An upper bound on the dimension of the Rauzy gasket

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1 Introduction

The Rauzy Gasket $\mathcal{G}$ is a compact subset of the standard 2-simplex, $\Delta = \{(x, y, z) \ : \ x, y, z \geq 0, \ x + y + z = 1\}$. It plays the role of an exceptional set in the theory of interval exchange transformations and other settings, and is the limit set of the iterated function scheme for the three weak projectivised linear maps $T_1, T_2, T_3 : \Delta \to \Delta$, defined by

$$T_1(x, y, z) = \left(\frac{1}{2-x}, \frac{y}{2-x}, \frac{z}{2-x}\right),$$
$$T_2(x, y, z) = \left(\frac{x}{2-y}, \frac{1}{2-y}, \frac{z}{2-y}\right),$$
$$T_3(x, y, z) = \left(\frac{x}{2-z}, \frac{y}{2-z}, \frac{1}{2-z}\right);$$

i.e., $\mathcal{G}$ is the smallest non-trivial closed set such that $\mathcal{G} = \bigcup_{j=1}^{3} T_j(\mathcal{G})$.

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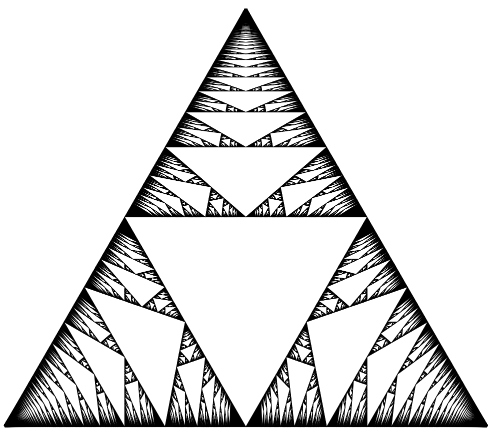

Figure 1: The Rauzy gasket

The gasket has an interesting history, appearing for the first time in 1991 in the work of Arnoux and Rauzy [1], in the context of interval exchange transformations, where it was conjectured that $\mathrm{Leb}(\mathcal{G}) = 0$. The gasket was rediscovered by Levitt in 1993, in a paper which also included a proof (due to Yoccoz) that $\mathrm{Leb}(\mathcal{G}) = 0$. The gasket $\mathcal{G}$ emerged for a third time in the work of De Leo and Dynnikov [4], this time in the context of Novikov's theory of magnetic fields on monocrystals (see [5] for the dichotomy between this and [1]). They gave an alternative proof that $\mathrm{Leb}(\mathcal{G}) = 0$ and proposed the stronger result $\dim_H(\mathcal{G}) < 2$. Novikov and Maltsev [12] also conjectured the stronger bound $\dim_H(\mathcal{G}) < 2$, which was rigorously established by Avila, Hubert and Skripchenko [3]. Empirical estimates in [4] suggest $\dim_H(\mathcal{G}) \approx 1.72$, and a lower bound was shown in [10]. Lastly, Fougeron used semiflows and thermodynamic techniques to show $\dim_H(\mathcal{G}) < 1.825$ [8]. Using completely elementary methods we show the following improved upper bound.

Theorem 1.1. $\dim_H(\mathcal{G}) \leq 1.7407$.

The Rauzy Gasket has a number of interesting recent applications. Gamburd, Magee and Ronan [9] showed asymptotic estimates for integer solutions of the Markov-Hurwitz equations featuring $\dim_H(\mathcal{G})$. Hubert and Paris-Romaskevich in [11] considered triangular tiling billiards, modelling refraction. The gasket $\mathcal{G}$ parameterises triangles admitting trajectories which escape non-linearly to infinity and closed orbits which approximate fractal-like sets.

In section 2 we give the technical result which leads to the bound in Theorem 1.1. This is formulated in terms of certain infinite matrices. In section 3 we give elementary preliminary bounds on the area and diameter of small triangles given as the images of Δ under compositions of the maps T_1, T_2 and T_3. In section 4 we use these to obtain a bound for the dimension, provided an associated sequence of real numbers X_n converges to zero. In sections 5 and 6 we present the core of the proof. In section 5, we use the estimates from section 3 to bound X_n in terms of expressions satisfying an iterative relation. In section 6 we use the renewal theorem to deduce that $X_n \to 0$ under the hypotheses of Theorem 2.8. Finally, in section 7 we apply Theorem 2.8 empirically to deduce the bound in Theorem 1.1.

A fuller account appears in [13].

2 A formal statement

The bound in Theorem 1.1 is a special case of a decreasing sequence of upper bounds, indexed by a parameter $m \in \mathbb{N}_{\geq 2}$. Each bound can be described using powers of an infinite matrix.

Definition 2.1 (An index set). *Let $\mathcal{V} = \bigcup_{k=1}^{m-1} \mathcal{V}_k$ denote the finite set where, for $k < m$,*

$$\mathcal{V}_k := \{1\}^k \times \{2\} \times \{1,2,3\}^{m-k} = \left\{ (1^k, 2, v_{k+2}, \ldots, v_{m+1}) \ : \ v_{k+2}, \ldots, v_{m+1} \in \{1,2,3\} \right\},$$

where we denote, e.g., $1^k = \overbrace{1, \ldots, 1}^{k}$. i.e., $\mathcal{V}$ is the family of strings of length $m+1$ beginning with a sequence of 1s of length k, followed by a 2, then a sequence of 1s, 2s and 3s of length $m-k$.

