A dynamical approach to validated numerics

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Abstract

We describe a method, using periodic points and determinants, for giving alternative expressions for dynamical quantities (including Lyapunov exponents and Hausdorff dimension of invariant sets) associated to analytic hyperbolic systems. This leads to validated numerical estimates on their values,

1 Introduction

1.1 An overview

There are many well known and important numerical invariants in the context of dynamical systems, including for example entropy, Lyapunov exponents, and the Hausdorff dimension of invariant sets. Even in the context of uniformly hyperbolic systems, many of these invariants do not have simple explicit expressions, nor are they easy to estimate. This leads naturally to the search for useful alternative expressions for each of these quantities which, in particular, lend themselves to accurate numerical evaluation.

Our aim in this account is to describe an approach developed over some years, based on periodic points for (hyperbolic) dynamical systems. In particular, in addition to presenting a general overview we will also give a general recipe for converting this information into formulae and useful numerical estimates for a variety of characteristic values, such as those mentioned above. Perhaps the most important aspect of this side of the work is that we obtain rigorous bounds on the errors. Given the data available, the idea is to minimise the error estimate, subject to the practical constraints imposed by the computational power available. There is always some flexibility in the choices, which we can exploit in order to obtain the best error estimate, i.e., the smallest bound on the error.

Typically, the quantities that we can expect to study in this approach are those that can be expressed in terms of the thermodynamic pressure. The main ingredient in our approach is a complex function (called the determinant) which packages together the data on periodic orbits and from which can be derived estimates on the pressure, and consequently the quantities of interest.

In our presentation, no specialist knowledge is required to implement the general result. However, for completeness we give a fairly complete outline of the proof (with some details on operator theory postponed to the appendix).

*The second author collaborated with A. Katok, G. Knieper and H. Weiss over 30 years ago on the regularity of topological entropy for Anosov flows. One of the methods used there was based on the characterisation of the entropy as a pole for the dynamical zeta function, which is a complex function closely related to the determinant central to ideas we pursue here.

1.2 An historical perspective

The starting point for our approach is the important work of Grothendieck in the 1950s on nuclear operators [17, 18]. This extended the classical theory of trace class operators and Fredholm determinants [12]. Although the impact of this theory in functional analysis and operator theory was well understood, it was not until 20 years later that Ruelle employed it with great effect in ergodic theory and dynamical systems, in his application to dynamical zeta functions [40]. This viewpoint was highly influential in the work of mathematical physicists (e.g., the well known study by Cvitanović in his monumental online tome [7]).

With the advent of modern computers, an important component in ergodic theory and dynamical systems has been the focus on *explicit computation* of quantities arising in the context of dynamical systems. Typically, these approaches are based on the quantity in question being expressed in terms of an associated Ruelle transfer operator, implicitly assumed to act on some appropriate space of functions, and then a finite dimensional approximation to the operator is used to reduce this to a finite dimensional matrix problem, to be solved numerically. In contrast, the approach we will describe is to exploit *real-analytic* properties of the underlying dynamical system by introducing Ruelle transfer operators with strong spectral properties (in particular nuclearity) which allows us to exploit the earlier circle of ideas initiated by Grothendieck.

1.3 A selection of applications

By way of an appetiser to this approach, we now list a cross section of actual and potential applications to a number of different areas in mathematics. We will elaborate these later, but for the purposes of motivation we list them here.

1. In number theory, a recent breakthrough in the understanding of the Markov and Lagrange spectra $M, L \subset (0, +\infty)$ from Diophantine approximation has been brought about by the work of Matheus & Moreira who were able to estimate the dimension of the difference $M \setminus L$ of these spectra, obtaining first a lower bound [34] and then an upper bound [35]. The methods, in both cases, involved the approximation of the dimension of certain fractal sets which would be amenable to the techniques developed in [25] for proving rigorous high quality bounds on the dimension (cf. Example 4.9 (b)).

2. In the field of spectral geometry, there is a strong tradition of computing eigenvalues of the Laplacian, dating back to pioneering work of Hejhal [20] using classical methods. However, McMullen's approximations for the lowest eigenvalue for certain infinite volume hyperbolic manifolds were based on the dimension of the limit set and using this viewpoint could be accurately estimated [32].

3. Within dynamical systems, among the most widely studied numerical quantities are Lyapunov exponents, measuring the exponential instability of solutions to various problems. There is also interest in estimating Lyapunov exponents in the theory of random matrix products; in particular, in information theory this leads to the computation of entropy rates for binary symmetric channels, related to examples of hidden Markov chains [19].

We will develop this last setting in the following subsection.

1.4 Illustrative Example: Lyapunov exponents for Bernoulli interval maps

We begin by specifying a suitable class of hyperbolic transformations. In particular, we want to assume that the systems we are studying are both real analytic and uniformly hyperbolic, for

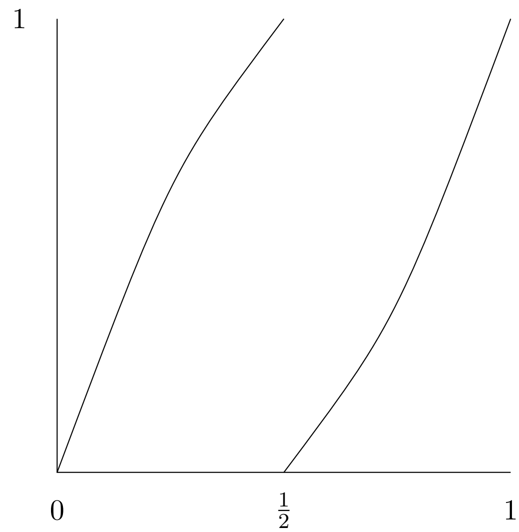

Figure 1: The graph of an expanding map of $\mathbb{R}/\mathbb{Z}$ represented on the unit interval.

example either a real analytic expanding map, or a real analytic hyperbolic diffeomophism or flow on a locally hyperbolic set.

Before formulating statements in greater generality, let us consider a very specific example of a one dimensional map. Let $X = \mathbb{R}/\mathbb{Z}$ be the unit circle and let $T : X \to X$ be a piecewise C^ω Bernoulli map of the interval which is expanding, i.e., there exists $\lambda > 1$ such that $|T'(x)| \geq \lambda$ for all x. In this setting it is well known that there is a unique T-invariant probability measure μ which is absolutely continuous with respect to Lebesgue measure (i.e., $\frac{d\mu}{dx} \in L^1(X)$) by the famous Lasota-Yorke theorem (see [28, §5.1]).

Example 1.1. *More concretely, suppose* $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ *is defined by* $T(x) = 2x + \epsilon \sin(2\pi x) \pmod 1$ *where* $|\epsilon| < \frac{1}{4\pi}$*, so in particular* $|T'(x)| > (2 - 2\pi\epsilon) > \frac{3}{2} > 1$ *for all* $x \in \mathbb{R}/\mathbb{Z}$*.*

The Lyapunov exponent of the unique absolutely continuous T-invariant probability measure μ is defined by

$$L(\mu) = \int \log |T'(x))| \, d\mu(x) \,, \tag{1}$$

and by the well known Rohlin identity this also equals the entropy $h(\mu)$.

We briefly summarise the method for estimating $L(\mu)$ in three steps.

Step 1 (Complex functions and coefficients). We wish to consider period-n points $T^n x = x$ and then define for $t \in \mathbb{R}$ the coefficients $a_1(t), a_2(t), \ldots$ using the Taylor expansion

$$\exp\left(- \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{T^n x = x} \frac{|(T^n)'(x)|^{-t}}{1 - ((T^n)'(x))^{-1}} \right) = 1 + \sum_{n=1}^{\infty} a_n(t) z^n + \cdots .$$

Step 2 (Coefficients and Lyapunov exponents). It can be shown that the Lyapunov exponent $L(\mu)$ given by (1) admits the alternative formulation

$$L(\mu) = \frac{-\sum_{n=1}^{\infty} a_n'(0)}{\sum_{n=1}^{\infty} n a_n(0)},$$

where both the numerator and denominator are absolutely convergent series. Truncating these series to give the computable quantity

$$L_N = \frac{-\sum_{n=1}^{N} a_n'(0)}{\sum_{n=1}^{N} n a_n(0)},$$

we note that

$$L_N \to L \text{ as } N \to +\infty,$$

and so for a large natural number N, the quantity L_N is an approximation to L.

Step 3 (Error bounds). The quality of the approximation to L given by L_N may be indicated heuristically by comparing how closely L_N and L_{N-1} agree, though more accurate errors in the approximation can be obtained using:

1. A certain value $\theta \in (0,1)$, which we refer to as the *contraction ratio*, measuring the extent to which a complex disc D is mapped inside itself by the inverse branches T_j of T;
2. The integrals $\beta_k = \frac{1}{r^{2k}} \int_0^1 \left| \sum_j T_j'(x + re^{2\pi i t})(T_j(x + re^{2\pi i t}))^k \right|^2 dt$, where $1 \leq k \leq L$; and
3. The weights $\alpha_k = \sqrt{\sum_{l=k+1}^{L} \beta_l^2}$ for $N \leq k \leq L$,

where $N < L$ are suitably chosen. In particular, we can then use these values to bound the coefficients $a_n(t)$, with $n > N$, for which it is impractical to explicitly compute them with effective error estimates. This will be explained in greater detail in §5.

