Constructing equilibrium states for Smale spaces

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In this article we shall consider the classical problem on the relationship between equilibrium states for different potentials. Moreover for any two Hölder continuous potentials, we shall give a geometric construction for transforming the Gibbs measure for one potential into the Gibbs measure for the other potential. The construction presents a new way to think about Gibbs measures complementing known constructions using, for example, periodic points [1] or homoclinic points [4].

We will work in the general setting of Smale spaces. Recall that a uniformly hyperbolic diffeomorphism has a local product structure by local stable and unstable manifolds (see [2]). A Smale space is an extension of the uniformly hyperbolic diffeomorphisms in the sense that we only have a compact metric space X and a homeomorphism $f : X \to X$ satisfying a local product structure determined by an appropriate bracket map $[\cdot, \cdot] : X \times X \to X$. Additionally, subshifts of finite type are a class of examples of Smale spaces and our work provides a unified approach to equilibrium states covering both uniformly hyperbolic diffeomorphisms and subshifts of finite type without any use of Markov partitions.

To describe our construction, let G_1 be a Hölder continuous potential and consider a measure $\mu^u_{G_1} = \mu^u_{x,G_1}$ supported on a piece of unstable manifold $W^u_\delta(x)$ with the conditional Gibbs property defined in §2. Intuitively the conditional Gibbs property gives a uniform bound on the measure of unstable Bowen balls of the form

$$\frac{\mu^u_{G_1}(B_{d_u}(y, n, \epsilon))}{e^{S_n G_1(y) - nP(G_1)}},$$

where $y \in W^u_\delta(x)$, $\epsilon > 0$ small, $S_n G_1(x) = \sum_{k=0}^{n-1} G_1(f^i x)$, $P(G_1)$ is the pressure and $B_{d_u}(y, n, \epsilon)$ denotes the Bowen ball in $W^u(x)$ with respect to the unstable metric on $W^u(x)$.

We can now give a brief overview of our construction for Smale spaces. Starting from a conditional Gibbs measure for a Hölder continuous function G_1. We then define a sequence of reference measures which are absolutely continuous with respect to $\mu^u_{G_1}$ and have the appropriately chosen density $e^{S_n G_2(y) - S_n G_1(y)}$ for a continuous G_2. Taking averaged pushforwards of the sequence of reference measures, the weak* convergent limits are equilibrium states for G_2. The precise statement can be found in Theorem 2.5. One way to view Theorem 2.5 is as a geometric method which transforms the Gibbs measure for G_1 into the Gibbs measure for G_2. The illustrative Example 2.7 provides an explicit calculation demonstrating the transformation of the $(1/2, 1/2)$-Bernoulli measure into the $(p, 1-p)$-Bernoulli measure using Theorem 2.5 for the full shift on two symbols.

Theorem 2.5 can also be viewed as a new way to construct the equilibrium state for any continuous function G_2 starting from the equilibrium state for a reference Hölder potential G_1. Moreover, Theorem 2.5 extends the construction in [9], where the authors study uniformly hyperbolic attractors and therefore exhibit the important property that there is an induced volume on unstable manifolds, to non-attracting uniformly hyperbolic systems.

The proof of Theorem 2.5 relies on a growth estimate on a piece of unstable manifold which relates the pressure of two continuous potentials G_1 and G_2. This result is of independent interest and its statement can be found in Lemma 3.1.

1 Definitions

We now state the definition of a Smale space which is based on §7 in Ruelle's book [11]. The definition has multiple technical conditions so we provide a couple of enlightening examples that illustrate these conditions.

Let X be a non-empty compact metric space with metric d. Assume there is an $\epsilon > 0$ and a map, $[\cdot,\cdot]$ with the following properties:

$$[\cdot,\cdot] : \{(x,y) \in X \times X \,:\, d(x,y) < \epsilon\} \to X$$

is a continuous map such that $[x,x] = x$ and

$$[[x,y],z] = [x,z], \tag{SS1}$$
$$[x,[y,z]] = [x,z], \tag{SS2}$$
$$f([x,y]) = [f(x), f(y)], \tag{SS3}$$

when the two sides of these relations are defined.

Additionally, we require the existence of a constant $0 < \lambda < 1$ such that, for any $x \in X$ we have the following two conditions: For $y, z \in X$ such that $d(x,y), d(x,z) < \epsilon$ and $[y,x] = x = [z,x]$, we have

$$d(f(y), f(z)) \leq \lambda d(y,z); \tag{SS4}$$

and for $y, z \in X$ such that $d(x,y), d(x,z) < \epsilon$ and $[x,y] = x = [x,z]$, we have

$$d(f^{-1}(y), f^{-1}(z)) \leq \lambda d(y,z). \tag{SS5}$$

Definition 1.1. *Let X be a compact metric space with metric d. Let $f : X \to X$ be a homeomorphism and $[\cdot,\cdot]$ have the properties $SS1 - SS5$ above. Then we define the Smale space to be the quadruple $(X, d, f, [\cdot,\cdot])$. If $f : X \to X$ is also topological mixing then we call $(X, d, f, [\cdot,\cdot])$ a mixing Smale space.*

In essence Smale spaces are systems that exhibit a local product structure given by $[\cdot,\cdot]$ and this product structure can be used to define local stable and unstable manifolds.

Definition 1.2. *For sufficiently small $\delta > 0$ one can define the stable and unstable manifolds through $x \in X$ by*

$$W^s_\delta(x) = \{y \in X \,:\, y = [x, y] \text{ and } d(x, y) < \delta\},$$
$$W^u_\delta(x) = \{y \in X \,:\, y = [y, x] \text{ and } d(x, y) < \delta\}.$$

From $SS4$ and $SS5$ we have that the stable and unstable manifolds are equivalently characterised in terms of the behaviour of forward and backward orbits,

$$W^s_\delta(x) = \{y \in X \,:\, d(f^n x, f^n y) \leq \delta, \forall n \geq 0\},$$

and

$$W^u_\delta(x) = \{y \in X \,:\, d(f^{-n} x, f^{-n} y) \leq \delta, \forall n \geq 0\}.$$

1.1 Examples

The conditions $SS1 - SS5$ are perhaps best understood with illustrating examples, namely hyperbolic diffeomorphisms and subshifts of finite type.

