THE UNIVERSAL PROPERTY OF THE STABLE MOTIVIC HOMOTOPY CATEGORY

DANIEL MARLOWE

1. Categorical setup

1.1. **Formal inversion.** Recall that a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ is the data of a co-Cartesian fibration $\mathcal{C}^{\otimes} \to N\mathrm{Fin}_{*}$, such that for each $n \geq 0$ and $0 \leq i \leq n$, the maps

$$\rho_i : \langle n \rangle \to \langle 1 \rangle, \quad j \mapsto \begin{cases} 0, 1, & j = i, \\ 0, & j \neq i \end{cases}$$

induce an equivalence ${\mathcal{C}^{\otimes}}_n \xrightarrow{\sim} \prod_{1 \leq i \leq n} {\mathcal{C}^{\otimes}}_1$; this is the same as a commutative algebra object in the symmetric monoidal ∞-category $\mathcal{C}\mathrm{at}_{\infty}^{\times}$ (with the cartesian monoidal structure), i.e. an object of $\mathrm{CAlg}(\mathcal{C}\mathrm{at}_{\infty}^{\times})$. Write $\mathcal{C} := {\mathcal{C}^{\otimes}}_1$ for the underlying category of $\mathcal{C}^{\otimes}$. An object $X \in \mathcal{C}$ is said to be $\otimes$-*invertible* if there is some $X^{*} \in \mathcal{C}$ with $X \otimes X^{*} \simeq \mathbb{1} \simeq X^{*} \otimes X$, or equivalently if the functor $- \otimes X : \mathcal{C} \to \mathcal{C}$ is an equivalence.
Given a small symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$, write $\mathrm{Mod}_{\mathcal{C}^{\otimes}}(\mathcal{C}\mathrm{at}_{\infty}^{\times})$ for the ∞-category of modules over $\mathcal{C}^{\otimes}$. There is a canonical equivalence

$$\mathrm{CAlg}(\mathrm{Mod}_{\mathcal{C}^{\otimes}}(\mathcal{C}\mathrm{at}_{\infty}^{\times})) \simeq \mathrm{CAlg}(\mathcal{C}\mathrm{at}_{\infty}^{\infty})_{\mathcal{C}^{\otimes}/},$$

and a forgetful functor $\mathrm{CAlg}(\mathrm{Mod}_{\mathcal{C}^{\otimes}}(\mathcal{C}\mathrm{at}_{\infty}^{\times})) \to \mathrm{Mod}_{\mathcal{C}^{\otimes}}(\mathcal{C}\mathrm{at}_{\infty}^{\times})$ (forgetting the algebra structure). This functor preserves limits, so we get by presentability a left adjoint

$$\mathrm{Free}_{\mathcal{C}^{\otimes}} : \mathrm{Mod}_{\mathcal{C}^{\otimes}}(\mathcal{C}\mathrm{at}_{\infty}^{\times}) \to \mathrm{CAlg}(\mathrm{Mod}_{\mathcal{C}^{\otimes}}(\mathcal{C}\mathrm{at}_{\infty}^{\times})),$$

associating to a $\mathcal{C}^{\otimes}$-module $\mathcal{D}^{\otimes}$ the free $\mathcal{C}^{\otimes}$-module generated by $\mathcal{D}$. Given now an object $X \in \mathcal{C}$, write

$$S_X := \{\mathrm{Free}_{\mathcal{C}^{\otimes}}(\mathcal{C}) \xrightarrow{\mathrm{Free}_{\mathcal{C}^{\otimes}}(-\otimes X)} \mathrm{Free}_{\mathcal{C}^{\otimes}}(\mathcal{C})\}.$$