Now that we have illustrated the general theme using the specific example of Lyapunov exponents for one dimensional expanding maps, we can turn to more general dynamical settings, and more general characteristic values. In the next section we describe the broad context of the results and consider the pressure function, from which many of the quantities we want to consider can be derived.

2 Hyperbolic maps and the pressure function

To set the scene, we begin by introducing two natural classes of discrete dynamical system, then consider the associated pressure function, which will be useful in providing the bridge between the dynamics the various quantities we wish to describe.

2.1 Hyperbolic maps

Let us begin with the discrete setting. Assume that $T : M \to M$ is either a smooth expanding map (perhaps on an invariant repeller $Y \subset M$) or Anosov diffeomorphism (see [28, §6.4]). Later we will consider the more general settings of repellers and the natural generalisation to flows. However, for the present we will restrict to the discrete cases above and, whenever more convenient, to the case of expanding maps.

Definition 2.1. *A partition $X = \cup_{i=1}^{k} X_i$ is called a Markov partition for the expanding map $T : X \to X$ if*

1. $\overline{\mathrm{int}(X_i)} \cap \overline{\mathrm{int}(X_j)}$ *for $i \neq j$;*

2. $X_i = \overline{\mathrm{int}(X_i)}$ *for* $i = 1, \cdots, k$*; and*

3. *each* $T(X_i)$ *for* $i = 1, \cdots, k$ *is a union of other elements of the partition.*

Eventually, we will want to assume that each of the restrictions $T|X_i$ ($i = 1, \cdots, k$) is real analytic, in the sense of having an analytic extension (via charts) to a complex neighbourhood U_i. However, to set up the definitions we only require that it be C^1.

In the case of Anosov diffeomorphisms the approach is somewhat similar, except that one uses Markov partitions for invertible maps. In the case of Anosov flows one can expect to use Markov Poincaré sections to reduce the analysis to the discrete case.

2.2 Pressure

The pressure function was introduced into the study of hyperbolic dynamical systems by Ruelle (see e.g. [41]). The importance of pressure stems from the fact that it yields a unifying concept to describe dynamical and geometric invariants. For example, it is well known that various dynamically and geometrically defined fractals (e.g. limit sets and Julia sets) have the property that their Hausdorff dimension is given by solving an associated *pressure equation* (usually known as the Bowen formula). More generally, there are a host of other dynamical quantities that can be expressed in terms of the pressure function, some of which are listed below in subsection 2.3.

To define pressure P, since we are considering hyperbolic maps we have the luxury of expressing this in terms of periodic orbits $T^n x = x$, for $n \geq 1$, and define $P : C^0\left(\coprod_i X_i\right) \to \mathbb{R}$ ($i = 1, \cdots, k$) on the disjoint union of the elements of the Markov Partition.

Definition 2.2. *The pressure of the continuous function g is given by*

$$P(g) := \limsup_{n \to +\infty} \frac{1}{n} \log \left(\sum_{T^n x = x} \exp \left(\sum_{j=0}^{n-1} g(T^i x) \right) \right),$$

and admits the alternative variational definition

$$P(g) = \sup \left\{ h(\mu) + \int g d\mu \, : \, \mu \textit{ is a } T\textit{-invariant probability measure} \right\}.$$

When g is Hölder continuous, there is a unique probability measure μ_g realising the above supremum [3].

Definition 2.3. *The measure μ_g is called the equilibrium measure (or Gibbs measure) for g.*

If $g = 0$ then $P(0) = h_{\mathrm{top}}(f)$ is the topological entropy, and the corresponding equilibrium measure μ_g is called the measure of maximal entropy (and the Bowen-Margulis measure in the case of Anosov systems).

Example 2.4. *For expanding maps $T : M \to M$, in the special case $g(x) = -\log |\mathrm{Jac}(D_x T)|$ then $P(g) = 0$, and the corresponding equilibrium measure μ_g is a T-invariant probability measure equivalent to the volume on M. For Anosov maps $T : M \to M$, in the special case that $g(x) = -\log |\mathrm{Jac}(D_x T|E^u)|$, where E^u is the unstable bundle, then $P(g) = 0$ and the corresponding equilibrium measure μ_g is called the Sinai-Ruelle-Bowen measure (or SRB-measure) (see [28], §20.4).*

It is this pressure function that often helps to relate periodic orbits to the quantities in which we are interested, and which we ultimately want to numerically estimate. A simple, but important, application is the following result due to Ruelle (see [41, p. 99]):

Lemma 2.5. *For any Hölder continuous functions $g_0, g : X \to \mathbb{R}$ the function*

$$t \mapsto P(g_0 + tg) \in \mathbb{R}, \text{ for } t \in \mathbb{R},$$

is analytic. Moreover

1. $\frac{dP(g_0+tg)}{dt}|_{t=0} = \int g d\mu_{g_0}$, *and*

2. $\frac{d^2P(g_0+tg)}{dt^2}|_{t=0} = \lim_{n\to+\infty} \frac{1}{n} \int \left(\sum_{i=0}^{n-1} g(T^i x)\right)^2 d\mu_{g_0}(x)$ *provided* $\int g d\mu_{g_0} = 0$.

The quantity in part 2 of Lemma 2.5 is often called the *variance*.

It is a feature of the method we use that we can obtain fairly explicit expressions for derivatives of pressure. In particular, those quantities that can be written in terms of the derivative expressions can therefore, in turn, be written in terms of periodic points.

2.3 Relating pressure to characteristic values

We can now consider a number of familiar quantities that we can write in terms of the pressure function. Below we list a few simple examples. Later we shall consider other applications, but for the present these three examples illustrate well our theme.

(I) Lyapunov exponents. For an expanding map $T : M \to M$ we can write the Lyapunov exponent for the absolutely continuous invariant measure μ by

$$L(\mu) = \int \log \|D_x T\| d\mu(x).$$

In the particular case of one dimension this reduces to the situation described in subsection 1.4.

The following follows immediately from part 1 of Lemma 2.5.

Lemma 2.6. *If we let $g_0(x) = -\log |\text{Jac}(D_x T)|$ and $g = -\log \|D_x T\|$ then $L(\mu) = \frac{d}{dt} e^{P(g_0+tg)}|_{t=0}$.*

(II) Variance. For an expanding map $T : M \to M$ we can write the variance for the absolutely continuous T-invariant measure μ and a Hölder continuous function $g : X \to \mathbb{R}$ with $\int g d\mu = 0$ defined by

$$\Sigma(g, \mu) := \lim_{n\to+\infty} \frac{1}{n} \int \left(\sum_{i=0}^{n-1} g(T^i x)\right)^2 d\mu(x).$$

The following follows immediately from part 2 of Lemma 2.5.

Lemma 2.7. *We can write*

$$\frac{d^2P(g_0 + tg)}{dt^2}|_{t=0} = \Sigma(g, \mu).$$

This plays an important role in the Central Limit Theorem [3], i.e., for any real numbers $a < b$ we have

$$\lim_{n\to+\infty} \mu\left(\left\{x \in X : \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} g(T^i x) \in [a, b]\right\}\right) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-u^2/2\sigma} du.$$

(III) Linear response. Let $T_\lambda : M \to M$ be a smooth family of expanding maps $(-\epsilon < \lambda < \epsilon)$ and let μ_{g_λ} be the associated absolutely continuous measure, arising as the Gibbs measure for $g_\lambda(x) = -\log \|DT_\lambda(x)\|$. Then by part 2 of Lemma 2.5 we have

$$\frac{\partial P(g_\lambda + tg)}{\partial t}|_{t=0} = \int g d\mu_{g_\lambda}.$$

Thus differentiating in λ formally gives:

$$\frac{d^2 P(g_\lambda + tg)}{dt\, d\lambda}|_{\lambda=0} = \frac{d}{d\lambda}\left(\int g d\mu_{g_\lambda}\right)|_{\lambda=0}.$$

(A slight subtlety here is that the differentiation of the pressure is easier with respect to the fixed expanding map $T_0 : M \to M$ and thus it is appropriate to introduce a family of conjugacies $\mu_\lambda : M \to M$ between T_0 and T_λ and to consider $\frac{d^2 P(g_0 \circ \pi_\lambda + tg\circ\pi_\lambda)}{dt d\lambda}|_{\lambda=0}$).

The above list does not exhaust the possible quantities that can be derived from the pressure, but gives a selection we hope illustrates our general approach.

In the next section, we will introduce a standard tool, the *transfer operator*, which allows us to analyse the pressure, and thus its many derivative properties, using basic ideas from linear operator theory. We will also describe the connection with a family of complex functions called determinants.

3 Transfer operators and determinants

A central object in thermodynamic formalism is the *transfer operator*, from which important dynamical and geometric invariants such as entropy, Lyapunov exponents, invariant measures, and Hausdorff dimension can be obtained.