Remark 2.2. The elements of $\mathcal{V}$ represent orbits in $\{1,2,3\}^{m+1}$ under the natural action of the dihedral group, excluding the orbits of $(1^m, 2)$ and (1^{m+1}).

More specifically, any two words $\underline{i}, \underline{v} \in \{1,2,3\}^{m+1}$ will be considered equivalent (written $\underline{i} \sim \underline{v}$) if there is some permutation π of $\{1,2,3\}$ such that $\pi(i_j) = v_j$ for all $j = 1, \ldots, m+1$. For example, if $i_1 \neq i_2$, (i_1, i_2^m) is equivalent to $(1, 2^m) =: \circledast$, which we consider as a distinguished element of $\mathcal{V}$.

For each $\underline{v} \in \mathcal{V}$ and $n \in \mathbb{N}$, there will be $2 \times 3^{n-m}$ strings $\underline{i} \in \{1,2,3\}^n$ for which the truncation $(i_1, \ldots, i_{m+1}) \in \{1,2,3\}^{m+1}$ is equivalent to $\underline{v}$.

For $n \geq 1$ and $\underline{i} = (i_1, \ldots, i_n) \in \{1,2,3\}^n$ we denote the composition $T_{\underline{i}} = T_{i_1} \circ \cdots \circ T_{i_n}$ and its image $\Delta_{\underline{i}} = T_{\underline{i}}(\Delta) \subset \Delta$. The following is easily seen [13].

Lemma 2.3. *Each $\Delta_{\underline{i}}$, $\underline{i} \in \{1,2,3\}^n$, $n \geq 1$ is again a triangle.*

As we shall see later, these 3^n small triangles provide a useful family of covers for $\mathcal{G}$.

We now formally define a finite matrix B, whose size depends on m and whose entries depend on δ, as follows. Since B acts as a weighted adjacency matrix, we first need a condition for adjacency.

Definition 2.4 (Adjacency)**.** *Given $\underline{v} = (1^k, 2, v_{k+1}, \ldots, v_{m+1}) \in \mathcal{V}$, we write $\underline{v} \mapsto_1 \underline{v}'$, where $\underline{v}' \in \mathcal{V} \cup \{m\}$ is as follows:*

a) If $k \leq m-2$, then $\underline{v}' = (1^{k+1}, 2, v_{k+1}, \ldots, v_m)$.

b) If $k = m-1$, then $\underline{v}' = m$.

Moreover, for $j = 2, 3$, we write $\underline{v} \mapsto_j \underline{v}'$, where $\underline{v}' = (1, 2^k, \ldots) \in \mathcal{V}_1$ is equivalent to $(j, 1^k, 2, v_{k+1}, \ldots, v_m)$.

This definition describes the non-zero indices of the matrix B, as follows.

Definition 2.5 (A finite matrix)**.** *Fix the value $\lambda := \frac{3}{2} - \frac{1}{\sqrt{3}} = 0.9226\ldots$ and $\delta > 0$. Then, we can consider the square matrix $B = B(\delta)$ indexed by $\mathcal{V} \cup \{m\}$, defined by*

$$B_{\underline{v},\underline{w}} = \begin{cases} \max\limits_{x \in \Delta_{\underline{w}}} (2 - x_j)^{-3\delta - \lambda(1-\delta)}, & \text{if } \underline{w} \mapsto_j \underline{v}; \\ 0, & \text{otherwise.} \end{cases}$$

(Note that $B_{\circledast,\underline{v}} = B_{\underline{v},m} = 0$ for all $\underline{v} \in \mathcal{V} \cup \{m\}$.)

Definition 2.6. *To define the infinite matrix, we introduce the following values, for $k \in \mathbb{N}$.*

$$a_k = \left(\frac{k+1}{2k+1}\right)^{3\delta + \lambda(1-\delta)} + \frac{1}{8^\delta}\left(\frac{k+1}{2k+1}\right)^{\lambda(1-\delta)} \qquad \text{and} \qquad b_k = \left(\frac{k+2}{k+3}\right)^{3\delta + \lambda(1-\delta)},$$

We now extend the finite matrix B to give an infinite matrix, D.

Definition 2.7 (An infinite matrix)**.** *Fixing $\delta > 0$, we define the infinite matrix D indexed by $\mathcal{V} \cup \{m, m+1, \dots\}$ as follows.*[1]

