I_j$ be the natural countable partition and fix $x_0 \in [0,1]$. For $\widehat{T}^n x = x_0$ let

$$N_j(x) = \#\{0 \leqslant i \leqslant n-1 : \widehat{T}^i(x) \in I_j\}.$$

Under these assumptions we have the following ([4], Theorem 1.5): There exists $C > 0$ such that

$$\#\{x : \exists n \geqslant 1 \text{ with } \log |(\widehat{T}^n)'(x)| < Y\} \sim Ce^Y \text{ as } Y \to +\infty$$

and there exists $\{\Lambda_j\}_j$ such that

$$\sum_{\log |(T^n)'(x)| < Y} N_j(x) \sim \Lambda_j C e^Y \text{ as } Y \to +\infty.$$

The associated statistical results are the following.

Theorem 3.2 (Large Deviations) *For each $x_0 \in [0,1]$ and all j and $\epsilon > 0$:*

$$\limsup_{Y \to +\infty} \frac{1}{Y} \log \left(\frac{\#\left\{x : \exists n \geqslant 1 \textit{ with } \widehat{T}^n x = x_0, \log |(\widehat{T}^n)'(x)| \leqslant Y, |N_j(x)/Q - \Lambda_j| > \epsilon \right\}}{\#\{x : \exists n \geqslant 1 \textit{ with } \widehat{T}^n x = x_0 : \log |(\widehat{T}^n)'(x)| < Y\}} \right) < 0.$$

Theorem 3.3 (Central Limit Theorem) *For each j there exists $\sigma_j^2 > 0$ such that for any $a \in \mathbb{R}$*

$$\frac{\#\left\{x : \exists n \geqslant 1 \textit{ with } \widehat{T}^n x = x_0, \log |(\widehat{T}^n)'(x)| < Y, N_j(x)/\sqrt{Y} \leqslant a \right\}}{\#\{x : \exists n \geqslant 1 \textit{ with } \widehat{T}^n x = x_0, \log |(\widehat{T}^n)'(x)| < Y\}} \to \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{a} e^{-u^2/2\sigma_j^2} du$$

as $Y \to +\infty$.

3.3 Modified Euclidean algorithms

For the purposes of motivation we recall that the classical Euclidean algorithm on pairs of integers $x > y$ is an algorithm which returns $y > x \pmod{y}$. This is an accelerated version of the slow (or subtractive) Euclidean algorithm which would return $x' = \max\{x - y, y\} > y' = \min\{x - y, y\}$. Dynamically, for the slow algorithm if we denote $\xi = y/x \in (0,1)$ then $\xi' = y'/x' = F(\xi)$ is the image under the Farey map

$$F(\xi) = \begin{cases} \frac{\xi}{1-\xi} \text{ if } 0 < \xi \leqslant \frac{1}{2} \\ \frac{1-\xi}{\xi} \text{ if } \frac{1}{2} \leqslant \xi \leqslant 1 \end{cases}$$

The transformation associated to the classical (accelerated) Euclidean is the standard Gauss map $G(\xi) = \frac{1}{\xi} \pmod 1$.

Finally, we can consider a slightly different form from results for the Euclidean algorithm on pairs $(x, y) \in \mathbb{N} \times \mathbb{N}$ for $x > y$ of the form

$$(x,y) \mapsto \begin{cases} A : (x-2y, y) & \text{if } x - 2y > y \\ B : (y, x-2y) & \text{if } y \geqslant x - 2y > 0 \\ C : (y, 2y-x) & \text{if } x - 2y \leqslant 0 \end{cases}$$

This is a Euclidean algorithm in the sense that starting from such (x, y) it will end in finitely many steps at some $(a, 0)$ or (a, a).

Example 3.4 *We can write* $(14, 5) \to (5, 4) \to (4, 3) \to (3, 2) \to (2, 1) \to (1, 0)$

This is referred to a slow algorithm. The associated faster algorithm comes from combining those steps. Romik considers inducing on $(1/5, 2/3)$. It might b natural to induce on $(1/3, 1/2)$

In particular, the statistical properties for $\widehat{T}$ translate into the statistical properties comparing the number of steps in the fast algorithm with the value y.

Theorem 3.5 (Large Deviations) *For all j and $\epsilon > 0$:*

$$\limsup_{Y \to +\infty} \frac{1}{Y} \log \left(\frac{\# \left\{ x : \exists n \geqslant 1 \text{ with } \widehat{T}^n x = x_0, \log |(\widehat{T}^n)'(x)| \leqslant Y, |N_j(x)/Q - \Lambda_j| > \epsilon \right\}}{\#\{x : \exists n \geqslant 1 \text{ with } \widehat{T}^n x = x_0 : \log |(\widehat{T}^n)'(x)| < Y\}} \right) < 0.$$

Theorem 3.6 (Central Limit Theorem) *For each j there exists $\sigma_j^2 > 0$ such that for any $a \in \mathbb{R}$*

$$\frac{\# \left\{ x : \exists n \geqslant 1 \text{ with } \widehat{T}^n x = x_0, \log |(\widehat{T}^n)'(x)| < Y, N_j(x)/\sqrt{Y} \leqslant a \right\}}{\#\{(x_0, y_0) : n(x_0, y_0) \leqslant Y \text{ and } 2 \log q < Y\}} \to \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{a} e^{-u^2/2\sigma_j^2} du$$

as $Y \to +\infty$.

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