Let us now restrict (for the present) to the case of expanding maps $T : X \to X$. The analyticity of the pressure, as well as other properties including the proof of Lemma 2.5, depend on the use of *transfer operators*. Eventually, we will want to consider operators acting on spaces of analytic functions, but for the purposes of defining them it suffices for the present to consider the Banach space of C^1 function $C^1(X, \mathbb{C})$ with the norm $\|f\| = \|f\|_\infty + \|Df\|_\infty$. The operators are then defined as follows:

Definition 3.1. *If $T : X \to X$ is a C^1 expanding map, and $g : X \to \mathbb{R}$ is C^1, then we define the transfer operator $\mathcal{L}_g : C^1(X, \mathbb{C}) \to C^1(X, \mathbb{C})$ by*

$$\mathcal{L}_g w(x) = \sum_{Ty=x} e^{g(y)} w(y)\,,$$

the summation being over the inverse images y of the point $x \in X$.

This operator preserves various function spaces and exhibits certain positivity properties which ensure that a Perron-Frobenius type theorem holds: when acting on $C^1(X, \mathbb{C})$, $\mathcal{L}_g$ has a leading eigenvalue which is simple, positive, and isolated. Moreover, the connection with the pressure comes from the following basic result due to Ruelle [41] (see also [3]).

Lemma 3.2. *The spectral radius of $\mathcal{L}_g$ is $e^{P(g)}$. In particular, $e^{P(g)}$ is a maximal isolated eigenvalue for $\mathcal{L}_g$.*

In particular, the differentiability (indeed analyticity) of the pressure follows by standard perturbation theory and the expression for the derivatives in the lemma follow by explicit manipulations.

Example 3.3. *In the special case that $g(x) = -\log|\mathrm{Jac}(D_xT)|$ then $P(g) = 0$ and $\mathcal{L}_g$ is known as the Ruelle-Perron-Frobenius operator. The eigenmeasure $m_g = \mathcal{L}_g^* m_g$ is normalised Lebesgue measure, and the equilibrium measure $\mu_g = h_g m_g$ is the unique T-invariant measure absolutely continuous with respect to Lebesgue measure, where $h_g = \mathcal{L}_g h_g$ is the maximal eigenfunction [3].*

Thus far we have been following a very traditional approach. However, now we introduce an extra ingredient.

3.1 Determinants and their coefficients

Let $T : X \to X$ be a C^1 expanding map. For any continuous function $G : X \to \mathbb{R}$ and each period-n point $T^n x = x$, $n \geq 1$, we can associate the weight

$$G^n(x) := \sum_{i=0}^{n-1} G(T^i x) \in \mathbb{R}.$$

Later we will want to assume that T and G are real analytic, but for the purposes of introducing the determinant we need only assume these weaker hyotheses. It is convenient to package up the information from individual periodic points into a single (generating) complex function.

Definition 3.4. *Given a continuous function $G : X \to \mathbb{R}$ we can formally define a function of the single complex variable by:*

$$D(z) = D_{G,T}(z) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{T^n x = x} \frac{\exp\left(\sum_{i=0}^{n-1} G(T^i x)\right)}{\det(I - [D(T^n)(x)]^{-1})} \right), \quad z \in \mathbb{C}\,.$$

The radius of convergence of the infinite series $D(z)$ is related to the pressure. More precisely, we can see that this converges to an analytic function provided the series converges, i.e., $|z| e^{P(G)} < 1$ where

$$e^{P(G)} = \lim_{n\to+\infty} \left| \sum_{T^n x = x} \frac{\exp\left(\sum_{i=0}^{n-1} G(f^i x)\right)}{\det(I - [D(T^n)(x)]^{-1})} \right|^{1/n} \left(= \lim_{n\to+\infty} \left| \sum_{T^n x = x} \exp\left(\sum_{i=0}^{n-1} G(f^i x) \right) \right|^{1/n} \right).$$

In particular, writing $D(z)$ as a power series

$$D(z) = 1 + \sum_{n=1}^{\infty} a_n z^n, \tag{3.1}$$

with coefficients $a_n = a_n(T, G)$ depending on T and G, we see it has radius of convergence at least $e^{-P(G)}$.

Example 3.5 (Expanding maps of the interval)**.** *In the particular case of an expanding map $T : X \to X$ of the interval X, given a continuous function $G : X \to \mathbb{R}$, the function $D(z)$ takes the simpler form:*

$$D(z) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{T^n x = x} \frac{\exp\left(\sum_{i=0}^{n-1} G(T^i x)\right)}{1 - 1/(T^n)'(x)} \right), \quad z \in \mathbb{C}\,.$$

This naturally leads to asking about the meromorphic extension of $D(z)$. To proceed further, we need to assume more regularity on the function G. This brings us to the following important result of Ruelle [40].

Lemma 3.6. *If T is real analytic then*

1. *$D_{G,T}(z)$ is analytic in all of $\mathbb{C}$.*
2. *The value $z = e^{-P(G)}$ is a simple zero for $D_{G,T}(z)$ in this extension.*

In particular, we see from part 1 of Lemma 3.6 that we can improve the result on the radius of convergence of the power series to $\lim_{n\to+\infty} |a_n|^{1/n} = 0$, i.e., for any $0 < \theta < 1$ there exists $C > 0$ such that $|a_n| \leq C\theta^n$. In fact, the original proof of Lemma 3.6 due to Ruelle, and inspired by work of Grothendieck, works in this way by giving estimates on the coefficients a_n. We will later describe quite precise bounds on the coefficients a_n which establishes part 1. We will return to this point in the next subsection.

Remark 3.7. If we assume that T and G are C^∞ then we would still have that $\lim_{n\to+\infty} |a_n|^{1/n} = 0$. However, as we shall see, in the analytic case we have more effective estimates on $|a_n|$.

3.2 Pressure and the characteristic quantities

In order to relate $D(z)$ back to the pressure, and thus the various dynamical quantities, we need to make different choices for G. More precisely, we can consider the special case of the function $G = g_0 + tg$ where $g_0, g : X \to \mathbb{C}$ and $t \in \mathbb{R}$. This leads to the following particular case of the previous definition.

Definition 3.8. *We formally define the determinant for g_0, g_1 to be the bi-complex function*

$$d_{g_0,g}(z,t) := \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{T^n x = x} \frac{\exp\left(\sum_{j=0}^{n-1} (g_0 + tg)(T^i x)\right)}{1 - 1/(T^n)'(x)}\right),$$

where $z, t \in \mathbb{C}$.

The relationship between the determinant and the pressure in Lemma 3.6 now implies the following.

Corollary 3.9. *Assume that g_0, g_1 are C^ω.*

1. *The function $d_{g_0,g}(z,t)$ has a bi-analytic extension to all of $\mathbb{C}^2$.*
2. *The value $z = e^{-P(g_0+tg)}$ occurs as a simple zero for $z \mapsto d_{g_0,g}(z,t)$.*

We can make the further simplifying assumption that $P(g_0) = 0$, where we can replace g_0 by $g_0 - P(g_0)$ if necessary. In particular, the first zero for $z \mapsto d_{g_0,g}(z,0)$ (where $t = 0$) occurs at $z = e^{-P(g_0)}$. We can use part 2 of Corollary 3.9, and the implicit function theorem, to write the derivative of the pressure as

$$\frac{dP(g_0 + tg)}{dt}|_{t=0} = \frac{\partial d_{g_0,g}(1,t)}{\partial t}|_{t=0} / \frac{\partial d_{g_0,g}(z,0)}{\partial z}|_{z=1}$$

in terms of partial derivatives of the determinant. Furthermore, in light of part 1 of Corollary 3.9, for each $t \in \mathbb{R}$ we can formally expand

$$d_{g_0,g}(z,t) = 1 + \sum_{n=1}^{\infty} a_n(t) z^n.$$

It is these values which we need to relate to the quantities in which we are interested, and which we ultimately want to numerically estimate.

This gives us the simplest definition of the numbers $(a_n(t))_{n=1}^\infty$, although we can explicitly expand these in terms of periodic points too. This reveals the following simple, but crucial, fact.

Lemma 3.10. *For each $n \in \mathbb{N}$, we can express the value $a_n(t)$ in terms of the periodic points of period at most n.*

In particular, this ensures that the coefficients $a_n(t)$ are relatively easy to estimate. Moreover, we can rapidly approximate $d_{g_0,g}(z,t) = 1 + \sum_{n=1}^\infty a_n(t) z^n$ by the truncated series

$$1 + \sum_{n=1}^{N} a_n(t) z^n$$

for N moderately large. The rapidity of the approximation is justified by the following result.

Corollary 3.11. *The coefficients $a_n = a_n(t)$ tend to zero at a super-exponential rate.*

When X is one dimensional then there exists $\theta \in (0,1)$ such that $|a_n| = O(\theta^{n^2})$ as $n \to \infty$. We will give very explicit estimates for the implied constants in the $O(\cdot)$ term.