$$D = \begin{pmatrix} B_{1,1} & \cdots & B_{1,N-1} & B_{1,N} + a_m & a_{m+1} & a_{m+2} & \cdots \\ B_{2,1} & \cdots & B_{2,N-1} & B_{2,N} & 0 & 0 & \cdots \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots \\ B_{N,1} & \cdots & B_{N,N-1} & B_{N,N} & 0 & 0 & \cdots \\ 0 & \cdots & 0 & b_m & 0 & 0 & \cdots \\ 0 & \cdots & 0 & 0 & b_{m+1} & 0 & \cdots \\ 0 & \cdots & 0 & 0 & 0 & b_{m+2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$
$$= \begin{pmatrix} 0 & \cdots & 0 & a_m & a_{m+1} & a_{m+2} & \cdots \\ B_{2,1} & \cdots & B_{2,N-1} & 0 & 0 & 0 & \cdots \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots \\ B_{N,1} & \cdots & B_{N,N-1} & 0 & 0 & 0 & \cdots \\ 0 & \cdots & 0 & b_m & 0 & 0 & \cdots \\ 0 & \cdots & 0 & 0 & b_{m+1} & 0 & \cdots \\ 0 & \cdots & 0 & 0 & 0 & b_{m+2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let $(D^k)_{i,j}$ be the entry corresponding to the (i,j)th entry of the kth power of D. Our main technical result is the following, which we will rephrase later in a more explicit way.

Theorem 2.8. *If $\delta > \frac{1-\lambda}{3-\lambda} = 0.037\dots$ satisfies $\sum_{k=1}^{\infty}(D^k)_{1,1} \leq 1$, then $\dim_H(\mathcal{G}) \leq 1 + \delta$.*

In particular, letting $m = 9$ and estimating numerically the value of $\delta \approx 0.7407$ (rounding up to four decimal places) giving equality in the hypothesis of Theorem 2.8 gives the bound in Theorem 1.1 in the introduction as a corollary (see §7).

3 Triangle estimates

In this section we collect together elementary but useful estimates for the triangles $\Delta_{\underline{i}}$.

Lemma 3.1 (Area estimate)**.** *If $\underline{i} = (i_1, \dots, i_n) \in \{1,2,3\}^n$ and $j \in \{1,2,3\}$ then, writing $j\underline{i} = (j, i_1, \dots, i_n)$, we have that*

$$\frac{\operatorname{area}(\Delta_{j\underline{i}})}{\operatorname{area}(\Delta_{\underline{i}})} \leq \max_{x \in \Delta_{\underline{i}}} (2 - x_j)^{-3}.$$

Proof. By a change of variables,

$$\operatorname{area}(\Delta_{j\underline{i}}) = \operatorname{area}(T_j \Delta_{\underline{i}}) = \int_{\Delta_{\underline{i}}} \operatorname{Jac} T_j(x) \,\mathrm{d}x \leq \max_{x \in \Delta_{\underline{i}}} \big(\operatorname{Jac} T_j(x)\big) \operatorname{area}(\Delta_{\underline{i}}).$$

[1] Here, $N-1$ is the cardinality of $\mathcal{V}$, and in the ordering of $\mathcal{V} \cup \{m\}$, we take $\circledast$ to be first and m last.

To complete the proof, we now show that $\operatorname{Jac} T_j(x) = (2-x_j)^{-3}$. If $j=1$, with respect to the orthogonal basis $(\frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}), \frac{1}{\sqrt{6}}(2\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}))$, the derivative map $DT_j(x)$ takes the following form:

$$DT_1(x) = \frac{1}{(2-x_1)^2} \begin{pmatrix} 1 & 0 \\ \frac{1}{\sqrt{3}}(x_3 - x_2) & 2 - x_1 \end{pmatrix}. \tag{1}$$

Thus, $\operatorname{Jac} T_1(x) = (2-x_1)^{-3}$. The other cases ($j \neq 1$) follow by symmetry. $\square$

Remark 3.2. In fact, one can deduce a simple formula for area($\Delta_{\underline{i}}$) from the precise form of the above jacobean (see [13] for more details).

We can apply a similar reasoning to the associated diameters.

Lemma 3.3 (Diameter estimate)**.** *If* $\underline{i} = (i_1, \ldots, i_n) \in \{1,2,3\}^n$ *and* $j \in \{1,2,3\}$ *then*

$$\frac{\operatorname{diam}(\Delta_{j\underline{i}})}{\operatorname{diam}(\Delta_{\underline{i}})} \leq \max_{x \in \Delta_{\underline{i}}} (2-x_j)^{-\lambda}.$$

Proof. We now consider the operator norm, $\|DT_j(x)\|$, of the derivatives $T_j(x)$ (i.e., with respect to the ambient metric in $\mathbb{R}^3$). Considering $j=1$, the matrix in (1) satisfies

$$\|DT_1(x)\| = \frac{1}{b\sqrt{2}} \sqrt{1 + a + b + \sqrt{(1+a+b)^2 - 4b}},$$

where $a = \frac{1}{3}(x_2 - x_3)^2$ and $b = (2-x_1)^2$. Using the simple bound $a \leq \frac{1}{3}(1-x_1)^2$, one has that, for all $x \in \Delta$,

$$\|DT_1(x)\|^2 \leq f(x_1) := \frac{2x_1^2 - 7x_1 + 8 + 2\sqrt{(1-x_1)^2\,(x_1^2 - 5x_1 + 7)}}{3(2-x_1)}.$$

We observe (with the help of a computer) that the function $t \mapsto \log(f(t))/\log(2-t)$ is increasing on $[0,1)$, and converges to -2λ as $t \to 1^-$. Therefore $\|DT_1(x)\| \leq \sqrt{f(x_1)} \leq (2-x_1)^{-\lambda}$. The result for $j=1$ then follows simply from the mean value theorem. The cases of $j \neq 1$ subsequently follow by symmetry. $\square$

In view of the previous lemmas, it is useful to have more explicit bounds involving $(2-x_j)^{-1}$ depending on where the point $x \in \Delta$ approximately lies. We henceforth denote

$$A_{n,k} = \{\underline{i} \in \{1,2,3\}^n \;:\; i_1 = \cdots = i_k \neq i_{k+1}\} \qquad (1 \leq k < n)$$

and work with the following simple bounds.