1.1.1 Locally maximal hyperbolic diffeomorphisms

Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism on a compact Riemannian manifold and let $X \subset M$ be a closed f-invariant set.

Definition 1.3. *The map $f : X \to X$ is called a (locally maximal) hyperbolic diffeomorphism if:*

1. *there exists a continuous splitting $T_X M = E^s \oplus E^u$ and $C > 0$ and $0 < \lambda < 1$ such that*
$$\|Df^n|E^s\| \leq C\lambda^n \text{ and } \|Df^{-n}|E^u\| \leq C\lambda^n$$
for $n \geq 0$;

2. *there exists an open neighbourhood U of X such that $X = \cap_{n\in\mathbb{Z}} f^n(U)$;*

The unstable manifold theory due to Hirsch and Pugh in [5] shows that uniformly hyperbolic systems are in fact Smale spaces.

1.1.2 Subshifts of finite type

Let A be a $k \times k$ matrix with entries consisting of zeros and ones and let $A(i, j)$ denote the (i, j)th entry of A.

Definition 1.4. *We define the one and two sided shift space Σ_A^+ and Σ_A, respectively, by*

$$\Sigma_A^+ = \{\underline{x} = (x_n)_0^\infty \in \{1, \dots, k\}^{\mathbb{Z}^+} \,:\, A(x_n, x_{n+1}) = 1, n \in \mathbb{Z}^+\},$$
$$\Sigma_A = \{\underline{x} = (x_n)_{-\infty}^\infty \in \{1, \dots, k\}^{\mathbb{Z}} \,:\, A(x_n, x_{n+1}) = 1, n \in \mathbb{Z}\}.$$

Define the two (one) sided shift map, $\sigma : \Sigma_A \to \Sigma_A$ ($\sigma : \Sigma_A^+ \to \Sigma_A^+$) by $\sigma(x_n) = x_{n+1}$.

When $A(i,j)=1$ for all $i,j \in \{1,\dots,k\}$, these are called full shifts.

For $\lambda \in (0,1)$ there is a metric on Σ_A defined by $d(\underline{x},\underline{y}) = \lambda^k$ where $k = \inf\{|n| : x_n \neq y_n\}$ (and on Σ_A^+ there is a metric $d(\underline{x},\underline{y}) = \lambda^k$ where $k = \inf\{n : x_n \neq y_n\}$).

Definition 1.5. *For each $m,n \in \mathbb{N}$, we denote by*

$$[i_{-m},\dots,i_n] = \{\underline{x} = (x_n)_{-\infty}^{\infty} \in \Sigma_A : x_{-m} = i_{-m},\dots,x_n = i_n\}$$

a cylinder in Σ_A where $i_{-m},\cdots,i_n \in \{1,\cdots,k\}$ and $A(i_j,i_{j+1}) = 1$ for $-m \leq j \leq n-1$. Similarly, for each $n \in \mathbb{N}$, we denote by

$$[i_0,\dots,i_n] = \{\underline{x} = (x_n)_0^{\infty} \in \Sigma_A^+ : x_0 = i_0,\dots,x_n = i_n\}$$

a cylinder *in Σ_A^+ of length n where $i_0,\cdots,i_n \in \{1,\cdots,k\}$ and $A(i_j,i_{j+1}) = 1$ for $0 \leq j \leq n-1$.*

For two sequences, $\underline{x},\underline{y} \in \Sigma_A$ such that $x_0 = y_0$, the product map $[\cdot,\cdot]$ is given by $[\underline{x},\underline{y}] = (\dots,y_{-2},y_{-1},x_0,x_1,x_2,\dots)$.

For the subshift of finite type an unstable manifold through $\underline{x} \in \Sigma_A$ is simply the elements of Σ_A which have the same past as $\underline{x}$. We will denote $\underline{x}^-$ by the sequences that have the past, $(x_n)_{-\infty}^0$ i.e., the terms agree for indices $n \leq 0$. Stable manifolds are similarly defined with a fixed future i.e., the terms agree for indices $n \geq 0$.

2 Constructing equilibrium states

We begin by recalling the following standard definition.

Definition 2.1. *Given a continuous function $G : X \to \mathbb{R}$*

$$P(G) := \sup\left\{h(\mu,f) + \int G d\mu : \mu = f\text{-invariant probability}\right\}$$

is the pressure *of G, where $h(\mu,f)$ denotes the entropy of μ. Any measure realizing this supremum is called an* equilibrium state *for G.*

For Smale spaces every continuous potential G has at least one equilibrium state [13]. If G is Hölder continuous then the equilibrium state is unique [11].

We require the following notion of a conditional Gibbs property.

Definition 2.2. *For $y \in W_\delta^u(x)$, $0 < \epsilon < \delta$ and $n \in \mathbb{N}$ we define the* unstable Bowen ball *of radius ϵ by*

$$B_{d_u}(y,n,\epsilon) = \{z \in W^u(x) : d_u(f^i z, f^i y) < \epsilon \text{ for } 0 \leq i \leq n-1\}$$

be the Bowen ball around $y \in W_\delta^u(x)$ in the induced unstable metric d_u on $W_\delta^u(x)$.

Let μ^u be a measure supported on a piece of unstable manifold centred at x. We say that it has the conditional Gibbs property for G if for every small $\epsilon > 0$ there is a constant $K = K(\epsilon) > 0$ such that, for every $y \in W^u_\delta(x)$ and $n \in \mathbb{N}$ we have,

$$K^{-1} \leq \frac{\mu^u(B_{d_u}(y, n, \epsilon))}{e^{S_nG(y) - nP(G)}} \leq K.$$

We write $\mu^u = \mu^u_G$ if this conditional property holds. We may also write $\mu^u_{x,G}$ when we need to emphasis the measure is supported on a piece of unstable manifold centred at x.

