General Large Deviations Theorems and their Applications

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X-compact metric space, $\mathcal{P}(X)$-probability measures on X with topology of weak convergence, $(\Omega_t, \mathcal{F}_t, P_t)$ – family of probability spaces: $t \in \mathbb{Z}$ or $t \in \mathbb{R}$, $\zeta^t : \Omega_t \to \mathcal{P}(X)$, $\zeta^t(\omega) = \zeta^t_\omega$ –family of measurable maps, and $r(t) \to \infty$ as $t \to \infty$.

Upper large deviation bound:

$$\limsup_{t\to\infty}(1/r(t)) \log P_t\{\zeta^t \in K\} \leq -\inf\{I(\nu) : \nu \in K\}, \; \forall K \text{ closed } \subset \mathcal{P}(X).$$

Lower large deviation bound:

$$\liminf_{t\to\infty}(1/r(t)) \log P_t\{\zeta^t \in G\} \geq -\inf\{I(\nu) : \nu \in G\}, \; \forall G \subset \mathcal{P}(X) \text{ open}.$$

Here I is a lower semi continuous convex functional to be identified.

General fact (Donsker-Varadhan): If both LD bounds hold true then $I(\mu) = \sup_{V\in C(X)}\left(\int V d\mu - Q(V)\right)$ if $\mu \in \mathcal{P}(X)$ where

$$Q(V) = \lim_{t\to\infty}(1/r(t)) \log \int \exp(r(t) \int V(x) d\zeta^t_\omega) dP_t(\omega).$$

It is called usually 2nd level of large deviations if

$$\zeta_\omega^t = \frac{1}{t}\int_0^t \delta_{Y_s(\omega)} ds, \ \zeta_\omega^t = \frac{1}{t}\sum_{s=0}^{t-1} \delta_{Y_s(\omega)}$$

where Y_s is a stochastic process, in particular, a dynamical system $Y_s(\omega) = F^s\omega$. For statistical mechanics applications;

$$\zeta_\omega^a = \frac{1}{|D(a)|}\sum_{q\in D(a)} \delta_{\theta_q\omega}$$

where $X = \Omega$ is a compact subset of $Q^{\mathbb{Z}^d}$, Q-finite set, $(\theta_q\omega)_m = \omega_{m+q}$, $\omega = (\omega_m,\ m \in \mathbb{Z}^d),\ D(a) = \{m : 0 \le m_i < a_i\}$.

A general theorem on large deviations

Theorem

Suppose that the limit

$$Q(V) = \lim_{t\to\infty}(1/r(t))\log\int\exp(r(t)\int V(x)d\zeta^t_\omega)dP_t(\omega)$$

exists for any $V \in \mathcal{C}(X)$. *Then the upper large deviations bound holds true with*

$$I(\mu) = \sup_{V\in C(X)}\left(\int Vd\mu - Q(V)\right) \text{ if } \mu \in \mathcal{P}(X)$$

and $I(\mu) = \infty$ *otherwise. If, in addition, there exists a sequence* $V_1, V_2, \dots, \in \mathcal{C}(X)$ *such that* $\overline{span\{V_i\}} = \mathcal{C}(X)$, *and for* $\forall n$, *any numbers* $\beta_1, \dots, \beta_n$, *and every function* $V = \beta_1 V_1 + \cdots + \beta_n V_n$ *there exists a unique measure* $\mu_V \in \mathcal{P}(X)$ *satisfying*

$$Q(V) = \int Vd\mu_V - I(\mu_V) = \sup_{\mu\in\mathcal{P}(X)}\left(\int Vd\mu - I(\mu)\right)$$

then the lower large deviation bound holds true, as well.

Convex analysis fact: Uniqueness of maximizing measure μ_V for $V = V(\beta_1, ..., \beta_n)$ is equivalent to differentiability of $Q(V)$ in $\beta = (\beta_1, ..., \beta_n)$.

Let $\Psi : \mathcal{P}(X) \to Y$ be continuous, Y be a Hausdorff space, ζ^t be as in General theorem. Then

$$P\{\Psi\zeta^t \in U\} = P\{\zeta^t \in \Psi^{-1}U\}$$

for $U \subset Y$ and $P \in \mathcal{P}(X)$. If LD holds true for ζ^t with the rate functional I then LD holds true for $\Psi\zeta^t$ with the rate $J(y) = \inf\{I(\nu) : \nu \in \Psi^{-1}y\}$. For occupational measures $\zeta^t_\omega = \frac{1}{t}\int_0^t \delta_{F^s\omega} ds$, $\omega \in X$ or $\zeta^t_\omega = \frac{1}{t}\sum_{s=0}^{t-1} \delta_{F^s\omega}$ and $\Psi\nu = \int g d\nu$ for some fixed continuous g we obtain LD for Cesáro averages $\frac{1}{t}\int_0^t g \circ F^s ds$ or $\frac{1}{t}\sum_{s=0}^{t-1} g \circ F^s$ with the rate function $J(y) = \inf\{I(\nu) : \int g d\nu = y\}$–called 1st level of LD. Can be obtained directly: if the limit

$$Q(\lambda) = \lim_{t\to\infty} \frac{1}{t} \log \int \exp(\lambda \sum_{s=0}^{t-1} g \circ F^s) dP$$

exists and is differentiable in λ then LD for $\frac{1}{t}\sum_{s=0}^{t-1} g \circ F^s$ holds true with the rate $J(y) = \sup_\lambda(\lambda y - Q(\lambda))$ (and similarly for $\frac{1}{t}\int_0^t g \circ F^s ds$).