Lemma 3.4. *If* $\underline{i} \in A_{n,k}$ *and* $i_1 = i, j, k \in \{1,2,3\}$ *are distinct then, for any* $\delta > 0$,

$$\max_{x \in \Delta_{\underline{i}}} (2-x_i)^{-1} \leq \frac{k+2}{k+3},$$

$$\max_{x \in \Delta_{\underline{i}}} (2-x_j)^{-1} \leq \frac{k+1}{2k+1}, \qquad \textit{and}$$

$$\max_{x \in \Delta_{\underline{i}}} \left\{(2-x_j)^{-3\delta} + (2-x_k)^{-3\delta}\right\} \leq \left(\frac{k+1}{2k+1}\right)^{3\delta} + \frac{1}{8^\delta}.$$

Proof. By symmetry, it suffices to consider $\underline{i} = (i_1, \ldots, i_n)$ with $i_1 = \cdots = i_k = 1$ and $i_{k+1} \neq 1$. Then $\Delta_{\underline{i}}$ is contained in

$$\mathrm{cl}\big(T^k(\Delta) \setminus T^{k+1}(\Delta)\big) = \left\{ x \in \Delta \,:\, \frac{k}{k+1} \leq x_1 \leq \frac{k+1}{k+2} \right\} \tag{2}$$

(cl denoting the topological closure), where we have used that $T^k(\Delta) = \{x \in \Delta \,:\, x_1 \geq \frac{k}{k+1}\}$. The first two bounds follow directly:

$$(2 - x_1)^{-1} \leq \left(2 - \frac{k+1}{k+2}\right)^{-1} = \frac{k+2}{k+3} \qquad \text{and} \qquad (2 - x_j)^{-1} \leq \left(2 - \frac{1}{k+1}\right)^{-1} = \frac{k+1}{2k+1}.$$

For the third bound, we observe that the function $f(x) := (2 - x_2)^{-3\delta} + (2 - x_3)^{-3\delta}$ is convex on Δ and thus takes its maximum on $T_1^k(\Delta)$ at one of its vertices:

$$\max_{\Delta_{\underline{i}}} f \leq \max_{T_1^k(\Delta)} f = \max\left\{ f \circ T_1^k(1,0,0), f \circ T_1^k(0,1,0), f \circ T_1^k(0,0,1) \right\}.$$

More explicitly, noting that $f \circ T_1^k(1,0,0) = f(1,0,0) \leq f \circ T_1^k(0,1,0) = f\left(\frac{k}{k+1}, \frac{1}{k+1}, 0\right) = f\left(\frac{k}{k+1}, 0, \frac{1}{k+1}\right) = f \circ T_1^k(0,0,1)$, one has

$$\max_{\Delta_{\underline{i}}} f \leq \left(2 - \frac{1}{k+1}\right)^{-3\delta} + 2^{-3\delta} = \left(\frac{k+1}{2k+1}\right)^{3\delta} + \frac{1}{8^\delta},$$

as required. □

4 Cover estimates

The upper bound on the dimension in Theorem 2.8 is based on finding a value $\delta \in (0,1)$ so that Lemma 4.1 below applies.

Its proof is simple and based on a simple sequence of open covers of $\mathcal{G}$, $\mathcal{U}_n$, each obtained by covering the set of nth level triangles, $\{\Delta_{\underline{i}} \,:\, |\underline{i}| = n\}$.

Lemma 4.1. *Assume $\delta > 0$ and that the sequence*

$$X_n := \sum_{|\underline{i}|=n} \mathrm{area}(\Delta_{\underline{i}})^\delta \,\mathrm{diam}(\Delta_{\underline{i}})^{1-\delta} \to 0 \qquad \text{as} \qquad n \to \infty.$$

Then $\dim_H(\mathcal{G}) \leq 1 + \delta$.

Proof. As illustrated in Figure 2, each triangle $\Delta_{\underline{i}}$ is contained in a rectangle with side lengths $b = \mathrm{diam}(\Delta_{\underline{i}})$ and $a = 2\,\mathrm{area}(\Delta_{\underline{i}})/\,\mathrm{diam}(\Delta_{\underline{i}})$, which can in turn be covered by $[2b/a]$ disks of diameter $2a$. That is, $\Delta_{\underline{i}}$ can be covered by $\mathrm{diam}(\Delta_{\underline{i}})^2/\,\mathrm{area}(\Delta_{\underline{i}})$ disks of diameter $4\,\mathrm{area}(\Delta_{\underline{i}})/\,\mathrm{diam}(\Delta_{\underline{i}})$. Denoting by $\mathcal{U}_n$ the union of the disks covering all of the triangles $\Delta_{\underline{i}}$, this gives

$$\sum_{U \in \mathcal{U}_n} \mathrm{diam}(U)^{1+\delta} \leq 4^{1+\delta} \sum_{|\underline{i}|=n} \left(\frac{\mathrm{diam}(\Delta_{\underline{i}})^2}{\mathrm{area}(\Delta_{\underline{i}})}\right) \left(\frac{\mathrm{area}(\Delta_{\underline{i}})}{\mathrm{diam}(\Delta_{\underline{i}})}\right)^{1+\delta} = 4^{1+\delta} X_n.$$