Remark 3.12. When X is d-dimensional (with $d \geq 2$) then there exists $0 < \theta < 1$ such that $|a_n| = O(\theta^{n^{1+1/d}})$ as $n \to \infty$.

Remark 3.13. More generally, we can assume we have a family of real analytic functions $g_1, \cdots, g_m : X \to \mathbb{C}$ and

$$d(z,t) := \exp\left(- \sum_{n=1}^{\infty} z^n \sum_{T^n x = x} \exp\left(\sum_{j=0}^{n-1} (g_0 + t_1 g_1 + \cdots + t_m g_m)(T^j x) \right) \right)$$

for $z \in \mathbb{C}$ and $t_1, \ldots, t_m \in \mathbb{R}$.

To summarize, we now have a method of approaching the pressure function which might be considered to have a simple analogy to that of estimating the largest eigenvalue of a matrix by computing the characteristic polynomial, which is a complex function whose zeros give the eigenvalues. This simple viewpoint relating the determinant and transfer operators ultimately leads to a surprisingly efficient method of computing pressure. Furthermore, we can accurately estimate the error terms (see §5).

Having related the various quantities which we want to estimate to the zero(s) of the determinant $d(z,t)$ (often via the pressure function, cf. subsection 2.3) we need to answer three key questions: *How can we use this formulation to get numerical estimates? Why does this approach lead to superior approximation estimates? How can we estimate the quantities with validated rigour?*

We can now turn to the practical problem of writing explicit expressions for the approximations to quantities we described in subsection 2.3, in terms of the first N coefficients in the expansion of the determinant.

3.3 Dynamical quantities and coefficients

We have established (in subsection 2.3) that many of the quantities that we want to estimate can be expressed in terms of pressure, and its derivatives, which in turn can be written in terms of the determinant and its derivatives. Since the determinant has a power series expansion it is a straightforward, but useful, exercise to write these expressions explicitly in terms of the derivatives of the coefficients. More precisely, let us write

$$A = \sum_{n=1}^{\infty} n a_n(0), \quad B = \sum_{n=1}^{\infty} n(n-1) a_n(0), \quad C = \sum_{n=1}^{\infty} a_n'(0), \quad D = \sum_{n=1}^{\infty} n a_n'(0), \quad E = \sum_{n=1}^{\infty} a_n''(0)$$

and the associated finite sums

$$A_N = \sum_{n=1}^{N} n a_n(0), \; B_N = \sum_{n=1}^{N} n(n-1) a_n(0), \; C_N = \sum_{n=1}^{N} a_n'(0), \; D_N = \sum_{n=1}^{N} n a_n'(0), \; E_N = \sum_{n=1}^{N} a_n''(0)$$

for $N \geq 1$.

A recurrent theme in our discussions is that we want to express the dynamical quantities in terms of A, B, C, D, E, etc., and then approximate these expressions by using instead the more computationally tractable quantities A_N, B_N, C_N, D_N, E_N.

Remark 3.14. In practical applications, even for quite simple examples, we might currently only expect to compute these values up to $N = 25$, say, in a reasonable time frame. However, with further technological advances, one might expect that this value can be improved.

To illustrate this principle, we can now reformulate the three key quantities described in subsection 2.3, and their approximations, in terms of these series and summations. We list these below.

(I) Lyapunov exponents. We can write the Lyapunov exponent for μ as

$$L(\mu) = -\frac{\sum_{n=1}^{\infty} a_n'(0)}{\sum_{n=1}^{\infty} n a_n(0)} = -\frac{C}{A},$$

and in particular the Lyapunov exponent can be approximated by the computable quantities

$$-\frac{C_N}{A_N}, \quad N \geq 1 .$$

(II) Variance. The variance is given by

$$\Sigma^2 = \left(\frac{C}{A}\right)^2 + \frac{1}{A}\left(B\left(\frac{C}{A}\right)^2 - 2DB\left(\frac{C}{A}\right) + E\right),$$

and in particular we can approximate the variance by

$$\left(\frac{C_N}{A_N}\right)^2 + \frac{1}{A_N}\left(B_N\left(\frac{C_N}{A_N}\right)^2 - 2D_N B_N\left(\frac{C_N}{A_N}\right) + E_N\right), \quad N \geq 1 .$$

(III) Linear response. We can write

$$\int g d\mu_T = -\frac{C}{A},$$

and in particular we can approximate the integral by

$$-\frac{C_N}{A_N}, \quad N \geq 1.$$

Finally, by replacing T by T_λ, differentiating both sides in λ and we get an expression for the linear response in terms of the (derivatives of the) coefficients a_n.

4 Rates of mixing and dimension

In this section we want to make a slight detour to introduce another two important quantities which, although not quite fitting into the same framework described in the previous section, can also be approximated using the determinant.

First we consider the rate(s) of mixing, which can be studied via the zeros of the determinant.

4.1 Rates of mixing

Let X be d-dimensional and let $T : X \to X$ be a C^ω conformal expanding map. More precisely, we can write the derivative $DT(x) = w(x)\Theta(x)$ where $w : X \to \mathbb{R}$ and $\Theta : X \to SO(d)$. In the particular case that $d = 1$ then the one dimensional map T is automatically conformal.

Let $g_0 : X \to \mathbb{R}$ be real analytic and let $\mu = \mu_{g_0}$ be the equilibrium-Gibbs measure associated to g_0.

Example 4.1. *As we observed in Example 2.4, when $g_0(x) = -\log|\mathrm{Jac}(T)(x)|$ the associated measure μ_{g_0} is the unique absolutely continuous invariant probability measure.*

The following is an important object in ergodic theory.

Definition 4.2. *Given a real analytic function g, we can consider the correlation function defined by*

$$c(n) = \int g \circ T^n g d\mu - \left(\int g d\mu\right)^2, \quad n \geq 1.$$

Since $T : (X, \mu) \to (X, \mu)$ is mixing we know that $c(n) \to 0$, as $n \to +\infty$. The (exponential) rate of mixing is given by the smallest value $0 < \lambda_1 < 1$ such that $c(n) = O(\lambda_1^n)$ for all such g and $n \geq 1$. Since λ_1 corresponds to the modulus of the second eigenvalue of the transfer operator, the connection to the determinant comes through the following simple lemma.

Lemma 4.3. *The rate of mixing $0 < \lambda_1 < 1$ is the reciprocal of the modulus $\rho > 1$ of the second smallest (in modulus) zero of $d_{g_0,g}(z, 0)$, i.e. $\lambda_1 = 1/\rho$.*

In particular, the value ρ in Lemma 4.3 is a zero of the (real valued) series

$$d_{g_0,g}(z, 0) = 1 + \sum_{n=1}^{\infty} z^n a_n(0)$$

and so in order to get rigorous bounds on ρ we can use the intermediate value theorem. More precisely, given $\epsilon_1, \epsilon_2 > 0$ and $N \in \mathbb{N}$ we look for bounds $\alpha_N < \rho < \beta_N$ by choosing α_N, β_N such that

$$1 + \sum_{n=1}^{N} \alpha_N^n a_n(0) \leq -\epsilon_1 \text{ and } 1 + \sum_{n=1}^{N} \beta_N^n a_n(0) \geq \epsilon_2$$

and

$$\epsilon_1 > \left|\sum_{n=N+1}^{\infty} \alpha_N^n a_n(0)\right| \text{ and } \epsilon_2 > \left|\sum_{n=N+1}^{\infty} \beta_N^n a_n(0)\right|.$$

Thus finding good bounds $\alpha_N < \rho < \beta_N$ reduces to:

1. getting good estimates on the coefficients $a_i(0)$ $(i = 1, \cdots, N)$; and

2. finding effective bounds on the tails $\sum_{n=N+1}^{\infty} a_n(0)$.

The first is a basic problem in practical computing. The second is a more challenging mathematical problem.

We can illustrate this with a simple example.

Example 4.4 (Lanford map, cf. [27, 30]). *Let $T : [0,1] \to [0,1]$ be defined by*

$$T(x) = 2x + \frac{1}{2}x(1-x) \pmod{1}.$$

Let μ denote the unique absolutely continuous T-invariant probability measure. We can now evaluate dynamical quantities such as the rate of mixing by approximating the infinite series for the determinant by finite truncations. For example, using periodic points up to period $N = 16$ we might estimate the first ten zeros (i.e. the ten zeros of smallest modulus) of the determinant by locating the first ten zeros of the degree-16 polynomial truncation (see Table 1).

n	zero z
1	1.0000000000000000000000033711203720152
2	1.72986531066431681927894069519181629
3	2.69226981834657374557299751785 28581
4	4.11324667567597777836456726729799 56
5	6.24548532057212911760331771248042 91
6	9.45389167173973265444734313321233 70
7	14.2827343364585244340003130808025 10
8	21.5492299945323271797579914086690 84
9	32.5162661027018034906756366309071 93
10	47.8291048470221875844677328981375 3

Table 1: Estimates on the first 10 zeros of the determinant

More accurate estimates on the second eigenvalue λ_1 of the transfer operator, i.e. the exponential rate of mixing, are obtained by approximating the reciprocal of the second zero of the determinant, given in Table 2, using degree-N truncations with N ranging up to $N = 25$.