Let $f : X \circlearrowleft$ be a continuous map of a compact metric space X and $B_x(\delta, n) = \{y : \max_{0 \le i \le n} dist(f^i x, f^i y) \le \delta\}$.

Proposition

Suppose that $m \in \mathcal{P}(X)$ and for some $\varphi \in C(X)$ and for all $n, \delta > 0,\ x \in X$ one has

$$((A_\delta(n))^{-1} \le m(B_x(\delta, n)) \exp\left(-\sum_{i=0}^{n-1} \varphi(f^i x)\right) \le A_\delta(n)$$

where $A_\delta(n) > 0$ satisfies

$$\lim_{n \to \infty} \frac{1}{n} \log A_\delta(n) = 0\,.$$

Then for any $V \in \mathcal{C}(X)$,

$$\lim_{n \to \infty} \frac{1}{n} \log \int_X \exp\left(n \int V d\zeta_x^n\right) dm(x) = \Pi(\varphi + V)$$

where Π is the topological pressure and $n \int V d\zeta_x^n = \sum_{i=0}^{n-1} V(f^i x)$.

Proposition implies upper bound of LD:

By the variational principle:

$$Q(V) = \Pi(\varphi + V) = \sup_{\mu - f\text{-invariant}} \left(\int V d\mu + \left(\int \varphi d\mu + h_\mu(f) \right) \right)$$
$$= \sup_{\mu \in \mathcal{P}(X)} \left(\int V d\mu - I(\mu) \right)$$

where

$$I(\mu) = \begin{cases} -\displaystyle\int \varphi d\mu - h_\mu(f) \text{ if } \mu \text{ is } f\text{-invariant}, \\ \infty \text{ otherwise.} \end{cases}$$

By Proposition, $0 = \Pi(\varphi) \geq \int \varphi d\mu + h_\mu(f)$, and so $I(\mu) \geq 0$. If h_μ is upper semicontinuous and since h_μ is affine then $I(\mu)$ is convex and lower semicontinuous and we have upper large deviation bound with the rate $I(\mu)$. Thus if $\zeta_x^n = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i x}$ then $\overline{\lim}_{n\to\infty} n^{-1} \log m\{x : \zeta_x^n \in K\} \leq -\inf_{\mu \in K} I(\mu)$ for any closed $K \subset \mathcal{P}(M)$.

Examples: subshifts of finite type, expanding and hyperbolic transformations

- Subshift of finite type: Here $X \subset \{1, ..., s\}^{\mathbb{N}}$, $(fx)_i = x_{i+1}$, $x = (x_0, x_1, ...)$ is the left shift, $X = \{x = (x_0, x_1, ...), \gamma_{x_i, x_{i+1}} = 1\}$, $\forall i \geq 0$ where $\Gamma = (\gamma_{ij}, i, j = 1, ..., s\}$ is a $s \times s$ matrix with 0 and 1 entries. Define the metric $d(x, \tilde{x}) = \exp(-\min\{i \geq 0 : x_i \neq \tilde{x}_i\})$. Then $B_x(\delta, n) = [x_0, ..., x_n] = \{\tilde{x} \in X : \tilde{x}_i = x_i \, \forall i \leq n\}$ provided $\delta = e^{-1}$. If m is a Gibbs measure with a continuous potential φ then conditions of Proposition are satisfied, and so upper LD bound holds true. If φ is Hölder continuous then by uniqueness of equilibrium states (maximizing measures in the variational principle) we see from the general theorem that lower LD bound holds true, as well.
- Expanding and Axiom A transformations: Here we can take as m not only Gibbs measures but in view of the Bowen-Ruelle volume lemma we can take m to be the normalized Riemannian volume. The function φ will be here the differential expanding coefficient (on the unstable subbundle: $\varphi(x) = -\ln|\text{Jac}_x^u f|$).