Example 2.3. *Let $f : X \to X$ be a uniformly hyperbolic diffeomorphism. It is shown by Leplaideur [6] that equilibrium states for Hölder continuous potentials have a local product structure (see Definition 2.13 [3]). Therefore, equilibrium states for Hölder potentials have conditional measures on unstable manifolds that satisfy the conditional Gibbs property.*

Example 2.4. *Consider the two sided subshift of finite type $\sigma_A : \Sigma_A \to \Sigma_A$. Bowen [2] shows we can replace G_1 acting on Σ_A by a homologous G_1' which only depends on $(x_i)_{i=0}^\infty$ without any change to the Gibbs measure μ_{G_1}. We can then define a continuous function G_1^+ on Σ_A^+ to be equal to G_1'. The Gibbs measure for G_1^+ on the one sided subshift of finite type restricted to the sequences, $\underline{y} \in \Sigma_A^+$ such that $x_0 = y_0$ and $A(x_0, y_1) = 1$ has the conditional Gibbs property for G_1.*

2.1 The main construction

We are now ready to state the main construction of this section.

Theorem 2.5. *Let $(X, d, f, [\cdot, \cdot])$ be a topologically mixing Smale space. Let $G_1 : X \to \mathbb{R}$ be a Hölder continuous potential and let $G_2 : X \to \mathbb{R}$ be a continuous potential. For μ_{G_1} a.e. $x \in X$ and $\delta > 0$ small, we can define a family of measures supported on $W^u_\delta(x)$ by*

$$\lambda_{n,G_2-G_1}(A) = \frac{\int_{W^u_\delta(x)\cap A} e^{S_nG_2(y)-S_nG_1(y)} d\mu^u_{G_1}(y)}{\int_{W^u_\delta(x)} e^{S_nG_2(y)-S_nG_1(y)} d\mu^u_{G_1}(y)}, \quad n \geq 1, \tag{2.1}$$

where $A \subset X$ a measurable set. Then the measures

$$\mu_{n,G_2-G_1} = \frac{1}{n} \sum_{i=0}^{n-1} f^i_* \lambda_{n,G_2-G_1}, \quad n \geq 1, \tag{2.2}$$

supported on $f^n W^u_\delta(x)$ have weak star accumulation points which are equilibrium measures for G_2. Moreover, when G_2 is a Hölder function then μ_{n,G_2-G_1} converges to the unique equilibrium state μ_{G_2}.

Example 2.6. *In the case where $f : X \to X$ is a mixing hyperbolic attractor and $G_1 = \varphi^{geo}$ is the geometric potential then μ_{G_1} is the SRB measure, $\mu^u_{G_1}$ is the induced volume on $W^u_\delta(x)$ and Theorem 2.5 recovers Theorem 1.2 in [9].*

Next we will see an illuminating example illustrating Theorem 2.5. We consider the full shift on two symbols and begin with a constant potential corresponding to the $(\frac{1}{2},\frac{1}{2})$-Bernoulli measure. Fixing $p\in(0,1)$ $(\neq 1/2)$ we show with an explicit calculation of μ_{n,G_2-G_1} that using Theorem 2.5 we can transform the $(\frac{1}{2},\frac{1}{2})$-Bernoulli measure into the $(p,1-p)$-Bernoulli measure.

The $(\frac{1}{2},\frac{1}{2})$-Bernoulli measure is a very well understood equilibrium state for the two sided subshift of finite type. Theorem 2.5 can be used to explicitly calculate the measure of cylinders for the equilibrium state of any other Hölder potentials.

Example 2.7. *Let $X=\{0,1\}^{\mathbb{Z}}$ and let $\sigma: X\to X$ be the full shift on two symbols given by $\sigma(x_n)_{n\in\mathbb{Z}}=(x_{n+1})_{n\in\mathbb{Z}}$. Let $G_1: X\to\mathbb{R}$ be the constant function $G_1=-\log 2$, then the associated unique equilibrium measure is the Bernoulli measure $\mu_{G_1}=\left(\frac{1}{2},\frac{1}{2}\right)^{\mathbb{Z}}$. For $p\in(0,1)$ not equal to $1/2$, we shall consider the locally constant potential, $G_2: X\to\mathbb{R}$ defined at $x=(x_n)_{n=-\infty}^{+\infty}$ by*

$$G_2(x)=\begin{cases}\log p & x_0=0\\ \log(1-p) & x_0=1.\end{cases}$$

Then the unique equilibrium measure associated to G_2 is the Bernoulli measure $\mu_{G_2}=(p,1-p)^{\mathbb{Z}}$. Given any point $x=(x_n)_{n=-\infty}^{\infty}\in X$,

$$W^u_\delta(x)=\{y=(y_n)_{n=-\infty}^{\infty} : y_i=x_i \text{ for } i\leq -1\}$$

and we can identify $W^u_\delta(x)=\{x_-\}\times X^+$ where $X^+=\{0,1\}^{\mathbb{Z}_+}$ and $x_-=(x_n)_{n=-\infty}^{-1}$. The conditional measure $\mu^u_{G_1}$ on X corresponds to the Bernoulli measure $\left(\frac{1}{2},\frac{1}{2}\right)^{\mathbb{Z}_+}$ on X^+. We can explicitly write

$$\begin{aligned} e^{S_nG_2(y)-S_nG_1(y)} &= \frac{1}{2^n}p^{\#\{0\leq i\leq n-1\,:\,y_i=0\}}(1-p)^{\#\{0\leq i\leq n-1\,:\,y_i=1\}}\\ &= \frac{\mu_{G_2}[y_0,\dots,y_{n-1}]}{\mu^u_{G_1}[y_0,\dots,y_{n-1}]}. \end{aligned}\tag{2.3}$$