We will address the rigorous bounds on the error later.

Remark 4.5. The rate of mixing has a more subtle generalisation of the following form. The speed of mixing is controlled by a sequence of complex numbers $\lambda_j \to 0$ ordered to be decreasing in modulus, and polynomials P_j. More precisely, for any $\delta > 0$ there exist M and $c_j = c_j(g)$ $(j = 1, \cdots, M)$ such that

$$c(n) = \sum_{j=1}^{M} c_j P_j(n) \lambda_j^n$$

where $P(n) = 1$ in the case of a single zero of the determinant. Since we can rewrite

$$c(n) = \int g \left(\mathcal{L}_{g_0}^n g\right) d\mu - \left(\int g d\mu\right)^2$$

the values λ_j are actually the *other* non-zero eigenvalues of the transfer operator. Furthermore,

$$P_j(n) = \begin{cases} 1 & \text{if } \lambda_j \text{ has multiplicity } m = 1 \\ \text{a polynomial of degree at most } n & \text{if } \lambda_j \text{ has multiplicity } > 1. \end{cases}$$

N	second eigenvalue λ_1 estimate
12	0. 5780796887515271422742765368788953299348846128812023109203951947004787498004165
13	0.5780796885356306834127405345836355663641109763750019611087170244976104563627485
14	0.5780796885371288506764371131157188309769151998850254045247866596035386808066373
15	0.5780796885371219470570630291328371225537224787114789966418506438634692131905786
16	0.5780796885371219681960432055118626393344991913606205477442507113706445878179303
17	0.5780796885371219681530107872274433995896003010891980049121721575572541941602200
18	0.5780796885371219681530690475964044549434630264578745046610737538545621059499499
19	0.5780796885371219681530689951228630661230046404665596121577924787108887084624467
20	0.5780796885371219681530689951543111607290086789469044407358342311002102577614591
20	0.5780796885371219681530689951543111607290086789469044407358342311002102577614591
21	0.5780796885371219681530689951542986173994750355886730998629996708085266331162364
22	0.5780796885371219681530689951542986207295902353572376170142239221705813115693187
23	0.5780796885371219681530689951542986207290016813780055122529428636713090405312973
24	0.5780796885371219681530689951542986207290017506309910801987018396260494147990458
25	0.5780796885371219681530689951542986207290017506255654011865278736341388970140860

Table 2: Estimates on the second eigenvalue λ_1 coming from the reciprocals of zeros for $1 + \sum_{n=1}^{N} z^n a_n(0)$, for $12 \leq N \leq 25$.

Furthermore, one can identify the eigenvalues in terms of the *other* zeros $\rho_j = 1/\lambda_j$ for $z \mapsto d_{f,z}(z, 1)$. All this said, in the case of the Lanford map the first 10 zeros of the determinant are simple.

4.2 Dimension of repellers

Let $T : X \to X$ be a C^ω conformal expanding map.

Definition 4.6. *A closed invariant set $Y \subset X$ is called a repeller if there is an open set U satisfying $Y \subset U \subset X$ such that $Y = \cap_{n=1}^{\infty} T^{-n} U$.*

We refer the reader to the book of Falconer for the definition and basic properties of Hausdorff dimension [10]. In the present context we have a convenient dynamical formulation. Let $g_0(x) = -\log |\mathrm{Jac}(T)(x)|$ and then consider the restriction $T : Y \to Y$. The following standard result relates the dimension $\dim_H(X)$ to the pressure function for the transformation $T : Y \to Y$ and function $sg_0 : Y \to \mathbb{R}$, for $s > 0$ (see [42]).

Lemma 4.7 (Bowen-Ruelle)**.** *We can characterise the Hausdorff dimension of the limit set by $s = \dim_H(Y)$ such that $P(sg_0) = 0$.*

We briefly recall some simple examples which fit into this setting.

Example 4.8. *If $T : J \to J$ is a hyperbolic rational map acting on its Julia set J, and $f = -s \log |T'|$, then the Hausdorff dimension $\dim_H(J)$ is given by the unique zero s of $P(-s \log |T'|)$ by Lemma 4.7. The same ideas apply to the Hausdorff dimension of limit sets of certain Fuchsian groups (see, for example, [4]).*

Lemma 4.7 allows us to define the dimension $\dim_H(Y)$ implicitly in terms of the determinant (defined in terms of periodic points for $T : Y \to Y$, i.e., those contained in Y). In particular, setting $g_0 = 0$, $g = -\log |T'|$ and $z = 1$ in Definition 3.8 we have $s = \dim_H(X)$ satisfies $d_{0,g}(1, s) = 0$.

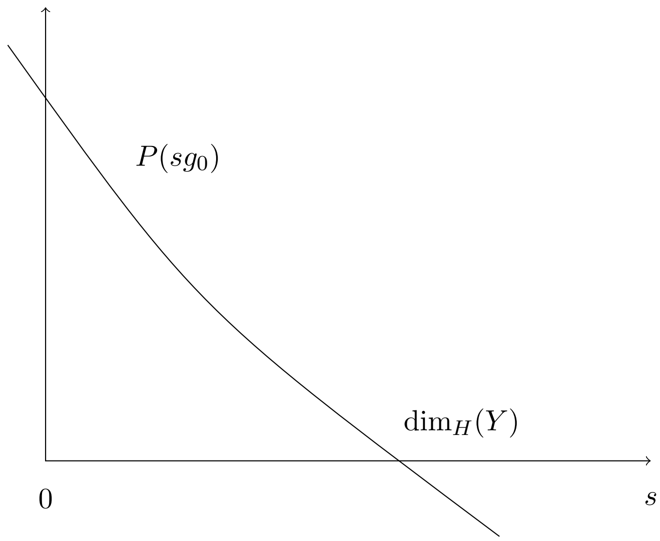

Figure 2: Graph of a pressure curve $s \mapsto P(sg_0)$

Equivalently, the Hausdorff dimension of the limit set is a solution $s = \dim_H(X)$ to the absolutely convergent series

$$1 + \sum_{n=0}^{\infty} a_n(s) = 0.$$

As before, when studying the rate of mixing in the previous subsection, in practice we can use the intermediate value theorem to get effective bounds $\alpha_N < \rho < \beta_N$ by choosing α_N, β_N such that

$$1 + \sum_{n=1}^{N} a_n(\alpha_N) \geq \epsilon_1 \text{ and } 1 + \sum_{n=1}^{N} a_n(\beta_N) \leq -\epsilon_2$$

where

$$\epsilon_1 > \sum_{n=N+1}^{\infty} a_n(\alpha_1) \text{ and } \epsilon_2 > \sum_{n=N+1}^{\infty} a_n(\alpha_2)$$

Thus, as before, finding good bounds comes down to:

1. getting good estimates on the coefficients $a_i(s)$ $(i = 1, \cdots, N)$; and
2. finding effective bounds on the tail $\sum_{n=N+1}^{\infty} a_n(s)$.

As we mentioned before, the first is a basic problem in computing and the second is a more challenging mathematical problem which we will address in §5.

We now turn to a class of deceptively simple examples.

Example 4.9 (Continued fractions and deleted digits)**.** *Consider a finite set $F \subset \mathbb{N}$ and the set $E_F \subset [0, 1]$ given by*

$$E_F = \{x = [x_1, x_2, \cdots] : x_1, x_2, \cdots \in F\},$$

i.e., the Cantor set of points whose continued fraction expansion has all coefficients lying in F. This can be viewed as a repeller for the map $T : E_F \to E_F$ defined by $T(x) = \frac{1}{x} \pmod 1$.

(a) In the case $F = \{1,2\}$ the set E_F is usually denoted E_2 and is of historical interest, with its Hausdorff dimension studied by Good [16] as far back as 1941, after even earlier work of Jarnik [24]. Using our algorithm we were able to compute the dimension accurately, and rigorously, to over 100 decimal places (see [25]), improving on the previous best rigorous estimate due to Falk & Nussbaum [11], using a subtler variant of Ulam's method.

(b) In the case $F = \{1,2,3,4,5\}$, the dimension $\dim_H(E_{\{1,2,3,4,5\}})$ appears as a crucial ingredient in the work of Huang [23], refining the work of Bourgain & Kontorovich [2] on a density one solution to the Zaremba Conjecture. Here we were able to use the algorithm to compute the dimension accurately, and rigorously, to 8 decimal places [26].

5 Rigorous error bounds

We now come to one of the most interesting and challenging aspects of the estimation problem: *finding rigorous upper bounds for the errors in the approximations.*

In the interests of clarity, and notational simplicity, we shall explain the ideas in the particular case of estimating the Hausdorff dimension of limit sets (corresponding to conformal iterated function schemes). The more general settings require variants of this basic approach.

In particular, we need to provide an estimate on the error when we truncate the series which comes from bounds on the terms $|a_n(t)|$ for large values of n. Our bounds will involve a number of variables in whose choice we have some limited flexibility. In particular, we can select these so as to optimise the error terms.