Let $f :\circlearrowleft$ be as in either of examples above. Let $\mathcal{O}_n$ be the set of all periodic points of f with period n, $\mathcal{O} = \cup_n \mathcal{O}_n$ and τ_x be the least period of $x \in \mathcal{O}$. Set $\zeta_x = \tau_x^{-1} \sum_{i=1}^{\tau_x} \delta_{f^i x}$ and $\mu_n = N_n^{-1} \sum_{x \in \mathcal{O}_n} \zeta_x$ where $N_n = \#\mathcal{O}_n$. Bowen: $\mu_n \Rightarrow \mu_{\max}$ as $n \to \infty$, $\mu_{\max}$ is the measure of maximal entropy: equidistribution of periodic orbits. For $\Gamma \subset \mathcal{O}$ let $\nu_n(\Gamma) = N_n^{-1} \#(\Gamma \cap \mathcal{O}_n)$. Then for any $g \in \mathcal{C}(M)$ (Bowen),

$$\lim_{n\to\infty} n^{-1} \log \int_{\mathcal{O}} \exp(n \int g d\zeta_x) d\nu_n(x)$$
$$= \lim_{n\to\infty} n^{-1} \log \left(N_n^{-1} \sum_{x \in \mathcal{O}_n} \exp \sum_{i=0}^{n-1} g(f^i x) \right)$$
$$= -h_{\text{top}}\ (f) + \Pi(g) = \sup_{\mu \in \mathcal{P}(M)} \left(\int g d\mu - \tilde{I}(\mu) \right)$$

where $\tilde{I}(\mu) = h_{\text{top}} - h_\mu$ if μ is f-invariant and $\tilde{I}(\mu) = \infty$, otherwise. General theorem and uniqueness of equilibrium states for Hölder functions yield

$$\limsup_{n\to\infty}\ n^{-1} \log \nu_n\{x \in \mathcal{O} : \zeta_x \in K\} \le -\inf\{\tilde{I}(\mu), \mu \in K\} \text{ and}$$
$$\liminf_{n\to\infty}\ n^{-1} \log \nu_n\{x \in \mathcal{O} : \zeta_x \in G\} \ge -\inf\{\tilde{I}(\mu) : \mu \in G\}$$

for any closed $K \subset \mathcal{P}(M)$ and open $\mathcal{G} \subset \mathcal{P}(M)$.

Large deviations from equidistribution in continuous time

Let $f^t : M \to M$ be a group of homeomorphisms of a compact metric space. Let CO be the set of all closed (periodic) orbits and $CO_\delta(t) \subset CO$ orbits with some period in the interval $[t-\delta, t+\delta]$. Let $\tau(\gamma)$ denotes the least period of $\gamma \in CO$. Set $\zeta_\gamma = (\tau(\gamma))^{-1} \int_0^{\tau(\gamma)} \delta_{f^s x} ds$ in the continuous time case and $\zeta_\gamma = (\tau(\gamma))^{-1} \sum_{i=1}^{\tau(\gamma)} \delta_{f^i x}$ in the discrete time case. Let $\mu_{t,\delta} = N_{t,\delta}^{-1} \sum_{\gamma \in CO_\delta(t)} \zeta_\gamma$ where $N_{t,\delta} = \#\{CO_\delta(t)\}$ is the number of elements in $CO_\delta(t)$. Bowen: under general conditions of expansiveness and specification $\mu_{t,\delta}$ weakly converges as $t \to \infty$ to the measure $\mu_{\max}$ with maximal entropy for f^t. For $\Gamma \subset CO$ set $\nu_{t,\delta}(\Gamma) = N_{t,\delta}^{-1} \# \{\Gamma \bigcap CO_\delta(t)\}$. Then $\lim_{t\to\infty} \nu_{t,\delta} \{\gamma \in CO : \zeta_\gamma \notin U_{\mu_{\max}}\} = 0$ for any neighborhood $U_{\mu_{\max}}$ of $\mu_{\max}$. A more precise statement is obtained via LD. Namely, for any $\delta > 0$ small enough,

$$\limsup_{t\to\infty} t^{-1} \log \nu_{t,\delta} \{\gamma \in CO : \zeta_\gamma \in K\} \leq -\inf\{I(\mu) : \mu \in K\}$$

for any closed $K \subset \mathcal{P}(M)$ while for any open $G \subset \mathcal{P}(M)$,

$$\liminf_{t\to\infty} t^{-1} \log \nu_{t,\delta} \{\gamma \in CO : \zeta_\gamma \in G\} \geq -\inf\{I(\mu) : \mu \in G\}$$

where $I(\mu) = h_{\text{top}}(f^1) - h_\mu(f^1)$ if $\mu \in \mathcal{P}(M)$ is f^t-invariant and $= \infty$, otherwise. In particular, this yields bounds of large deviations from the equidistribution for closed geodesics on negatively curved manifolds.