where we recall, $[y_0,\cdots,y_{n-1}]=\{(z_k)_{k=-\infty}^{\infty} : z_i=y_i \text{ for } 0\leq i\leq n-1\}$. By the definition of λ_n we have that

$$\sigma^i_*\lambda_n(A)=\frac{\int_{\sigma^{-i}A\cap W^u_\delta(x)} e^{S_nG_2(y)-S_nG_1(y)}d\mu^u_{G_1}(y)}{\int_{W^u_\delta(x)} e^{S_nG_2(y)-S_nG_1(y)}d\mu^u_{G_1}(y)}\tag{2.4}$$

where we have the simplifications, $P(G_1)=P(G_2)=0$ and

$$\begin{aligned}\int_{W^u_\delta(x)} e^{S_nG_2(y)-S_nG_1(y)}d\mu^u_{G_1}(y) &= \sum_{[y_0,\dots,y_{n-1}]}\mu^u_{G_1}([y_0,\dots,y_{n-1}])\frac{\mu_{G_2}[y_0,\dots,y_{n-1}])}{\mu^u_{G_1}[y_0,\dots,y_{n-1}])}\\ &= 1.\end{aligned}$$

Consider the set $A = [z_{-M}, \dots z_{-1}, z_0, z_1, \dots, z_N]$, for $M, N \in \mathbb{N}$. We will calculate $\sigma_*^i \lambda_n(A)$ for $n \in \mathbb{N}$ and $n \gg N + M$. Notice that for $i \geq M$,

$$\sigma^{-i}(A) = \bigcup_{[y_0, \dots, y_{i-M-1}]} [y_0, \dots, y_{i-M-1}, z_{-M}, \dots, z_N].$$

We have that $S_n G_1$ and $S_n G_2$ are constant on $[y_0, \dots, y_{n-1}]$ so we can rewrite the integral in equation (2.4) as a sum over the cylinders of the same length. For ease of reading, when the intersection is non-empty, let

$$\begin{aligned} \sigma^{-i}(A) \cap [y_0, \dots, y_{n-1}] &= [y_0, \dots, y_{i-M-1}, z_{-M}, \dots z_N, y_{i+N+1}, \dots, y_{n-1}], \\ &=: \sigma^{-i}_{y_0, \dots, y_{n-1}}(A). \end{aligned}$$

for $M \leq i < n - N$, We can now simplify equation (2.4) using equation (2.3) as follows.

$$\begin{aligned} \sigma_*^i \lambda_n(A) &= \sum_{\sigma^{-i}_{y_0, \dots, y_{n-1}}(A)} \mu^u_{G_1}(\sigma^{-i}_{y_0, \dots, y_{n-1}}(A)) \frac{\mu_{G_2}(\sigma^{-i}_{y_0, \dots, y_{n-1}}(A))}{\mu^u_{G_1}(\sigma^{-i}_{y_0, \dots, y_{n-1}}(A))} \\ &= \sum_{\sigma^{-i}_{y_0, \dots, y_{n-1}}(A)} \mu_{G_2}(\sigma^{-i}_{y_0, \dots, y_{n-1}}(A)) \\ &= \sum_{[y_0, \dots, y_{n-1}]} \mu_{G_2}(\sigma^{-i}(A) \cap [y_0, \dots, y_{n-1}]) \\ &= \mu_{G_2}(A). \end{aligned}$$

Therefore,

$$\begin{aligned} \mu_n(A) &= \frac{1}{n} \sum_{i=0}^{n-1} \sigma_*^i \lambda_n(A) \\ &= \frac{1}{n} \sum_{i=0}^{M-1} \sigma_*^i \lambda_n(A) + \frac{1}{n} \sum_{i=M}^{n-N-1} \sigma_*^i \lambda_n(A) + \frac{1}{n} \sum_{i=n-N}^{n-1} \sigma_*^i \lambda_n(A) \\ &= \frac{1}{n} \sum_{i=0}^{M-1} \sigma_*^i \lambda_n(A) + \frac{n - (N+M)}{n} \mu_{G_2}(A) + \frac{1}{n} \sum_{i=n-N}^{n-1} \sigma_*^i \lambda_n(A) \\ &\xrightarrow{n \to \infty} \mu_{G_2}(A). \end{aligned}$$

This is consistent with Theorem 2.5, we have practised alchemy, transforming μ_{G_1} into μ_{G_2}.

This example also hints at an interesting feature. In the construction of the SRB measure for hyperbolic attractors [10] there is no need to average the pushforwards of the induced volume on $W^u_\delta(x)$. Example 2.7 shows that even for the full shift on two symbols, there is a continuous potential such that $\sigma_*^n \lambda_n$ does not converge to the required equilibrium state. This can be seen with the following calculation.

$$\begin{aligned}
\sigma_*^n \lambda_n(A) &= \sum_{[y_0,\dots,y_{n-M-1}]} \mu_{G_1}^u([y_0,\dots,y_{n-M-1},z_{-M},\cdots,z_N]) \frac{\mu_{G_2}([y_0,\dots,y_{n-M-1}z_{-M},\cdots,z_N])}{\mu_{G_1}^u([y_0,\dots,y_{n-M-1}z_{-M},\cdots,z_N])} \\
&= \mu_{G_1}^u([z_0,\dots,z_N])\mu_{G_2}([z_{-M},\dots,z_{-1}]) \\
&\neq \mu_{G_2}(A).
\end{aligned}$$

It is an interesting question to ask whether the averaging in (2.2) is required in the setting of uniformly hyperbolic attractors. Answering this would have important consequences for the rate of convergence to the equilibrium state for G_2.

3 Growth of unstable manifolds for Smale spaces

The proof of Theorem 2.5 relies on the following growth rate result of unstable manifolds.