The result follows from the standard definition of the Hausdorff dimension (see, e.g., [6]). □

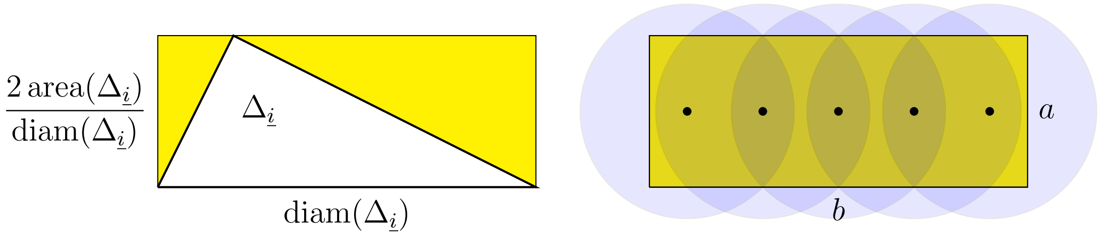

Figure 2: Left: covering $\Delta_{\underline{i}}$ by a rectangle. Right: covering a rectangle by open disks.

Remark 4.2. Perhaps surprisingly, this lemma appears to give a significant improvement on working exclusively with either $\mathrm{diam}(\Delta_{\underline{i}})$ or $\mathrm{area}(\Delta_{\underline{i}})$ (as in [8] and [1]).

To apply Lemma 4.1 we want to bound X_n from above using a partition of $\{1,2,3\}^n$. We then apply different bounds to the triangles $\phi_j(\Delta_{\underline{i}})$ according to which element of the partition the index $\underline{i}$ lies in.

Definition 4.3 (Partitioning up the sequences)**.** *Fixing $m < n$, we can partition*

$$\{1,2,3\}^n := \bigcup_{\underline{v}\in\mathcal{V}} A_{n,\underline{v}} \cup \bigcup_{k=m}^{n-1} A_{n,k} \cup \bigcup_{j=1}^{3} \{(j^n)\},$$

where $A_{n,k}$ is as above and $A_{n,\underline{v}} = \{\underline{i} \in \{1,2,3\}^n : (i_1,\ldots,i_{m+1}) \sim \underline{v}\}$, recalling the equivalence relation $\sim$ on page 3.

This partition naturally gives the following components of X_n, for $n > m$.

Definition 4.4. *For $n > m$ and any $\alpha \in \mathcal{V} \cup \{m,\ldots,n-1\}$, we write*

$$X_{n,\alpha} := \sum_{\underline{i}\in A_{n,\alpha}} \mathrm{area}(\Delta_{\underline{i}})^{\delta}\,\mathrm{diam}(\Delta_{\underline{i}})^{1-\delta}.$$

We now derive bounds on X_{n+1} using estimates on $X_{n,\underline{v}}$ and $X_{n,k}$.

5 Inductive bounds on $X_{n,\underline{v}}$ and $X_{n,k}$

We now turn to the problem of showing that $X_n \to 0$ as $n \to \infty$, which will enable us to apply Lemma 4.1. Our method, broadly speaking, is based on obtaining inductive bounds bounding

1. $X_{n+1,k+1}$ in terms of $X_{n,k}$, and
2. $X_{n+1,\underline{v}}$ in terms of the $X_{n,\underline{w}}$ and $X_{n,k}$.

These will prove useful in applying the renewal theorem in the next section.

5.1 Bounds of terms indexed by numbers

The next simple lemma is independent of m, and features the elements b_k.

Lemma 5.1 (Number Lemma). *Fix $\delta \in (0,1)$. Then, for all $n > k$,*

$$X_{n+1,k+1} \leq b_k X_{n,k}.$$

Proof. Since $j\underline{i} \in A_{n+1,k+1}$ if and only if $\underline{i} \in A_{n,k}$ and $j = i_1$, we may write

$$\begin{aligned} X_{n+1,k+1} &= \sum_{\underline{i} \in A_{n,k}} \operatorname{area}(\Delta_{i_1\underline{i}})^\delta \operatorname{diam}(\Delta_{i_1\underline{i}})^{1-\delta} \\ &\leq \sum_{\underline{i} \in A_{n,k}} \left(\frac{k+2}{k+3}\right)^{3\delta} \left(\frac{k+2}{k+3}\right)^{\lambda(1-\delta)} \operatorname{area}(\Delta_{\underline{i}})^\delta \operatorname{diam}(\Delta_{\underline{i}})^{1-\delta} \\ &= b_k X_{n,k}, \end{aligned}$$

where we have applied Lemmas 3.1 and 3.3. □

5.2 Bounds of terms indexed by words

We can bound $X_{n+1,\underline{v}}$, $n > m$, with $\underline{v} \in \mathcal{V} \cup \{m\}$, using terms at level n with the matrix B and the coefficients a_k, as formulated in the next lemma.