Let Γ be a finite directed graph with vertices $V = \{1, ..., m\}$ and an adjacency matrix $B = (b_{ij})_{0\leq i,j\leq m}$, i.e. $b_{ij} = 1$ iff an arrow goes from i to j. Now the space of paths of length n has the form

$$X_B(n) = \{x = (x_0, x_1, ..., x_n);\ x_i \in V\ \forall i = 0, ..., n \text{ and } b_{x_i x_{i+1}} = 1\ \forall i = 0, ..., n-1\}$$

with $X_B = X_B(\infty) \subset V^{\mathbb{N}}$ taken with the product topology. Let $\Pi_n(a, b)$ be the set of all $(x_0, x_1, ..., x_n) \subset X_B(n)$ with $x_0 = a$ and $x_n = b$, $\Pi_n = \cup_a \Pi_n(a, a)$ and $\zeta_x^n = \frac{1}{n}\sum_{i=0}^{n-1}\delta_{\sigma^i x}$. Then for any continuous function g on X_B,

$$\lim_{n\to\infty}\frac{1}{n}\log\int(\exp\int g d\zeta_x^n)d\eta_n(x) = \lim_{n\to\infty}\frac{1}{n}\log\big(|\Pi_n(a,b)|^{-1}$$
$$\times\sum_{\alpha\in\Pi_n(a,b)} e^{S_n g(x_\alpha^{(n)})}\big) = \sup_{\nu\in\mathcal{I}_B}\big(\int g d\nu - (h_{\text{top}} - h_\nu(\sigma))\big).$$

where $\mathcal{I}_B \subset \mathcal{P}(X_B)$ is the space of shift invariant measures. Then for any $a, b \in V$,

$$\limsup_{n\to\infty}\frac{1}{n}\log\big(|\Pi_n(a,b)|^{-1}|\{\alpha\in\Pi_n(a,b) : \zeta_{x_\alpha}^n \in K\}|\big) \leq -\inf_{\nu\in K} I(\nu)$$

for each closed $K \subset \mathcal{P}(X_B)$ while for each open $U \subset \mathcal{P}(X_B)$,

$$\liminf_{n\to\infty}\frac{1}{n}\log\big(|\Pi_n(a,b)|^{-1}|\{\alpha\in\Pi_n(a,b) : \zeta_{x_\alpha}^n \in U\}|\big) \geq -\inf_{\nu\in U} I(\nu)$$

where $x_\alpha, \alpha = (\alpha_0, ..., \alpha_n)$ is in the cylinder C_α, $I(\nu) = h_{\text{top}}(\sigma) - h_\nu(\sigma)$ if $\nu \in \mathcal{I}_B$ and $I(\nu) = \infty$, otherwise. Remains true for Π_n in place of $\Pi_n(a, b)$.

Here Y_n is a Markov chain on a compact X with transition probabilities $P(x, dy)$ satisfying Doeblin's condition and $T_V g(x) = \int e^{V(y)} g(y) P(x, dy)$. Then

$$\log \lambda_V = \lim_{n\to\infty} n^{-1} \log T_V^n 1(x) = \lim_{n\to\infty} n^{-1} \log E_x \exp\left(\sum_{k=1}^{n} V(Y_k)\right)$$

where λ_V is the principal eigenvalue of T_V. Knowing that $\log \lambda_V$ satisfies the Donsker-Varadhan variational formula and uniqueness of the maximizing measure μ_V in this formula for each $V \in \mathcal{C}(X)$ we derive for $\zeta_\omega^n = n^{-1} \sum_{k=1}^n \delta_{Y_k(\omega)}$ and $x \in X$,

$$\limsup_{n\to\infty} n^{-1} \log P_x\{\omega : \zeta_\omega^n \in K\} \le -\inf\{I(\mu), \mu \in K\} \text{ and}$$
$$\liminf_{n\to\infty} n^{-1} \log P_x\{\omega : \zeta_\omega^n \in G\} \ge -\inf\{I(\mu), \mu \in G\}$$

for any closed $K \subset \mathcal{P}(X)$ and open $G \subset \mathcal{P}(X)$, where

$$I(\mu) = -\inf_{u>0} \int \log\left(\frac{Pu}{u}\right) d\mu, \qquad Pu(x) = \int u(y) P(x, dy).$$

We consider a difference equation

$$x^\epsilon_{k+1} - x^\epsilon_k = \epsilon B(x^\epsilon_k, F^k\omega),\ x^\epsilon_0 = x \in \mathbb{R}^d$$

where $B(x,\omega)$ is a bounded Lipschitz in x and continuous in ω vector field on $\mathbb{R}^d \times \Omega$, $F : \Omega \to \Omega$ is continuous. First, we study LD on the 2nd level, namely, for

$$\zeta^{\epsilon,T} = \zeta^{\epsilon,T}_\omega = T^{-1}\int_0^T \delta_{(s,F^{[s\epsilon^{-1}]}\omega)} ds$$

which is a probability measure on $[0\,T] \times \Omega$. By the General Theorem we have to consider the limit

$$Q_F(V) = \lim_{\epsilon\to 0} \epsilon \log \int_\Omega \exp(\epsilon^{-1}T^{-1}\int_0^T V(t, F^{[t\epsilon^{-1}]}\omega)dt)dP(\omega)$$