5.1 The contraction ratio θ for expanding maps

Let us assume that we can associate a Markov partition $\mathcal{P} = \{P_1, \cdots, P_K\}$. In the particular case that $T : X \to X$ is an expanding map we can consider

1. charts and the complexification of the maps T to neighbourhoods $\mathbb{C}^d \supset U_i \supset P_i$ ($i = 1, \cdots, K$); and
2. associated contractions $\psi_{ij} : U_i \to U_j$ wherever $T(\mathrm{int}(U_j)) \supset \mathrm{int}(U_i)$.

In more fortunate situations we can assume that we have *Bernoulli* contractions where $U = U_i$ ($i = 1, \cdots, K$) are identical and $\psi_{ij} = \psi_i$ ($i = 1, \cdots, K$) are independent of j. (This applies in the case of Example 1.1 and the Lanford map in Example 4.4)

Definition 5.1. *Choose $0 < \Theta_i < 1$ such that we can choose complex polydiscs $P_i \subset B(c_i, r_i) \subset U_i$ where*

$$B(c_i, r_i) = \{\underline{z} \in \mathbb{C} : |z_i - c_i| < r_i, \quad i = 1, \cdots, K\}$$

such that

$$\overline{\psi_{ij}(B(c_i, r_i))} \subset B(c_j, \Theta_j r_j).$$

Let $\theta := \max_i\{\Theta_i^{1/K}\}$. In the particular case of Bernoulli contractions we can take $\theta := \max_i\{\Theta_i\}$.

Remark 5.2. It may not always be possible to extend the contractions analytically to discs about elements of the partitions. However, in that case we can consider instead the more refined partition by elements $P_{i_0} \cap T^{-1} P_{i_1} \cap \cdots \cap T^{-n} P_{i_n}$, for suitable n.

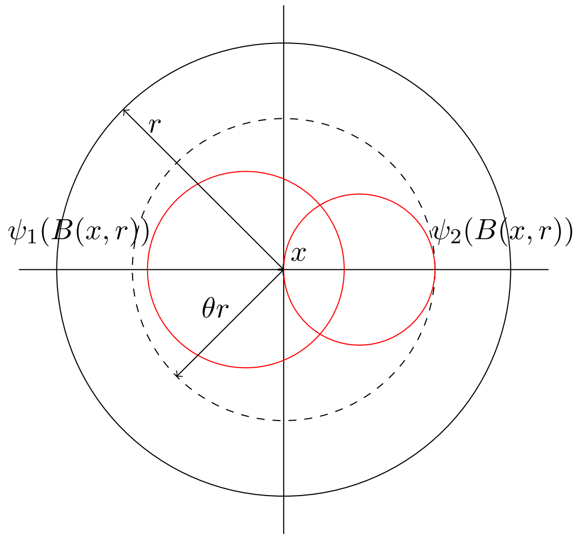

Figure 3: The choice of $0 < \theta < 1$ for two contractions $\psi_1, \psi_2 : B(c,r) \to B(c,r)$ with $B(c,\theta r) \supset \psi_1(B(c,r)) \cup \psi_2(B(c,r))$.

Example 5.3. *For an expanding map of the interval then we can take $d = 1$. For a Bernoulli expanding map T we would like to take $k = 1$ (i.e., a single disc $B(c,r)$ and contractions $\psi_i : B(c,r) \to B(c,r)$ for $i = 1, \cdots, k$) arising from the inverse branches of T. The P_i will be a partition of the interval into subintervals and the discs $P_i \subset U_i \subset \mathbb{C}$ extend into the complex plane.*

First parameter choice: Choose a real number $0 < \theta < 1$.

There is no canonical choice of θ and one can try to arrange the partition and the polydiscs so as to minimise the possible choices. This can be achieved either by trial and error, or by simple calculus.

In the next subsection we begin to elaborate on the description of the error bounds mentioned in subsection 1.4.

5.2 Bounds on the determinant coefficients

The key to obtaining validated estimates on characteristic values is to get accurate bounds on the coefficients a_n, especially for large $n \geq 1$.

Second parameter choice: Choose a natural number $N > 0$.

This should be chosen as large as is practicable. Typically this will depend on the time, computer memory and computing power available to compute periodic points. For ℓ contractions, to compute a_n for $1 \leq n \leq N$ would require making estimates on up to of the order of ℓ^N periodic points. This involves quantities β_k, t_m and B_k which we define below.

i) Let $\epsilon_1 = \epsilon_1(N) > 0$ be the corresponding error bound, by which we mean that for $1 \leq n \leq N$ the coefficient a_n can be computed to a guaranteed accuracy of no more than ϵ_1.

Third parameter choice: Choose a natural number $L > N$.

Typically this choice will depend on the time, computer memory and computing power available to compute integrals. This will involve us in numerically integrating (to appropriate precision) approximately $L - N$ integrals.

ii) Let $\epsilon_2 = \epsilon_2(L) > 0$ be the bound on the tail

$$\sum_{k=L}^{\infty} \|\mathcal{L}q_k\|^2 \leq \frac{C\theta^L}{1-\theta} < \epsilon_2$$

where

$$\|\mathcal{L}q_k\|^2 = r_i^{-2k} \int_0^1 |\sum_j \psi_j'(c_j + r_i e^{2\pi i t})(c_i + r_i \psi_j(e^{2\pi i t}))^k|^2 dt \leq C\theta^n$$

and where:

(a) $C = K^2 \max_j \|\psi_j'\|_\infty^s$; and

(b) $q_k(z) = (z - c_i)^k r_i^{-k}$, with $i = i(k) \pmod K$,

where K is the number of discs needed. (In the Bernoulli case we have the integral around the same curve).

iii) Let $\epsilon_3 = \epsilon_3(L, N) > 0$ be the bound on rigorous computational estimates $(\beta_k)_{k=N}^L$ such that

$$\beta_k := \|\mathcal{L}q_k\|^2 = r_i^{-2k} \int_0^1 \left| \sum_j \psi_j'(c_j + r_i e^{2\pi i t})(c_i + r_i \psi_j(e^{2\pi i t}))^k \right|^2 dt \text{ for } N < k < L$$

up to an error of ϵ_3.

We can now use these choices to define a sequence (t_m) for the next set of bounds.

Definition 5.4. *We can define a sequence of positive real numbers*

$$t_m := \begin{cases} \left(\sum_{k=m+1}^L \beta_k + (L-m)\epsilon_3 + \epsilon_2\right)^{1/2} & \text{for } m \leq L \\ C\theta^m & \text{for } m > L. \end{cases}$$

for $C > 0$ as above.

In particular, these numbers will tend to zero as $m \to \infty$. We can combine the values of t_n by introducing the following definition of B_k.

Definition 5.5. *For $1 \leq k \leq L$, define positive real numbers*

$$B_k := \sum_{m_1 < \cdots < m_k \leq L} t_{m_1} \cdots t_{m_k}.$$

Typically, the coefficients B_k will tend to zero quite quickly. Moreover, B_k is defined up to an error

$$\epsilon_4 := \epsilon_3 \left(\max\{t_i\}\right)^{L-1} L^k.$$

The quantities β_k, t_m and B_k are used in giving the following bounds on the coefficients a_n for the determinant.

Theorem 5.6 (Coefficient bounds)**.** *Let* $c = \frac{1}{\prod_{k=1}^{\infty}(1-\theta^j)}$*. Then*

1. *for* $N < n \leq L$ *we can bound,*

$$|a_n| \leq \gamma := c \sum_{k=1}^{n} (B_k + \epsilon_4)(\theta^L C)^{n-k}; \text{ and}$$

2. *for* $n > L$ *we can bound*

$$|a_n| \leq \xi(\theta^L C)^n \text{ where } \xi := c \left(\sum_{k=1}^{L} (B_k + \epsilon_4)(\theta^L C)^{-k} \right).$$

In particular, we get effective bounds for $|a_n|$ when $n > N$ (combining these different bounds for $N < n \leq L$ and $n > L$).

Remark 5.7. In part 1 of Proposition 5.6 we have used the simply proven bound

$$\sum_{r_1,\cdots,r_{n-k}=L}^{\infty} C^{n-k}\theta^{r_1+\cdots+r_{n-k}} \leq c(\theta^L C)^{n-k}.$$

The proof of Theorem 5.6 follows the same lines as the arguments in [25, 26, 27]. For the convenience of the reader we give a brief account of the underlying operator theory ideas in the proof in the Appendix.

5.3 Applying the bounds

There are two different ways that the bounds in Theorem 5.6 might be applied to estimate the accuracy of our approximations to the relevant characteristic values, depending on the quantity in question.