Lemma 5.2 (Word Lemma). *Let $\delta \in (0,1)$, $n > m$ and $\underline{v} \in \mathcal{V} \cup \{m\}$. Then, recalling $\circledast = (1, 2^n)$, there exists a positive sequence c_n such that $\sum_n c_n < \infty$ and*

$$\begin{aligned} X_{n+1,\circledast} &\leq c_n + \sum_{k=m}^{n-1} a_k X_{n,k} \qquad \textit{and} \\ X_{n+1,\underline{v}} &\leq \sum_{\underline{w} \in \mathcal{V}} B_{\underline{v},\underline{w}} X_{n,\underline{w}} \qquad \textit{if } \underline{v} \neq \circledast. \end{aligned}$$

Proof. For the first inequality, note that $j\underline{i} \in A_{n+1,\circledast}$ if and only if $j \neq i_1$ and $\underline{i} \in \bigcup_{k=m}^{n-1} R_k \cup \{(1^n), (2^n), (3^n)\}$. Therefore, using Lemmas 3.1 and 3.3 and symmetry,

$$\begin{aligned} X_{n+1,1} &= 6 \operatorname{area}(\Delta_{(2,1^n)})^\delta \operatorname{diam}(\Delta_{(2,1^n)})^{1-\delta} + \sum_{k=m}^{n-1} \sum_{\underline{i} \in A_{n,k}} \sum_{j \neq i_1} \operatorname{area}(\Delta_{j\underline{i}})^\delta \operatorname{diam}(\Delta_{j\underline{i}})^{1-\delta} \\ &\leq c_n + \sum_{k=m}^{n-1} \sum_{\underline{i} \in A_{n,k}} \sum_{j \neq i_1} \operatorname{area}(\Delta_{\underline{i}})^\delta \operatorname{diam}(\Delta_{j\underline{i}})^{1-\delta} \left(\frac{k+1}{k+2}\right)^{\lambda(1-\delta)} \\ &\leq c_n + \sum_{k=m}^{n-1} \sum_{\underline{i} \in A_{n,k}} \operatorname{area}(\Delta_{\underline{i}})^\delta \operatorname{diam}(\Delta_{\underline{i}})^{1-\delta} \left(\frac{k+1}{k+2}\right)^{\lambda(1-\delta)} \left(\left(\frac{k+1}{k+2}\right)^{3\delta} + \frac{1}{8^\delta}\right) \\ &= c_n + \sum_{k=m}^{n-1} a_k X_{n,k}, \end{aligned}$$

where $c_n := 6\,\mathrm{area}(\Delta_{(2,1^n)})^\delta\,\mathrm{diam}(\Delta_{(2,1^n)})^{1-\delta}$.

Regarding the second inequality: The combinatorics above and in the previous proof show that $\underline{i} \in \bigcup_{k=m}^{n-1} A_{n,k} \cup \{(1^n),(2^n),(3^n)\}$ implies $j\underline{i} \notin A_{n+1,\underline{v}}$ for every $\underline{v} \in \mathcal{V} \cup \{m\} \setminus \{\circledast\}$. Fixing such a $\underline{v}$ and considering the contrapositive, one sees that $j\underline{i} \in A_{n+1,\underline{v}}$ implies that $\underline{i} \in A_{n,\underline{w}}$ and $\underline{w} \mapsto_{j'} \underline{v}$ for some $\underline{w}, j'$. Assuming $(i_1,\ldots,i_{m+1}) = \underline{w}$ for simplicity, we have $j' = j$ and $\Delta_{\underline{i}} \subset \Delta_{\underline{w}}$. Hence, by Lemmas 3.1 and 3.3,

$$\frac{\mathrm{area}(\Delta_{j\underline{i}})^\delta\,\mathrm{diam}(\Delta_{j\underline{i}})^{1-\delta}}{\mathrm{area}(\Delta_{\underline{i}})^\delta\,\mathrm{diam}(\Delta_{\underline{i}})^{1-\delta}} \le \max_{\underline{x}\in\Delta_{\underline{w}}}(2-x_j)^{-3\delta-\lambda(1-\delta)} =: B_{\underline{v},\underline{w}}.$$

Thus, we may write the following.

$$\begin{aligned} X_{n+1,\underline{v}} = \sum_{j=1}^{3} \sum_{\substack{\underline{w}\in\mathcal{V}:\\ \underline{w}\mapsto_j\underline{v}}} \sum_{\underline{i}\in A_{n,\underline{w}}} \mathrm{area}(\Delta_{j\underline{i}})^\delta\,\mathrm{diam}(\Delta_{j\underline{i}})^{1-\delta} &\le \sum_{\substack{\underline{w}\in\mathcal{V}:\\ B_{\underline{v},\underline{w}}\neq 0}} \sum_{\underline{i}\in A_{n,\underline{w}}} B_{\underline{v},\underline{w}}\,\mathrm{area}(\Delta_{\underline{i}})^\delta\,\mathrm{diam}(\Delta_{\underline{i}})^{1-\delta} \\ &= \sum_{\underline{w}\in\mathcal{V}} B_{\underline{v},\underline{w}} X_{n,\underline{w}}, \end{aligned}$$

as required.