The full subcategory of S_X-local objects in $\mathrm{CAlg}(\mathcal{C}\mathrm{at}_{\infty}^{\infty})_{\mathcal{C}^{\otimes}/}$ identifies with the full subcategory $\mathrm{CAlg}(\mathcal{C}\mathrm{at}_{\infty}^{\infty})^{X}_{\mathcal{C}^{\otimes}/}$ on objects for which the structure map $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ sends X to a $\otimes$-invertible object. Again by presentability, there is an adjunction

$$\mathrm{CAlg}(\mathcal{C}\mathrm{at}_{\infty}^{\infty})_{\mathcal{C}^{\otimes}/} \underset{}{\overset{\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes}, X)}}{\rightleftarrows}} \mathrm{CAlg}(\mathcal{C}\mathrm{at}_{\infty}^{\infty})_{\mathcal{C}^{\otimes}/}.$$

In particular, there is a universal functor

$$\iota : \mathcal{C}^{\otimes} \to \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes}, X)}(\mathcal{C}) =: \mathcal{C}^{\otimes}[X^{-1}]$$

with ιX invertible in $\mathcal{C}^{\otimes}[X^{-1}]$, and such that restriction along ι induces an equivalence

$$\mathrm{Fun}_{\mathcal{C}^{\otimes}}(\mathcal{C}^{\otimes}[X^{-1}], \mathcal{D}^{\otimes}) \to \mathrm{Fun}_{\mathcal{C}^{\otimes}}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$$

for any $\mathcal{C}^{\otimes}$-algebra $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ in $\mathrm{CAlg}(\mathcal{C}\mathrm{at}_{\infty}^{\times})^{X}_{\mathcal{C}^{\otimes}/}$. We can upgrade this to the presentable setting with the following observation: the forgetful functor $\mathrm{CAlg}(\mathcal{C}\mathrm{at}_{\infty}^{\times}) \to \mathcal{C}\mathrm{at}_{\infty}^{\times}$ admits a left adjoint $\mathrm{free}^{\otimes} : \mathcal{C}\mathrm{at}_{\infty}^{\times} \to \mathrm{CAlg}(\mathcal{C}\mathrm{at}_{\infty}^{\times})$, and we write $*^{\otimes} := \mathrm{free}^{\otimes}(\Delta^0)$. An object of $\mathcal{C}^{\otimes}$ is the data of a functor

$$*^{\otimes} \to \mathcal{C}^{\otimes},$$

and a $\mathcal{C}^\otimes$-algebra $f : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ has fX $\otimes$-invertible if and only if there is a factorisation, for $* \in \Delta^0$ the unique object,

$$\begin{array}{ccc} *^\otimes & \longrightarrow & \mathcal{L}^\otimes_{(*^\otimes,*)}(*^\otimes) \\ \downarrow X & & \downarrow \\ \mathcal{C}^\otimes & \xrightarrow{f} & \mathcal{D}^\otimes. \end{array} \tag{1}$$

The pushout (in $\mathcal{C}\mathrm{at}^\times_\infty$)

$$\mathcal{C}^\otimes \coprod_{*^\otimes} \mathcal{L}^\otimes_{(*^\otimes,*)}(*^\otimes),$$

is such that a monoidal functor $\mathcal{C}^\otimes \to \mathcal{D}^\otimes$ sends X to a $\otimes$-invertible object if and only if it factors through the map

$$\mathcal{C}^\otimes \to \mathcal{C}^\otimes \coprod_{*^\otimes} \mathcal{L}^\otimes_{(*^\otimes,*)}(*^\otimes),$$

and this factorisation is unique up to contractible choice. Accordingly, there is a canonical equivalence

$$\mathcal{C}^\otimes \coprod_{*^\otimes} \mathcal{L}^\otimes_{(*^\otimes,*)}(*^\otimes) \simeq \mathcal{C}^\otimes[X^{-1}].$$