for any $V_t(\omega) = V(t,\omega) \in \mathcal{C}([0, T] \times \Omega)$. For the same classes of dynamical systems as we obtained LD above we get that

$$Q_F(V) = \int_0^T \Pi(\varphi + T^{-1}V_t)dt = \sup_{\eta\in\mathcal{P}([0,T]\times M)} (\int V d\eta - I_{0T}(\eta))$$

where φ is the potential of the Gibbs measure P (or P is the normalized volume on Ω), $I_{0T}(\eta) = \int_0^T I(\eta_t)dt$ if $d\eta = T^{-1}d\eta_t dt$ and $= \infty$, otherwise, while $I(\eta_t) = -\int \varphi d\eta_t - h_{\eta_t}(F)$ if η_t is F-invariant and $= \infty$, otherwise.

We obtain LD for time changed sequences $z_t^\varepsilon = x_{[t/\varepsilon]}^\varepsilon$. let $w_t^\varepsilon(\omega) = x + \int_0^t B(w_s^\varepsilon(\omega), F^{[s/\varepsilon]}\omega)ds$. Then we get by induction that for some $C > 0$,

$$\max_{0 \leq k \leq T\varepsilon^{-1}} |z_{k\varepsilon}^\varepsilon - w_{k\varepsilon}^\varepsilon| \leq C2^{CT}\varepsilon$$

and so we can deal with w^ε in place of z^ε. Next, we apply a contraction principle argument. On the subspace $\mathcal{M} \subset \mathcal{P}([0, T]) \times \Omega)$ of measures μ such that $d\mu = T^{-1} d\mu_t dt$, $\mu_t \in \mathcal{P}(M)$, $t \in [0, T]$ define the map $\Psi_x\, \mathcal{M} \rightarrowtail \mathcal{C}_{0T}(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ by $\varphi = \Psi_x \mu$ with

$$\varphi = x + \int_0^t \int_\Omega B(\varphi_s, \omega) d\mu_s(\omega) ds$$

which is well defined since B is Lipshitz continuous in the first variable, where $\mathcal{C}_{0T}(\mathbb{R}^d) = \{\varphi : \varphi_t \in \mathbb{R}^d$ continuous in $t \in [0, T]\}$. Then Ψ is a continuous map if one takes the topology of weak convergence on $\mathcal{P}([0, T] \times \Omega)$ and the metric

$$\rho_{0T}(\varphi, \tilde{\varphi}) = \sup_{0 \leq t \leq T} \text{dist}(\varphi_t, \tilde{\varphi}_t)$$

on $\mathcal{C}_{0T}(\mathbb{R}^d)$.

Clearly, $w^\epsilon = \Psi_x \zeta^{\epsilon,T}$ and we obtain the large deviations bounds for w^ϵ with the rate functional

$$S_{0T}(\varphi) = \inf_\eta \{I_{0T}(\eta) :\ \Psi_x \eta = \varphi\}$$

This functional can be written also in the following form

$$S_{0T}(\varphi) = \int_0^T \inf\{I(\mu) :\ d\varphi_t/dt = \bar{B}_\mu(\varphi_t),\ \mu \in \mathcal{P}(M)\} dt.$$

where $\bar{B}_\mu(x) = \int B(x,y) d\mu(y)$. If we set
$\Phi^a_{0T} = \{\varphi \in \mathcal{C}_{0T}(\mathbb{R}^d) :\ \varphi_0 = x, S_{0T}(\varphi) \le a\}$ then these large deviations bounds can be written in the form: for any $a, \beta, \lambda > 0$, each $\delta > 0$ small enough, and every $\varphi \in \mathcal{C}_{0T}(\mathbb{R}^d)$, $\varphi_0 = x$ there exists $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$,

$$P\{\rho_{0T}(w^\epsilon, \varphi) < \beta,\ w_0^\epsilon = x\} \ge \exp\left\{-\frac{1}{\epsilon}(S_{0T}(\varphi) + \lambda)\right\}$$

and

$$P\{\rho_{0T}(w^\epsilon, \Phi^a_{0T}(x)) \ge \beta,\ w_0^\epsilon = x\} \le \exp\left\{-\frac{1}{\epsilon}(S_{0T}(\varphi) + \lambda)\right\}.$$

Another approach to the lower bound

Used without uniqueness of equilibrium state as for $\mathbb{Z}^d$-actions, $d > 1$. Let $(\Omega, \mathcal{B})$ be a measurable space and $\mathcal{P}(\Omega)$ be the space of probability measures defined on $\mathcal{B}$. Suppose that $\mathcal{F} \subset \mathcal{B}$ and $\nu, \mu \in \mathcal{P}(\Omega)$. Define the Kullback-Leibler information by

$$H^{\mathcal{F}}(\nu|\mu) = \mu(p^{\mathcal{F}}_{\nu,\mu} \log p^{\mathcal{F}}_{\nu,\mu})$$

if $\nu \overset{\mathcal{F}}{\prec} \mu$ with Radon-Nikodym derivative $p^{\mathcal{F}}_{\nu,\mu}$ and $H^{\mathcal{F}}(\nu|\mu) = \infty$, otherwise. Let $\zeta^n : \Omega \to \mathcal{P}(\Omega)$, $n = 1, 2, \ldots$ be a sequence of measurable maps where $\mathcal{P}(\Omega)$ is taken with some measurable structure.