Finally, a direct computation of the vertices of $\Delta_{(2,1^n)} = T_2T_1^n(\Delta)$ shows that its diameter and area are proportional to n^{-1} and n^{-2} respectively, as $n \to \infty$. Hence c_n is proportional to $n^{-1-\delta}$ as $n \to \infty$, and is thus summable for every $\delta > 0$, completing the proof □

6 Renewal Theorem

We now use the iterative bounds of the last section to show that $X_n \to 0$ as $n \to \infty$, under the hypotheses of Theorem 2.8. This uses the following mild adaptation of the classical renewal theorem of Feller [7, p.330].

Theorem 6.1. *Given sequences $(Y_n)_{n=0}^\infty$, $(\mu_n)_{n=1}^\infty$ and $(\nu_n)_{n=1}^\infty$ which are non-negative and satisfy $\sum_{k=1}^\infty \mu_k < 1$, $\sum_{k=1}^\infty \nu_k < \infty$, and for each $n \in \mathbb{N}$,*

$$Y_n \le \nu_n + \sum_{k=1}^{n} \mu_k Y_{n-k},$$

we have that $Y_n \to 0$ as $n \to \infty$.

We apply this theorem with $Y_n = X_{n+m,\circledast}$, to give Theorem 2.8, which can be rewritten as follows.

Theorem 6.2. *If $\delta > \frac{1-\lambda}{3-\lambda} = 0.037\ldots$ satisfies*

$$\sum_{k=1}^{\infty} (B^k)_{m,\circledast} \sum_{k=m}^{\infty} a_k \prod_{i=m}^{k-1} b_k \le 1, \tag{3}$$

then $\dim_H(\mathcal{G}) \le 1 + \delta$.

Proof. Assume additionally that (3) holds with a strict inequality (the case of equality follows in the limit, since the LHS is decreasing in δ), and that $\delta < 1$ (the conclusion being otherwise trivial). Applying Lemma 5.1 repeatedly in the first estimate of Lemma 5.2 gives

$$X_{n+1,\circledast} \le c_n + \sum_{k=m}^{n-1} a_k \prod_{i=m}^{k-1} b_i X_{n+m-k,m}.$$

Moreover, the second estimate of Lemma 5.2 extends inductively to give, for all $\hat{n} > m$,

$$X_{\hat{n},m} \le \sum_{k=1}^{\hat{n}-m-1} (B^k)_{m,\circledast} X_{\hat{n}-k,\circledast} + \sum_{\substack{\underline{v}\in\mathcal{V}: \\ \underline{v}\neq\circledast}} (B^{\hat{n}-m-1})_{m,\underline{v}} X_{m+1,\underline{v}}.$$

Putting these two together gives the renewal-style inequality

$$X_{n+1,\circledast} \le \nu_n + \sum_{k=1}^{n-m} \mu_k X_{n+1-k,\circledast},$$

with coefficients

$$\mu_k = \sum_{i+j=k} (B^i)_{m,\circledast} a_{m+j} \prod_{l=m}^{m+j-1} b_l$$

and remainder term

$$\nu_n = c_n + \sum_{k=m}^{n-1} a_k \prod_{i=m}^{k-1} b_i \sum_{y\in\mathcal{V}-\{\circledast\}} (B^{n-k-1})_{m,y} X_{m+1,y}.$$

The hypothesis on the coefficients required by the renewal theorem is precisely the strict (3):

$$\sum_{k=1}^{\infty} \mu_n = \sum_{k=0}^{\infty} (B^k)_{m,\circledast} \sum_{k=m}^{\infty} a_k \prod_{i=m}^{k-1} b_i < 1.$$

Regarding the remainder terms: By elementary combinatorics, B^{m+1} has only one zero row and zero column, so (3) also ensures that the spectral radius $\rho = \rho(B) < 1$. Thus, there exists $C > 0$ such that

$$\sum_{n=1}^{\infty} \nu_n - c_n \le C \sum_{n=1}^{\infty} \sum_{k=m}^{n-1} a_k \prod_{i=m}^{k-1} b_i \rho^{n-k} = C \sum_{k=m}^{\infty} a_k \prod_{i=m}^{k-1} b_i \sum_{n=1}^{\infty} \rho^n.$$

Using that $\rho < 1$ and

$$a_k \prod_{i=m}^{k-1} b_i = \mathcal{O}\big(k^{-3\delta-\lambda(1-\delta)}\big) \qquad (k \to \infty),$$

we see that the right hand side is summable, since $\delta > \frac{1-\lambda}{3-\lambda}$. That is, $\sum_n \nu_n < \infty$, and thus the renewal theorem gives that $X_{n,\circledast} \to 0$ as $n \to \infty$.

Finally, we use this to bound X_n via the following lemma.