Lemma 3.1. *Let $(X,d,f,[\cdot,\cdot])$ be a mixing Smale space. Let $G_1 : X \to \mathbb{R}$ Hölder and $G_2 : X \to \mathbb{R}$ continuous. For a.e.(μ_{G_1}) $x \in X$ and $\delta > 0$ sufficiently small,*

$$P(G_2) - P(G_1) = \lim_{n\to\infty} \frac{1}{n} \log \int_{W_\delta^u(x)} e^{S_n(G_2-G_1)(y)} d\mu_{G_1}^u(y).$$

Before we prove Lemma 3.1, we recall the following simple property.

Lemma 3.2. *Let $G : X \to \mathbb{R}$ be a continuous potential. For any $\tau > 0$, there is an $\epsilon > 0$ small enough such that, for any $x \in X$ and $n \in \mathbb{N}$, if $d_n(x,y) < \epsilon$ then*

$$|S_nG(x) - S_nG(y)| \leq n\tau. \tag{3.1}$$

In the proof of Lemma 3.1 we will use Bowen's definition of the pressure (see for example [13]) using spanning and separated sets which is equivalent to the definition given in Definition 2.1 by the variational principle [12].

Proof of Lemma 3.1. To get an upper bound on the growth rate in Lemma 3.1 we proceed as follows. Given $\epsilon > 0$ and $n \geq 1$, we want to create an $(n,\kappa\epsilon)$-separated set for some $\kappa \in (0,1)$ independent of n and ϵ. To this end we can choose a maximal number of points $y_i \in f^n W_\delta^u(x)$ $(i = 1,\cdots,N = N(n,\epsilon))$ so that $d_u(y_i,y_j) > \epsilon/2$ whenever $i \neq j$ (where d_u is the induced distance on $f^n W_\delta^u(x)$). By the definition of the Smale space, the map $f^n : W_\delta^u(x) \to f^n W_\delta^u(x)$ is locally distance expanding and thus, in particular, the points $x_i = f^{-n} y_i$ $(i = 1,\cdots,N = N(n,\epsilon))$ form an $(n,\kappa\epsilon)$-separated set.

Now we have constructed $\{x_i\}$, we can relate these points to an integral. Let $B_{d_u}(y,n,\epsilon)$ denote the Bowen ball contained within the unstable manifold with respect to the induced metric d_u, then

$$\begin{aligned}
\sum_{i=1}^N e^{S_nG_2(x_i)} &= \sum_{i=1}^N \int_{B_{d_u}(x_i,n,\epsilon)} e^{S_nG_2(x_i)}\mu_{G_1}^u(B_{d_u}(x_i,n,\epsilon))^{-1}d\mu_{G_1}^u(y),\\
&\geq e^{-n\tau}\sum_{i=1}^N \int_{B_{d_u}(x_i,n,\epsilon)} e^{S_nG_2(y)}\mu_{G_1}^u(B_{d_u}(x_i,n,\epsilon))^{-1}d\mu_{G_1}^u(y),\\
&\geq e^{-n\tau+nP(G_1)}K^{-1}\sum_{i=1}^N \int_{B_{d_u}(x_i,n,\epsilon)} e^{S_nG_2(y)-S_nG_1(x_i)}d\mu_{G_1}^u(y),\\
&\geq e^{-2n\tau+nP(G_1)}K^{-1}\sum_{i=1}^N \int_{B_{d_u}(x_i,n,\epsilon)} e^{S_nG_2(y)-S_nG_1(y)}d\mu_{G_1}^u(y),\\
&\geq e^{-2n\tau+nP(G_1)}K^{-1}\int_{W_\delta^u(x)} e^{S_nG_2(y)-S_nG_1(y)}d\mu_{G_1}^u(y).
\end{aligned}$$

In particular: Line 2 uses Lemma 3.2 for G_2; Line 3 uses the upper bound of the conditional Gibbs property of $\mu_{G_1}^u$; Line 4 uses Lemma 3.2 for G_1; and Line 5 follows from the maximality of $\{y_i\}$, in particular $W_\delta^u(x) \subset \cup_i B_{d_u}(x_i,n,\epsilon)$. Then letting $K(n) = e^{-2n\tau}K^{-1}$ gives

$$\frac{1}{n}\log Z_{1,G_2}(n,\kappa\epsilon) \geq \frac{1}{n}\log\left(K(n)\int_{W_\delta^u(x)} e^{S_nG_2(y)-S_nG_1(y)+nP(G_1)}d\mu_{G_1}^u(y)\right).$$

Taking a limit as $n\to\infty$ and $\epsilon\to 0$,

$$P(G_2) \geq -2\tau + \lim_{n\to\infty}\frac{1}{n}\log\int_{W_\delta^u(x)} e^{S_nG_2(y)-S_nG_1(y)+nP(G_1)}d\mu_{G_1}^u(y).$$

Since $\tau > 0$ is arbitrarily small then,

$$P(G_2) \geq \lim_{n\to\infty}\frac{1}{n}\log\int_{W_\delta^u(x)} e^{S_nG_2(y)-S_nG_1(y)+nP(G_1)}d\mu_{G_1}^u(y).$$

Before starting on the proof of the lower bound we present a simple result.

Lemma 3.3. *For any $\epsilon > 0$ there exists an $m > 0$ such that $f^mW_\delta^u(x)$ is ϵ-dense in X. In particular, we can assume that $X = \cup_{y\in f^mW_\delta^u(x)}W_\epsilon^s(y)$.*

Proof. This is an immediate consequence of the topological mixing assumption and the local product structure for Smale spaces. □