Proposition

Suppose that there exists a measurable set $U \subset \mathcal{P}(\Omega)$ and a sequence of σ-algebras $\mathcal{F}_n \subset \mathcal{B}$, $n = 1, 2, \ldots$ such that $\{\omega : \zeta^n_\omega \in U\} \in \mathcal{F}_n$ for all $n = 1, 2, \ldots$ and

$$\lim_{n\to\infty} \nu\{\zeta^n \in U\} = 1 .$$

If $r(n) \to \infty$ as $n \to \infty$ and

$$h = \limsup_{n\to\infty} (r(n))^{-1} H^{\mathcal{F}_n}(\nu|\mu)$$

then

$$\liminf_{n\to\infty} (r(n))^{-1} \log \mu\{\zeta^n \in U\} \geq -h.$$

Let $f : X \circlearrowleft$, ν be a f-invariant ergodic measure and $\zeta_x^n = n^{-1} \sum_{i=0}^{n-1} \delta_{f^i x}$. Then by Birkhoff's ergodic theorem $\nu\{x : \zeta_x^n \in \mathcal{U}\} \to 1$ as $n \to \infty$ for any open $\mathcal{U} \supset \nu$. Let ξ be a finite partition and $\mathcal{F}_n$ be the finite algebra generated by $\xi^n = \bigvee_{k=0}^{n-1} f^{-k}\xi$. Then

$$H^{\mathcal{F}_n}(\nu|\mu) = \sum_{A \in \xi^n} \nu(A) \log \nu(A) - \sum_{A \in \xi^n} \nu(A) \log \mu(A).$$

If $\mu(A) \sim \text{const} \exp\left(\sum_{i=0}^{n-1} \varphi(f^i x)\right)$ for any $x \in A$, $A \in \xi^n$, and $n > 0$ (Gibbs measure) for certain continuous functions φ then

$$\lim_{n\to\infty} n^{-1} H^{\mathcal{F}_n}(\nu|\mu) = -h_\nu(f) - \int \varphi d\nu\,.$$

If, in addition, for any f-invariant measure ν there exists a sequence of ergodic f-invariant measures ν_n which converge weakly to ν and $h_{\nu_n}(f) \to h_\nu(f)$ as $n \to \infty$ then it follows from Proposition above that for any open $\mathcal{U} \subset \mathcal{P}(X)$,

$$\liminf_{n\to\infty} n^{-1} \log \mu\{x : \zeta_x^n \in \mathcal{U}\} \geq -\inf\{I(\nu) : \nu \in \mathcal{U}\}$$

with $I(\nu)$ as before $= -\int \varphi d\nu - h_\mu(f)$ if ν is f-invariant.

Subshifts of finite type for $\mathbb{Z}^d$-actions

Q is a finite alphabet (spins), $Q^{\mathbb{Z}^d}$ is considered with product topology: it is the space of all maps $\omega : \mathbb{Z}^d \to Q$ (configurations)

$\theta_m,\ m \in \mathbb{Z}^d$ shifts of $Q^{\mathbb{Z}^d}$, $(\theta_m\omega)_n = \omega_{n+m}$ where $\omega_k \in Q$ is the value of ω on k.

Ω is a closed θ_m-invariant subset of $Q^{\mathbb{Z}^d}$ of permissible configurations, (Ω, θ) is called a subshift. It is a subshift of finite type if $\exists$ a finite (window) $F \subset \mathbb{Z}^d$ and $\Xi \subset Q^F$ such that

$$\Omega = \Omega_{(F,\Xi)} = \left\{\omega \in Q^{\mathbb{Z}^d} : (\theta_m\omega)_F \in \Xi \text{ for every } m \in \mathbb{Z}^d\right\},$$

where ω_R is the restriction of $\omega \in Q^{\mathbb{Z}^d}$ to $R \subset \mathbb{Z}^d$.

Weak specification property (in the sense of dynamical systems): $\exists N$ such that for any subsets $R_i \subset \mathbb{Z}^d$ which are N apart and for any $\xi_i \in \Omega_{R_i}$ (this is the restriction of Ω to R_i which gives permissible configurations on R_i) one can find $\omega \in \Omega$ such that $\omega_{R_i} = \xi_i$.