(a) Explicit values. Assume that we have an expression that can be written in terms of $d(z,t)$ and its derivatives $\frac{\partial^{i+j}}{\partial z^i \partial s^j} d(z,s)$ $(i, j \geq 0)$. The value of $d(z,t)$ can be approximated using the preceding estimates

$$\left| d(z,t) - \left(1 + \sum_{n=1}^{N} a_n(t) z^n \right) \right| \leq N|z|^N \epsilon_1 + \sum_{n=N+1}^{M} \gamma_n |z|^n + \xi \frac{|z|(\theta^L C)^L}{1 - |z|(\theta^L C)}.$$

More generally, we can bound

$$\left| \frac{\partial^{i+j}}{\partial z^i \partial s^j} d(z,s) - \sum_{n=1}^{N} z^n \frac{n!}{(n-i)!} \frac{\partial^i}{\partial s^i} a_n(s) \right|$$

where we bound $\frac{\partial^i}{\partial s^i} d(z,s)$ using Cauchy's theorem.

These estimates can be applied, for example, to computing the error terms in the estimates on Lyapunov exponents, variance and linear response as in subsection 3.3.

(b) Implicit values. If we are seeking a zero of $d(z,s)$ then provided we can choose $z_1 < z_2$ close and with validated bounds

$$d(z_1, s) < 0 < d(z_2, s) \text{ or } d(z_2, s) < 0 < d(z_1, s)$$

using $z = z_1$ or $z = z_2$, then there must be a zero in the interval $[z_0, z_1]$.

This approach can be used to estimate errors in computing the rates of mixing and Hausdorff dimension as described in section 4. We will illustrate this in a specific instance in the next section.

6 A worked example: Hausdorff dimension of the set E_2

To illustrate how Theorem 5.6 can be applied in practice, we want to consider a particular concrete problem. In particular, we will describe its use in estimating the Hausdorff dimension of a specific Cantor set (see [25]).

Recall from Example 4.9 that E_2 is the subset of $[0,1]$ consisting of those reals whose continued fraction expansion contains only the numbers 1 and 2. In other words, if

$$T_1(x) := \frac{1}{1+x} \quad \text{and} \quad T_2(x) := \frac{1}{2+x}$$

then E_2 is the corresponding limit set (i.e. the smallest non-empty closed set X such that $T_1(X) \cup T_2(X) = X$). Let us consider the estimation of the Hausdorff dimension of E_2, referring to [25] for full details.

Defining $\underline{i} = (i_1, \cdots, i_n) \in \{1,2\}^n$ and $|\underline{i}| = n$, and letting $x_{\underline{i}} = T_{\underline{i}}(x_{\underline{i}})$ be the fixed point for

$$T_{\underline{i}} = T_{i_1} \circ \cdots \circ T_{i_n} : [0,1] \to [0,1]\,,$$

we have the determinant

$$d(z,t) := \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{|\underline{i}|=n} \frac{|(T_{\underline{i}})'(x_{\underline{i}})|^t}{1 - (T_{\underline{i}})'(x_{\underline{i}})} \right),$$

and $\dim_H(E_2)$ is the value t such that $d(1,t) = 0$. This value is then approximated using the following four steps:

1. For each t approximate $z \mapsto d(z,t)$ by a polynomial $z \mapsto d_N(z,t)$;
2. Set $z = 1$ and consider $t \mapsto d_N(z,1)$;
3. Solve for $t_N = t$: $d_N(1,t) = 0$;
4. Then $t_N \to \dim_H(E_2)$ as $N \to +\infty$.

In particular, for each $N \in \mathbb{N}$ we obtain an approximation t_N to $\dim_H(E_2)$. The sequence t_N gives an intuitive estimate on the quality of the approximation in terms of the difference $|t_N - t_{N-1}|$.

We can write the series expansion

$$d(z,t) = 1 + \sum_{n=1}^{\infty} a_n(t) z^n = \underbrace{1 + \sum_{n=1}^{N} a_n(t) z^n}_{=: d_N(z,t)} + \underbrace{\sum_{n=N+1}^{\infty} a_n(t) z^n}_{=: \epsilon_N(z,t)}$$

for some $N \geq 1$, and take for the approximating polynomial

$$d_N(z,t) = 1 + \sum_{n=1}^{N} a_n(t) z^n$$

where N is chosen to be sufficiently large that (with $z = 1$ and $0 \leq t \leq 1$) the error ϵ_N is small, but sufficiently small that the terms $a_n(t)$, $n = 1, 2, \cdots, N$ can be calculated in a reasonable time. In the present setting one might choose $N = 25$, say.

We can explain part of the mechanism for effective estimates on the coefficients a_n $(n \geq 1)$ as follows.

Step 1. Choose $z_0 \in \mathbb{R}$ and $r > 0$ such that

$$D = \{z \in \mathbb{C} : |z - z_0| < r\} \supset [0,1] \text{ and } T_1 D, T_2 D \subset D.$$

For example, we could let $z_0 = 1$ and $r = \frac{3}{2}$. Consider the family of transfer operators defined on analytic functions $w : D \to \mathbb{C}$ by

$$\mathcal{L}_t w(z) = \frac{1}{(z+1)^{2t}} w\left(\frac{1}{z+1}\right) + \frac{1}{(z+2)^{2t}} w\left(\frac{1}{z+2}\right), \quad t \in \mathbb{R}.$$

Step 2. Let $q_k(z) = \frac{(z-z_0)^k}{r^k}$, for $z \in D$ and $k \geq 1$. We can then define

$$t_m = \left(\sum_{k=m-1}^{\infty} \|\mathcal{L}(q_k)\|^2\right)^{\frac{1}{2}} (m \geq 1) \text{ where } \|q_k\|^2 = \int_0^1 |q_k(z_0 + re^{2\pi i t})|^2 dt \quad (k \geq 1).$$

Step 3. We can bound the coefficients a_n $(n > N)$ by

$$|a_n| \leq \sum_{m_1 < \cdots < m_n} t_{m_1} t_{m_2} \cdots t_{m_n}.$$

We will give more details of the underlying operator theory in the appendix.

Given $M > 0$ (in the present setting one can choose $M = 600$) we can numerically estimate $\|\mathcal{L}(q_k)\|$ for $k \leq M$, and we can trivially bound $\|\mathcal{L}(q_k)\|$ for $k > M$. Combining these various bounds gives the estimate.

7 Future directions

In this note we have discussed how the periodic point algorithm works and how the error terms can be efficiently estimated. However, there may be further scope to fine-tune the underlying analysis and estimates, perhaps by using different Hilbert spaces or other operators. This leads to the first very general question.

Objective 7.1. *Can we improve the approach to computing the dynamical determinant and, in particular, the error estimates?*

We have illustrated the general approach with a number of applications and examples. However, we now want to propose some further potential applications.

7.1 Dynamical invariants and Lyapunov exponents

We have already discussed the theoretical use of our method to rigorously estimate certain dynamical quantities. However this has only been done practically in only a small number of cases (e.g., Lyapunov exponents, variance). However, it remains a task yet to be completed to explore how to estimate rigorously other quantities (e.g., resonances, linear response) for simple test examples (such as the Lanford map).

Objective 7.2. *Apply this approach to compute more dynamical invariants in simple examples.*

Lyapunov exponents also occur naturally in the theory of random matrix products, which has been an active area of research since the pioneering work of Kesten and Furstenberg in the 1960s. For example, given $k \times k$ square matrices $A_1, \cdots, A_k$ *with positive entries* we define the *Lyapunov exponent* by

$$\lambda = \lim_{n \to +\infty} \frac{1}{d^n} \sum_{i_1, \cdots, i_n \in \{1, \cdots, d\}} \frac{\log \|A_{i_1} \cdots A_{i_n}\|}{n}.$$

There was an implementation of the basic algorithm in [37],

Objective 7.3. *Can the error bounds in the computation of λ be made more effective?*

An interesting application is to the case of binary symmetric channels in information theory. In this context there are positive matrices and λ is related to a useful value called the Entropy Rate.

7.2 Connections to number theory

We have already mentioned the application to the density one Zaremba conjecture (see [26]). However, we now want to describe a different application to number theory.

Given an irrational number α we can associate the number

$$\mu(\alpha) = \limsup_{p,q \to +\infty} |q^2| \left| \alpha - \frac{p}{q} \right|$$

(i.e., the best constant in diophantine approximation for α). The *Lagrange spectrum* is defined to be the set $\mathcal{L} = \{1/\mu(\alpha) : \alpha \in \mathbb{R} - \mathbb{Q}\}$. On the other hand, we can consider those binary quadratic forms $f(x, y) = ax^2 + bxy + cy^2$ $(a, b, c \in \mathbb{R})$ with discriminant $D(f) = b^2 - 4ac > 0$ and denote

$$\lambda(f) = \inf \left\{ |f(x, y)| : (x, y \in \mathbb{Z}^2 - \{(0, 0)\}) \right\} / \sqrt{D(f)}.$$

The *Markov spectrum* is defined to be the set $\mathcal{M} = \{1/\lambda(f) : \alpha \in \mathbb{R} - \mathbb{Q}\}$. It is known that $\mathcal{L} \subset \mathcal{M}$ and Matheus & Moreira showed that $0.513 \cdots < \dim_H(\mathcal{M} - \mathcal{L}) < 0.98 \cdots$ [35]. Moreover, in their article they conjecture that the bounds can be improved to $\dim_H(\mathcal{M} - \mathcal{L}) < 0.88$, based on empirical estimates using our algorithm.