To get a lower bound on the growth rate in Proposition 3.1, given $\epsilon > 0$ and $n \geq 1$ we want to construct a well chosen $(n,2\epsilon)$-spanning set. We begin by choosing a suitable covering of $f^{n+m}W_\delta^u(x)$ by ϵ-balls

$$B_{d_u}(y_i,\epsilon) : i = 1,\cdots,N := N(n+m,\epsilon)$$

contained within the unstable manifold with respect to the induced metric d_u and let $A_\epsilon := \{y \in f^{n+m}W^u_\delta(x) : \nexists z \in W^u(f^{n+m}x)\backslash f^{n+m}W^u_\delta(x)$ with $d_u(y,z) < \epsilon/2\}$. We can choose a maximal set $S = \{y_1, \cdots, y_{N(n+m,\epsilon)}\}$ with the additional property that $d_u(y_i, y_j) > \epsilon/2$ for $i \neq j$ and $y_i \in A_\epsilon$. By our choice of S we have that

$$A_\epsilon \subset \bigcup_{i=1}^{N(n+m,\epsilon)} B_{d_u}(y_i, \epsilon/2).$$

By the triangle inequality we have that

$$f^{n+m}W^u_\delta(x) \subset \bigcup_{i=1}^{N(n+m,\epsilon)} B_{d_u}(y_i, \epsilon).$$

Since $B_{d_u}(f^{-(n+m)}(y_i), n+m+1, \frac{\epsilon}{4}) \cap B_{d_u}(f^{-(n+m)}(y_j), n+m+1, \frac{\epsilon}{4}) = \emptyset$ for $i \neq j$, then the disjoint union satisfies,

$$\bigcup_{i=1}^{N(n+m,\epsilon)} B_{d_u}(f^{-(n+m)}(y_i), n+m+1, \epsilon/4) \subset W^u_\delta(x). \tag{3.2}$$

We again use the property that $f^n : f^m W^u_\delta(x) \to f^{n+m}W^u_\delta(x)$ locally expands distance along the unstable manifold. In particular, this means that the preimages $x_i := f^{-n}y_i \in f^m(W^u_\delta(x))$ $(i = 1, \cdots, N)$ form an $(n, 2\epsilon)$-spanning set. [To see this we use Lemma 3.3, for any point $z \in X$ we can choose a point $y \in f^m(W^u_\delta(x))$ with $z \in W^s_\epsilon(y)$ and observe that $d(f^j z, f^j y) < \epsilon$ for $0 \leq j \leq n$.] We can then choose an x_i such that $d_n(y, x_i) < \epsilon$ since f^n is locally expanding along unstable manifolds. In particular, by the triangle inequality

$$d(f^j z, f^j x_i) \leq d(f^j z, f^j y) + d(f^j y, f^j x_i) < 2\epsilon$$

for $0 \leq j \leq n-1$.

We will now use the construction of the points $\{x_i\}$ to get the required lower bound. We first require the following simple inequality

$$\begin{aligned} e^{S_n G_2(x_i)} &= e^{S_{n+m}G_2(f^{-m}(x_i)) - S_m G_2(f^{-m}(x_i))} \\ &\leq e^{S_{n+m}G_2(f^{-m}(x_i)) + m||G_2||_\infty}. \end{aligned}$$

For ease of notation, set $\overline{B}(x_i) = B_{d_u}(f^{-m}(x_i), n+m+1, \frac{\epsilon}{4})$. Therefore,

$$\begin{aligned}
\sum_{i=1}^{N} e^{S_n G_2(x_i)} &= \sum_{i=1}^{N} \int_{\overline{B}(x_i)} e^{S_n G_2(x_i)} \mu^u_{G_1}(\overline{B}(x_i))^{-1} d\mu^u_{G_1}(y) \\
&\leq e^{m||G_2||_\infty} \sum_{i=1}^{N} \int_{\overline{B}(x_i)} e^{S_{n+m} G_2(f^{-m}(x_i))} \mu^u_{G_1}(\overline{B}(x_i))^{-1} d\mu^u_{G_1}(y) \\
&\leq e^{m||G_2||_\infty + (n+m)\tau} \sum_{i=1}^{N} \int_{\overline{B}(x_i)} e^{S_{n+m} G_2(y)} \mu^u_{G_1}(\overline{B}(x_i))^{-1} d\mu^u_{G_1}(y), \\
&\leq e^{m||G_2||_\infty + (n+m)\tau + (n+m+1)P(G_1)} K \sum_{i=1}^{N} \int_{\overline{B}(x_i)} e^{S_{n+m} G_2(y) - S_{n+m+1} G_1(f^{-m}(x_i))} d\mu^u_{G_1}(y) \\
&\leq e^{m||G_2||_\infty + 2(n+m+1)\tau + (n+m+1)P(G_1)} K \sum_{i=1}^{N} \int_{\overline{B}(x_i)} e^{S_{n+m} G_2(y) - S_{n+m+1} G_1(y)} d\mu^u_{G_1}(y).
\end{aligned}$$

Moreover, by (3.2) we can bound

$$\sum_{i=1}^{N} \int_{\overline{B}(x_i)} e^{S_{n+m} G_2(y) - S_{n+m} G_1(y)} d\mu^u_{G_1}(y) \leq \int_{W^u_\delta(x)} e^{S_{n+m} G_2(y) - S_{n+m} G_1(y)} d\mu^u_{G_1}(y).$$

Letting $L(n) = e^{m||G_2||_\infty + 2(n+m)\tau + P(G_1) + ||G_1||_\infty} K$, we have

$$Z_{0,G_2}(n, 2\epsilon) \leq L(n) \int_{W^u_\delta(x)} e^{S_{n+m} G_2(y) - S_{n+m} G_1(y) + (n+m)P(G_1)} d\mu^u_{G_1}(y)$$

and thus

$$P(G_2) \leq 2\tau + \lim_{n\to\infty} \frac{1}{n+m} \log \int_{W^u_\delta(x)} e^{S_{n+m} G_2(y) - S_{n+m} G_1(y) + (n+m)P(G_1)} d\mu^u_{G_1}(y).$$

Again $\tau > 0$ can be chosen arbitrarily small and so

$$P(G_2) \leq \lim_{n\to\infty} \frac{1}{n} \log \int_{W^u_\delta(x)} e^{S_n G_2(y) - S_n G_1(y) + nP(G_1)} d\mu^u_{G_1}(y).$$

This concludes the proof. □

4 Proof of Theorem 2.5

In this section we will complete the proof of Theorem 2.5. The proof follows the general lines of the proof of Theorem 1.2 in [9].