Shift invariant interaction potential:

$\Phi = \left\{\Phi_\Lambda,\right\}$, $\Phi_\Lambda : \Omega_\Lambda \longrightarrow R$, defined for all $\Lambda \subset \mathcal{A}$–collection of nonempty finite sets, assuming

$$\|\Phi\| = \sum_{\Lambda:0\in\Lambda\in\mathcal{A}} |\Phi_\Lambda| < \infty, \text{ where } |\Phi_\Lambda| = \sup_{\xi\in\Omega_\Lambda} |\Phi_\Lambda(\xi)|$$

$$\text{and } \Phi_{\Lambda-m}(\theta_m\xi) = \Phi_\Lambda(\xi),\ \forall\Lambda \in \mathcal{A},\ \forall\xi \in \Omega_\Lambda.$$

Energy functions and partition functions:

$$U_\Lambda^\Phi(\xi)=\sum_{X\subset\Lambda}\Phi_X(\xi_X),\ U_{\Lambda,\eta}^\Phi(\xi)=\sum_{X\subset\mathcal{A}:X\cap\Lambda\neq\phi}\Phi_X\left((\xi\vee\eta)_X\right),\xi\vee\eta\in\Omega,$$

$$Z_\Lambda^\Phi=\sum_{\xi\in\Omega_\Lambda}\exp\left(-U_\Lambda^\Phi(\xi)\right),\ Z_\Lambda^\Phi(\eta)=\sum_{\xi\in\Omega_\Lambda:\xi\vee\eta\in\Omega}\exp\left(-U_{\Lambda,\eta}^\Phi(\xi)\right).$$

Gibbs measure on Ω:

$\mu\in\mathcal{P}(\Omega)$ satisfies

$$\mu\left(\Xi_\Lambda(\xi)\,|\,\mathcal{B}_{\Lambda^c}\right)(\eta)=\left(Z_\Lambda^\Phi(\eta)\right)^{-1}\exp\left(-U_{\Lambda,\eta}^\Phi(\xi)\right),$$

for $\forall\xi\in\Omega_\Lambda,\ \eta\in\Omega_{\Lambda^c},\ \xi\vee\eta\in\Omega,\ \forall\Lambda\in\mathcal{A},\ \Xi_\Lambda(\xi)=\{\omega\in\Omega:\omega_\Lambda=\xi\}$, where $\Lambda^c=\mathbb{Z}^d\setminus\Lambda$ and $\mathcal{B}_{\Lambda^c}$ is the Borel σ-algebra on Ω_{Λ^c}.

Limit in the sense of van Hove:

We write $\Lambda_\gamma\nearrow\infty$, where γ belongs to a directed set Γ and we consider limits along Γ, if

$$\lim_{\gamma\in\Gamma}|\Lambda_\gamma|=\infty \text{ and } \lim_{\gamma\in\Gamma}\frac{|(\Lambda_\gamma+a)\setminus\Lambda_\gamma|}{|\Lambda_\gamma|}=0.$$

Set $\zeta_\omega^\Lambda = |\Lambda|^{-1} \sum_{m\in\Lambda} \delta_{\theta_m \omega}$.

Theorem

Let (Ω, θ) be a subshift of finite type satisfying the weak specification. Then for any interaction Φ as above and any Gibbs measure μ,

$$\limsup_{\Lambda \nearrow \infty} |\Lambda|^{-1} \log \mu\{\omega : \zeta_\omega^\Lambda \in K\} \leq - \inf_{\nu \in K} I^\Phi(\nu)$$

$\forall K$ closed $\subset \mathcal{P}(\Omega)$ while for any open $G \subset \mathcal{P}(\Omega)$,

$$\liminf_{\Lambda \nearrow \infty} |\Lambda|^{-1} \log \mu\{\omega : \zeta_\omega^\Lambda \in G\} \geq - \inf_{\nu \in G} I^\Phi(\nu)$$

where $I^\Phi(\nu) = \Pi(A^\Phi) - \int A^\Phi d\nu - h_\nu$ if $\nu \in \mathcal{P}(\Omega)$ is shift invariant and $= \infty$, otherwise. Here $A^\Phi(\omega) = -\sum \Big\{ |R|^{-1} \Phi_R(\omega_R) : R \subset \mathbb{Z}^d$ is finite and $0 \in R \Big\}$,

$$\Pi(A^\Phi) = \lim_{\Lambda \nearrow \infty} \frac{\log Z_\Lambda^\Phi}{|\Lambda|} = \lim_{\Lambda \nearrow \infty} |\Lambda|^{-1} \log \left(\sum_{\xi \in \Omega_\Lambda} \exp \Big(\sum_{m \in \Lambda,\, \forall \omega^\xi \in \Xi_\Lambda(\xi)} A^\Phi(\theta_m \omega^\xi) \Big) \right)$$

is the pressure and $h_\nu = \lim_{\Lambda \nearrow \infty} |\Lambda|^{-1} H_\Lambda(\nu)$ with $H_\Lambda(\nu) = -\sum_{\xi \in \Omega_\Lambda} \nu(\Xi_\Lambda(\xi)) \log \nu(\Xi_\Lambda(\xi))$ being the (mean) entropy.