Objective 7.4. *Obtain improved rigorous bounds on $\dim_H(\mathcal{M} - \mathcal{L})$.*

This involves rigorously computing the Hausdorff dimension of limit sets associated to iterated function systems $\{\phi_i : I \to I\}$, but with a Markov condition, i.e., there is a 0-1 matrix A and compositions $\phi_i \circ \phi_j$ are only allowed if $A(i, j) = 1$. Whereas the basic algorithm still applies in this setting, the major complication is to get effective error estimates.

7.3 Spectral geometry and the Selberg zeta function

Given a compact surface V, with a metric ρ of constant negative curvature, we can associate the Selberg zeta function defined by

$$Z_\rho(s) = \prod_{n=0}^{\infty} \prod_{\gamma} \left(1 - e^{-s(n+s)l(\gamma)}\right), \quad s \in \mathbb{C},$$

where γ denotes a closed geodesic of length $l(\gamma)$. To relate this to our analysis, we recall that we can associate a piecewise C^ω expanding map of the circle (using the Bowen-Series approach) and then the zeta function can be written in terms of the determinants $\det(I - \mathcal{L}_s)$ of the associated transfer operators $\mathcal{L}_s$ (see e.g. [36, 40]).

The zeros of $Z_\rho(s)$ have a spectral interpretation, in terms of eigenvalues of the Laplacian on (V, ρ), but these can be well estimated using other techniques. Other special values such as $Z_\rho'(0)$ can be written in terms $\frac{d}{ds}\det(I - \mathcal{L}_s)|_{s=0}$ and this is proportional to the much studied *determinant of the Laplacian*, originally defined in terms of the spectrum of the Laplacian (see e.g. [8, 14, 44]).

Objective 7.5. *Can we get useful bounds on $Z_\rho'(0)$?*

There is a well-known problem of Sarnak to show that there is a (local) minimum for the determinant that occurs at very symmetric hypergeometric surfaces which could be addressed using this approach.

In a different direction, the Weil-Petersson metric is a classical distance on the space of such Riemann metrics ρ. There is a particularly useful thermodynamic interpretation for the Weil-Petersson metric due to McMullen in terms of the second derivative of the associated thermodynamic pressure in the context above of the piecewise C^ω expanding maps.

Objective 7.6. *Can one get effective estimates on the Weil-Petersson metric?*

This could then be used to explore empirically Weil-Petersson metric on the space of metrics. Moreover, there are higher dimensional analogues of Weil-Petersson metrics in [5] which could be similarly analysed.

A Appendix: Operator theory and coefficient bounds

We now give a little more detail on how the bounds on the coefficients a_n arise using operator theory.

A.1 Operator theory

Assume for notational convenience that $T : X \to X$ is a Bernoulli expanding map of the unit interval X. Assume that we can choose an open disc $X \subset U \subset \mathbb{C}$ so that T extends analytically to U. In the particular case that $D = \{z \in \mathbb{C} : |z - z_0| < r\}$ then we can let $\mathcal{H}$ be the Hardy Hilbert space of analytic functions on U for which the norm $\|\cdot\|$ is given by

$$\|f\|^2 = \sup_{\rho < r} \int_0^1 |f(z_0 + \rho e^{2\pi i t})|^2 dt < \infty, \quad f \in \mathcal{H}.$$

Example A.1. *If $D = \left\{z \in \mathbb{C} : |z - \frac{1}{2}| < 1\right\}$ then $f \in \mathcal{H}$ presented in the form*

$$f(z) = \sum_{n=0}^{\infty} b_n \left(z - \frac{1}{2}\right)^n$$

has norm $\|f\|^2 = \sum_{n=0}^{\infty} |b_n|^2$.

Let $\mathcal{L} = \mathcal{L}_{G,T} : \mathcal{H} \to \mathcal{H}$ be a transfer operator. We can then assume that $\mathcal{L}$ is a trace class operator and a simple Lidskii-type identity relates periodic orbits to the spectrum

$$\text{trace}(\mathcal{L}^n) = \sum_{T^n x = x} \frac{\exp\left(\sum_{i=0}^{n-1} G(T^i x)\right)}{1 - 1/(T^n)'(x)}$$

and then

$$D(z) = \det(I - z\mathcal{L}) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \text{trace}(\mathcal{L}^n)\right)$$

where both sides depend on G and T [39]. The basic idea is to use the operator description of $D(z)$ to get estimates on the a_n via the approximation numbers s_l $(l \geq 1)$ defined by

$$s_l := \inf\{\|\mathcal{L} - K\| : K \text{ bounded linear operator of finite rank } l\}, \quad l \geq 1\,.$$

For definiteness, let $(e_n)_{n=0}^{\infty}$ be a complete orthonormal family for $\mathcal{H}$. For any

$$f = \sum_{n=0}^{\infty} c_n e_n \in \mathcal{H} \text{ with } \|f\| = \sqrt{\sum_{n=0}^{\infty} |c_n|^2} = 1$$

and each $l \geq 1$ we can define $K_l f = \sum_{n=0}^{l-1} c_n \mathcal{L}(e_n) \in \mathcal{H}$. In particular, we can bound

$$\begin{aligned} s_l \leq & \|(\mathcal{L} - K_l)f\| \leq \sum_{n=l}^{\infty} |c_n| \|\mathcal{L}(e_n)\| \\ & \leq \sqrt{\sum_{l=n}^{\infty} |c_n|^2} \sqrt{\sum_{n=l}^{\infty} \|\mathcal{L}(e_n)\|^2} \leq \sqrt{\sum_{n=l}^{\infty} \|\mathcal{L}(e_n)\|^2} \end{aligned} \tag{A1}$$

using the Cauchy-Schwarz inequality.

Example A.2 (Example A.1 revisited). *If $D = \{z \in \mathbb{C} : |z - \frac{1}{2}| < 1\}$ then we can take $e_n(z) = (z - \frac{1}{2})^n / \sqrt{n}$, for $n \geq 0$.*

We can now bound the coefficients a_n $(n > N)$ by

$$|a_n| \leq \sum_{m_1 < \cdots < m_n} s_{m_1} s_{m_2} \cdots s_{m_n}, \tag{A2}$$

where the summation is over distinct natural numbers $m_1, \cdots, m_n \in \mathbb{N}$ such that $m_1 < \cdots < m_n$, using [15, Cor. VI.2.6].

Given $M > 0$ (in the present setting one can choose $M = 600$) we can numerically estimate $\|\mathcal{L}(e_k)\|$ for $k \leq M$, and we can trivially bound $\|\mathcal{L}(e_k)\|$ for $k > M$.

A.2 Basic convergence

The basic convergence (which follows from work of Ruelle [40], after Grothendieck) only requires the following bound (referred to in [25, 26, 27] as an *Euler bound*, after [9]):

If there exists $C > 0$ and $0 < r < 1$ with $s_l \leq C r^l$ for $l \geq 1$ then

$$|a_n| \leq \frac{C^n r^{n(n+1)/2}}{(1-r)(1-r^2)\cdots(1-r^n)} = O(r^{n(n+1)/2}) \tag{A3}$$

for $n \geq 1$.

In particular, the coefficients a_n tend to zero fast enough to ensure absolute convergence of the series describing the various quantities of interest (cf. Part 1 of Corollary 3.9 and Corollary 3.11).

A.3 Better estimates

Assume that we can compute numerically $a_1, \cdots, a_N$ for some N. We could then use the above bound (A3) to bound the terms a_n for $n > N$ (although this wasn't the original purpose of this approach nor is it necessarily a very efficient approach). The basis of the improved approach involves:

1. again computing $a_1, a_2, \cdots, a_N$ using the periodic points of period at most N (where N is chosen depending on the limitations of our computer).
2. bounding $|a_n|$, for $n \geq N$ using (A2) and different bounds on the s_l:
   (a) for $N < l \leq M$ we can bound s_l using (A1) (where M is chosen depending on the limitations of our computer); and
   (b) For $M < l$ we can use the bound $s_l \leq Cr^l$.

There are further refinements possible, but this explains the gist of the approach (see [25]).

A.4 Generalisations

Finally, we can describe a plan for how this method can be extended to Anosov diffeomorphisms and flows.

If we have a C^ω Anosov diffeomorphism T then it is more appropriate to write

$$D(z) = D_{G,T}(z) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{T^n x = x} \frac{\exp\left(\sum_{i=0}^{n-1} G(T^i x)\right)}{\det(I - (DT)^{-1}(x))}\right), \quad z \in \mathbb{C},$$

An extra feature here is that we only need to take $G_t(x) = tg(x)$ because of the contributions from $\det(I - (DT)^{-1}(x))$.

One can adapt this approach using either the less fashionable device of Markov partitions and the approach of Rugh, or using anisotropic spaces of functions when applicable (for example on the torus in the context of Faure-Roy). The key point is simply to find a setting to which the functional analysis applies.

For C^ω Anosov flows it is perhaps simpler to use Markov Poincaré sections to accommodate the operator theory being used.

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