Proof. We begin by observing that If we were to replace the potential G_2 by $G_2 + P(G_1)$ then the measures $\lambda_{n,G_2-G_1} = \lambda_{n,G_2-G_1+P(G_1)}$. Thus when we write λ_{n,G_2-G_1} we are really considering $\lambda_{n,G_2-G_1+P(G_1)}$.

By Alaoglu's theorem on the weak star compactness of the space of probability measures, the measures μ_{n,G_2-G_1} have a weak star convergent subsequence to some measure μ. Moreover, for any continuous $F : X \to \mathbb{R}$ we can bound

$$\begin{aligned}
&\left| \int F d\mu_{n,G_2-G_1} - \int F \circ f d\mu_{n,G_2-G_1} \right| \\
&= \left| \frac{1}{n} \sum_{k=0}^{n-1} \int F \circ f^k d\lambda_{n,G_2-G_1} - \frac{1}{n} \sum_{k=0}^{n-1} \int F \circ f^{k+1} d\lambda_{n,G_2-G_1} \right| \\
&\leq \frac{2\|F\|_\infty}{n} \to 0 \text{ as } n \to +\infty
\end{aligned}$$

and, in particular, one easily sees that μ is f-invariant.

For convenience, we denote

$$Z_n^{G_2,G_1} = \int_{W_\delta^u(x)} e^{S_nG_2(y) - S_nG_1(y) + nP(G_1)} d\mu_{G_1}^u(y)$$

and for $A \subset X$ let,

$$K_{n,A}^{G_2,G_1} = \int_{W_\delta^u(x) \cap A} e^{S_nG_2(y) - S_nG_1(y) + nP(G_1)} d\mu_{G_1}^u(y).$$

Definition 4.1. *Given a finite partition $\mathcal{P} = \{P_i\}_{i=1}^N$ we say that it has size $\epsilon > 0$ if $\sup_i \{\mathrm{diam}(P_i)\} < \epsilon$.*

By Lemma 3.2, for any $\tau > 0$ there is an $\epsilon > 0$ small enough, such that if we choose a partition $\mathcal{P}$ of size $\epsilon > 0$, then for all $x, y \in A \in \bigvee_{i=0}^{n-1} f^{-i}\mathcal{P}$, we have,

$$|S_nG_k(x) - S_nG_k(y)| \leq n\tau \tag{4.1}$$

for $k = 1, 2$.

Choosing a partition of size $\epsilon > 0$, for each element of the refined partition we can choose an $x_A \in A \in \bigvee_{i=0}^{n-1} f^{-i}\mathcal{P}$. We now find a convenient form for the integral $\int_X G_2 d\mu_{n,G_2-G_1}$. First we can write

$$\int_{W_\delta^u(x)} G_2(y) d\lambda_{n,G_2-G_1}(y) = \frac{e^{nP(G_1)}}{Z_n^{G_2,G_1}} \int_{W_\delta^u(x)} e^{S_n(G_2-G_1)(y)} G_2(y) d\mu_{G_1}^u(y)$$

and then

$$\int_{f^i(W_\delta^u(x))} G_2(y) df_*^i\lambda_{n,G_2-G_1}(y) = \frac{e^{nP(G_1)}}{Z_n^{G_2,G_1}} \int_{W_\delta^u(x)} e^{S_n(G_2-G_1)(y)} G_2(f^i(y)) d\mu_{G_1}^u(y).$$

Recalling the definition of μ_{n,G_2-G_1} we can write

$$\begin{aligned}
\int_X G_2(y)d\mu_{n,G_2-G_1}(y) &= \frac{e^{nP(G_1)}}{nZ_n^{G_2,G_1}}\int_{W^u_\delta(x)} e^{S_n(G_2-G_1)(y)}S_nG_2(y)d\mu^u_{G_1}(y)\\
&= \frac{e^{nP(G_1)}}{nZ_n^{G_2,G_1}}\sum_{A\in\bigvee_{i=0}^{n-1}f^{-i}\mathcal{P}}\int_{W^u_\delta(x)\cap A} e^{S_n(G_2-G_1)(y)}S_nG_2(y)d\mu^u_{G_1}(y)\\
&\geq \frac{e^{nP(G_1)}}{nZ_n^{G_2,G_1}}\sum_{A\in\bigvee_{i=0}^{n-1}f^{-i}\mathcal{P}}\Big(S_nG_2(x_A)-n\tau\Big)\int_{W^u_\delta(x)\cap A} e^{S_n(G_2-G_1)(y)}d\mu^u_{G_1}(y)\\
&= -\tau+\frac{1}{n}\sum_{A\in\bigvee_{i=0}^{n-1}f^{-i}\mathcal{P}}\frac{K_{n,A}^{G_2,G_1}}{Z_n^{G_2,G_1}}S_nG_2(x_A). \qquad (4.2)
\end{aligned}$$

We next consider the entropy of μ_{n,G_2-G_1}. For $A\in\bigvee_{i=0}^{n-1}T^{-i}\mathcal{P}$, consider

$$\begin{aligned}
\log\int_{W^u_\delta(x)\cap A} e^{S_n(G_2-G_1)(y)}d\mu^u_{G_1}(y) &\leq \log\left(e^{2n\tau}\int_{W^u_\delta(x)\cap A} e^{S_n(G_2-G_1)(x_A)}d\mu_{G_1}(y)\right)\\
&= 2n\tau+S_n(G_2-G_1)(x_A)+\log\mu^u_{G_1}(W^u_\delta(x)\cap A).
\end{aligned}$$