Let, again, (Ω,θ), $\Omega=\Omega_{(F,\Xi)}$ be a subshift of finite type. For $a=(a_1,\dots,a_d)\in\mathbb{Z}^d$, $a_i>0$, $1\le i\le d$ set $\Lambda(a)=\{i\in\mathbb{Z}^d: 0\le i_k<a_k,\ 1\le k\le d\}$ and we write $a\to\infty$ if $a_1,\dots,a_d\to\infty$. Let also $\mathbb{Z}^d(a)$ be the subgroup of $\mathbb{Z}^d$ generated by $(a_1,0,...,0),...,(0,...,0,a_d)$. The set of a-periodic points: $\Pi_a=\{\omega\in\Omega:\ \mathbb{Z}^d(a)\omega=\omega\}$. The weak specification: $\exists N>0$ such that $\forall\, R_i\subset\mathbb{Z}^d$ which are N apart and $\forall$ permissible configurations ξ_i on R_i $\exists\omega\in\Omega$ such that $\omega_{R_i}=\xi_i$. The strong specification: $\exists N>0$ such that $\forall\, R_i\subset\Lambda(a)$ which are N apart and $\forall$ permissible configurations ξ_i on R_i $\exists\omega\in\Pi_a$ such that $\omega_{R_i}=\xi_i$. Set $\zeta^a_\omega=\zeta^{\Lambda(a)}_\omega$ where, again, $\zeta^\Lambda_\omega=|\Lambda|^{-1}\sum_{m\in\Lambda}\delta_{\theta^m\omega}$. Define $\nu_a(\Gamma)=|\Pi_a|^{-1}|\Gamma\cap\Pi_a|$, $\Gamma\subset\Omega$ which is the uniform distribution on Π_a.

Theorem

Suppose that (Ω,θ) is a subshift of finite type satisfying the strong specification. Then for any closed $K\subset\mathcal{P}(\Omega)$ and for any open $G\subset\mathcal{P}(\Omega)$,

$$\limsup_{a\to\infty}|\Lambda(a)|^{-1}\log\nu_a\{\omega:\zeta^a_\omega\in K\}\le-\inf_{\eta\in K}J(\eta)\ \textit{and}$$
$$\liminf_{a\to\infty}|\Lambda(a)|^{-1}\log\nu_a\{\omega:\zeta^a_\omega\in G\}\ge-\inf_{\eta\in G}J(\eta)$$

where $J(\eta)=h_{\mathrm{top}}-h_\eta$ if η is shift invariant and $=\infty$, otherwise, and $h_{\mathrm{top}}=\sup\{h_\eta:\eta$ is shift invariant$\}$ is the topological entropy of the subshift.

$X = [-1,1]$, $f_a : X \circlearrowleft$, $f_a x = 1 - ax^2$, $0 < a < 2$. Conditions: for $\lambda = \frac{9}{10}\log 2$ and $\alpha = \frac{1}{100}$,

- $f = f_a$ with a close enough to 2; f is topologically mixing on $[f^2 0, f0]$;
- $|(f^n)'(f0)| \geq e^{\lambda n}$ $\forall n \geq 0$; $|f^n 0| \geq e^{-\alpha\sqrt{n}}$ $\forall n \geq 1$.

Then f has an acip μ and the set of parameters a satisfying the above has positive Lebesgue measure. Set $\delta_x^n = \frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i x}$, $\lambda(\nu) = \int \log|f'|d\nu$ and let $h(\nu)$ be the entropy of an f-invariant $\nu \in \mathcal{P}(X)$.

Theorem

let $F(\nu) = h(\nu) - \lambda(\nu)$ if $\nu \in \mathcal{P}(X)$ is f-invariant and $= \infty$, otherwise and define $I(\nu) = -\inf_G \sup\{F(\eta) : \eta \in G\}$, where $\inf$ is over all open neighborhoods of ν (lower semi-continuous regularization of $-F$). Then

$$\liminf_{n\to\infty} \log \mu\{x \in X : \delta_x^n \in G\} \geq -\inf\{I(\nu) : \nu \in G\} \text{ and}$$
$$\limsup_{n\to\infty} \log \mu\{x \in X : \delta_x^n \in K\} \leq -\inf\{I(\nu) : \nu \in K\}$$

for any open $G \subset \mathcal{P}(X)$ and closed $K \subset \mathcal{P}(X)$ (with respect to the topology of weak convergence).

by the general Donsker-Varadhan result:
$\lim_{n\to\infty} \frac{1}{n}\log\int \exp(\sum_{i=0}^{n-1}\varphi\circ f^i)d\mu = \sup_{\nu\in\mathcal{P}(X)}(\int \varphi d\nu - I(\nu))$ $\forall \varphi \in (X)$.

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