Lemma 6.3. *Given $m \in \mathbb{N}$ and $\delta \in (0,1)$, there exists $C > 0$ such that $X_n \leq C X_{n+m+1,\circledast}$.*

Proof. For each $\underline{i} \in \{1,2,3\}^n$, the word $\underline{i}' = (1, 2^m, i_1, \ldots, i_n) \in A_{n+m+1,\circledast}$. This word satisfies

$$\frac{\operatorname{area}(\Delta_{\underline{i}'})^\delta \operatorname{diam}(\Delta_{\underline{i}'})^{1-\delta}}{\operatorname{area}(\Delta_{\underline{i}})^\delta \operatorname{diam}(\Delta_{\underline{i}})^{1-\delta}} = \frac{\operatorname{area}(T_1 T_2^m \Delta_{\underline{i}})^\delta \operatorname{diam}(T_1 T_2^m \Delta_{\underline{i}})^{1-\delta}}{\operatorname{area}(\Delta_{\underline{i}})^\delta \operatorname{diam}(\Delta_{\underline{i}})^{1-\delta}} \geq K^{(1+\delta)(m+1)},$$

where we define K by bounding the *minimum* singular value of $DT_j(x)$ uniformly over $x \in \Delta$, applying the estimates of section 3:

$$\left\| D(T_j^{-1})\big(T_j(x)\big) \right\| = \frac{\operatorname{Jac} T_j(x)}{\|DT_j(x)\|} \geq (2 - x_j)^{\lambda - 3} \geq 2^{\lambda - 3} =: K.$$

In particular, with $C = K^{(1+\delta)(m+1)}$

$$X_{n+m+1,\circledast} \geq \sum_{|\underline{i}|=n} \operatorname{area}(\Delta_{\underline{i}'})^\delta \operatorname{diam}(\Delta_{\underline{i}'})^{1-\delta} \geq C X_n,$$

as required. □

Thus, since $X_{n,\circledast} \to 0$ as $n \to \infty$ implies that $X_n \to 0$ as $n \to \infty$, applying Lemma 4.1 completes the proof of Theorem 2.8. □

7 Numerical estimates

The usefulness of Theorem 2.8 depends on our ability to check the hypothesis for a given candidate value of δ. However, since the size of $\mathcal{V}$ (and hence of B) is exponentially increasing in m, such a check requires an exponentially increasing amount of computer time and memory, and with diminishing returns. Fortunately, for relatively small values of m, we have bounds on the dimension which improve upon the known bounds.

We now present the values δ_m, giving equality in (3) (for each fixed value of m), in the table below. These were calculated using Wolfram Mathematica on a Lenovo ThinkPad X220; the 9th estimate was confirmed in less than 90 seconds.

m	$\delta_m + 1$
2	1.8285
3	1.7978
4	1.7764
5	1.7624

m	$\delta_m + 1$
6	1.7534
7	1.7475
8	1.7435
9	1.7407

Table 1: Upper bounds $1 + \delta_m$ on $\dim_H(\mathcal{G})$ for different choices of m, rounded upwards to four decimal places.

Remark 7.1. The elementary nature of this method suggests that a similar approach might be used in other related examples of sets and bounds on their dimension.

References

[1] P. Arnoux and F. Rauzy. Représentation géométrique de suites de complexité $2n + 1$. *Bull. Soc. Math. France* **119** (1991), no. 2, 199–215.

[2] P. Arnoux and J.-C. Yoccoz. Construction de diffeomorphismes pseudo-Anosov. *C. R. Acad. Sci. Paris Sér. I Math.* **292** (1981), no. 1, 75–78.

[3] A. Avila, P. Hubert and A. Skripchenko. On the Hausdorff dimension of the Rauzy gasket. *Bull. Soc. Math. France* **144** (2016), no. 3, 539–568.

[4] R. De Leo and I. Dynnikov. Geometry of plane sections of the infinite regular skew polyhedron, *Geom. Dedicata* **138** (2009), 51–67.

[5] I. A. Dynnikov, P. Hubert and A. Skripchenko. Dynamical systems around the Rauzy gasket and their ergodic properties. arXiv:2011.15043

[6] K. Falconer. *Fractal Geometry*, Wiley: New York, 1990

[7] W. Feller. *An Introduction to Probability Theory and Its Applications*, vol 2, Wiley: New York, 1971.

[8] C. Fougeron. Dynamical properties of simplicial systems and continued fraction algorithms. arXiv:2001.01367

[9] A. Gamburd, M. Magee and R. Ronan. An asymptotic formula for integer points on Markoff-Hurwitz varieties. *Ann. of Math. (2)* **190** (2019), no. 3, 751–809.

[10] R. Gutiérrez-Romo and C. Matheus. Lower bounds on the dimension of the Rauzy gasket. *Bull. Soc. Math. France* **148** (2020), no. 2, 321–327.

[11] P. Hubert and O. Paris-Romaskevich. Triangle tiling billiards and the exceptional family of their escaping trajectories: circumcenters and Rauzy gasket. *Exp. Math.* (2019).

[12] A. Maltsev and S. P. Novikov. Dynamical systems, topology, and conductivity in normal metals. *J. Statist. Phys.* **115** (2004), no. 1–2, 31–46.

[13] B. Sewell. *Equidistribution of infinite interval substitution schemes, explicit resonances of Anosov toral maps, and the Hausdorff dimension of the Rauzy gasket.* PhD thesis. Available from `www.warwick.ac.uk/fac/sci/maths/people/staff/sewell`