Since $\mathcal{P}$ has size ϵ then $W^u_\delta(x)\cap A\subset B_{d_u}(x_A,n,\epsilon)$. Using the conditional Gibbs property of $\mu^u_{G_1}$ we have,

$$\mu^u_{G_1}(W^u_\delta(x)\cap A)\leq Ke^{S_nG_1(x_A)-nP(G_1)}.$$

In particular, this shows

$$\begin{aligned}
\log K_{n,A}^{G_2,G_1} &\leq nP(G_1)+2n\tau+S_n(G_2-G_1)(x_A)+\log K+S_nG_1(x_A)-nP(G_1)\\
&= S_nG_2(x_A)+\log K+2n\tau, \qquad (4.3)
\end{aligned}$$

where $K>0$ is independent of n and A. Working from the definition of the entropy we can write

$$\begin{aligned}
H_{\lambda_{n,G_2-G_1}}\left(\bigvee_{r=0}^{n-1}f^{-h}\mathcal{P}\right) &= -\sum_{A\in\bigvee_{r=0}^{n-1}f^{-h}\mathcal{P}}\lambda_{n,G_2-G_1}(A)\log\lambda_{n,G_2-G_1}(A)\\
&= -\sum_{A\in\bigvee_{r=0}^{n-1}f^{-h}\mathcal{P}}\frac{K_{n,A}^{G_2,G_1}}{Z_n^{G_2,G_1}}\log\frac{K_{n,A}^{G_2,G_1}}{Z_n^{G_2,G_1}}\\
&= \log Z_n^{G_2,G_1}-\sum_{A\in\bigvee_{r=0}^{n-1}f^{-h}\mathcal{P}}\frac{K_{n,A}^{G_2,G_1}}{Z_n^{G_2,G_1}}\log K_{n,A}^{G_2,G_1},
\end{aligned}$$

where the last equality uses that, by definition $\sum_{A\in\bigvee_{r=0}^{n-1}f^{-h}\mathcal{P}}K_{n,A}^{G_2,G_1}=Z_n^{G_2,G_1}$.

Using equation (4.3) we have the lower bound

$$H_{\lambda_{n,G_2-G_1}}\left(\bigvee_{r=0}^{n-1} f^{-h}\mathcal{P}\right) \geq Z_n^{G_2,G_1} - \sum_{A\in\bigvee_{r=0}^{n-1} f^{-h}\mathcal{P}} \frac{K_{n,A}^{G_2,G_1}}{Z_n^{G_2,G_1}}\Big(S_nG_2(x_A) + \log K + 2n\tau\Big)$$

$$= Z_n^{G_2,G_1} - \log K - 2n\tau - \sum_{A\in\bigvee_{r=0}^{n-1} f^{-h}\mathcal{P}} \frac{K_{n,A}^{G_2,G_1}}{Z_n^{G_2,G_1}} S_nG_2(x_A). \quad (4.4)$$

Putting together (4.2) and (4.4),

$$\begin{aligned} &H_{\lambda_{n,G_2-G_1}}\left(\bigvee_{r=0}^{n-1} f^{-h}\mathcal{P}\right) + n\int_X G_2(y)d\mu_{n,G_2-G_1}(y) \\ &\geq Z_n^{G_2,G_1} - \log K - 2n\tau - \sum_{A\in\bigvee_{r=0}^{n-1} f^{-h}\mathcal{P}} \frac{K_{n,A}^{G_2,G_1}}{Z_n^{G_2,G_1}} S_nG_2(x_A) \\ &\quad - n\tau + \sum_{A\in\bigvee_{i=0}^{n-1} f^{-i}\mathcal{P}} \frac{K_{n,A}^{G_2,G_1}}{Z_n^{G_2,G_1}} S_nG_2(x_A) \\ &= Z_n^{G_2,G_1} - \log K - 3n\tau. \end{aligned}$$

We can now use this and an entropy estimate due to Misiurewicz [7] (stated in Lemma 4.5 [8]) to write

$$\begin{aligned} q\log Z_n^{G_2,G_1} - qn\int_X Gd\mu_{n,G_2-G_1} - q(\log K + 3n\tau) &\leq qH_{\lambda_{n,G_2-G_1}}\left(\bigvee_{r=0}^{n-1} f^{-h}\mathcal{P}\right) \\ &\leq nH_{\mu_{n,G_2-G_1}}\left(\bigvee_{i=0}^{q-1} f^{-i}\mathcal{P}\right) + 2q^2\log \mathrm{Card}(\mathcal{P}), \end{aligned}$$

which we can rearrange to get,

$$\frac{\log Z_n^{G_2,G_1}}{n} - \frac{\log K + 3n\tau}{n} - \frac{2q\log\mathrm{Card}(\mathcal{P})}{n} \leq \frac{H_{\mu_{n,G_2-G_1}}\left(\bigvee_{i=0}^{q-1} f^{-i}\mathcal{P}\right)}{q} + \int_X G_2 d\mu_{n,G_2-G_1}.$$

Letting $n_k \to +\infty$,

$$\begin{aligned} P(G_2) &= \lim_{k\to\infty} \frac{\log Z_{n_k}^{G_2,G_1}}{n_k} \\ &\leq \lim_{k\to\infty}\left(\frac{H_{\mu_{n_k,G_2-G_1}}\left(\bigvee_{i=0}^{q-1} f^{-i}\mathcal{P}\right)}{q} + \int_X G_2 d\mu_{n_k,G_2-G_1}\right) + 3\tau \\ &= \frac{H_{\mu}\left(\bigvee_{i=0}^{q-1} f^{-i}\mathcal{P}\right)}{q} + \int_X G_2 d\mu + 3\tau, \end{aligned}$$

where we assume without loss of generality that the boundaries of the partition have zero measure. Letting $q \to \infty$,

$$P(G_2) \leq h_\mu(\mathcal{P}) + \int_X G_2 d\mu + 3\tau. \tag{4.5}$$

Therefore, since τ can be chosen arbitrarily and μ is an f-invariant probability measure, we see from the variational principle that the inequalities in equation (4.5) are actually equalities (since $h_\mu(\mathcal{P}) \leq h(\mu)$) and therefore we conclude that the measure μ is an equilibrium state for G_2. □

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