This diagram (1) factors by the universal monoidal property of presheaves as

$$\begin{array}{ccc} *^\otimes & \longrightarrow & \mathcal{L}^\otimes_{(*^\otimes,*)}(*^\otimes) \\ \downarrow j & & \downarrow j \\ \mathcal{P}^\otimes(*^\otimes) & \longrightarrow & \mathcal{P}^\otimes(\mathcal{L}^\otimes_{(*^\otimes,*)}(*^\otimes)) \\ \downarrow & & \downarrow \\ \mathcal{C}^\otimes & \xrightarrow{f} & \mathcal{D}^\otimes, \end{array}$$

where the dashed arrows are given by left Kan extension. For $\mathcal{C}^\otimes$ a presentably symmetric monoidal category (a commutative algebra object in $\mathrm{Pr}^{\mathrm{L},\otimes}$) and $X \in \mathcal{C}$, we define $\mathcal{C}^\otimes[X^{-1}]$ as the pushout in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}$

$$\mathcal{C}^\otimes[X^{-1}] := \mathcal{C}^\otimes \coprod_{\mathcal{P}^\otimes(*^\otimes)} \mathcal{P}^\otimes(\mathcal{L}^\otimes_{(*^\otimes,*)}(*^\otimes)).$$

This can again be identified as the image of $\mathcal{C}^\otimes$ under the left adjoint $\mathcal{L}^{\mathrm{Pr}}_{(\mathcal{C}^\otimes,X)}$ to the fully faithful restriction functor

$$\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L},\otimes})_{\mathcal{C}^\otimes[X^{-1}]/} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L},\otimes})_{\mathcal{C}^\otimes/}.$$

1.2. **Stabilisation.** Suppose $\mathcal{C}$ is an ∞-category, and (G,U) an adjoint pair $\mathcal{C} \underset{U}{\overset{G}{\rightleftarrows}} \mathcal{C}$. The stabilisation of $\mathcal{C}$ with respect to (G,U) is the limit in $\mathcal{CAT}_\infty$

$$\mathrm{Stab}_{(G,U)}(\mathcal{C}) := \lim(\dots \xrightarrow{G} \mathcal{C} \xrightarrow{G} \mathcal{C} \xrightarrow{G} \mathcal{C}),$$

and for free we get a functor $\Omega^\infty_{(G,U)} : \mathrm{Stab}_{(G,U)}(\mathcal{C}) \to \mathcal{C}$. In the case $\mathcal{C}$ is finitely (co)complete with final object $*$, the stabilisation $\mathrm{Stab}(\mathcal{C})$ of $\mathcal{C}_{*/}$ with respect to the pair

$$\mathcal{C}_{*/} \underset{\Omega}{\overset{\Sigma}{\rightleftarrows}} \mathcal{C}_{*/}$$

is a stable ∞-category, and restriction along Ω^∞ induces an equivalence, for any stable ∞-category $\mathcal{D}$,

$$(\Omega^\infty)^* : \mathrm{Fun}^{\mathrm{ex}}(\mathrm{Stab}(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{D}).$$

In the case $\mathcal{C}$ is presentable, the limit can equivalently be taken in Pr^{R}, and we get for free an adjoint $\Sigma^\infty : \mathcal{C} \to \mathrm{Stab}(\mathcal{C})$, with the universal property that for $\mathcal{D}$ a stable presentable ∞-category, the restriction

$$(\Sigma^\infty)^* : \mathrm{Fun}^L(\mathrm{Stab}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$$

is an equivalence.
There is an analogous construction in the symmetric monoidal setting: for $\mathcal{C}^\otimes \in \mathrm{CAlg}(\mathcal{C}\mathrm{at}_\infty^\times)$, $X \in \mathcal{C}$, and $M \in \mathrm{Mod}_{\mathcal{C}^\otimes}(\mathcal{C}\mathrm{at}_\infty^\times)$, we have an endofunctor $- \otimes X : M \to M$ induced by the functor

$$\mathcal{C}^\otimes \to \mathrm{End}(M)^\otimes$$

classifying the $\mathcal{C}^\otimes$-action. We wish to find some universal approximation to M on which X acts as an equivalence; if M is $\mathcal{C}$ itself, the first obstruction to this comes from the observation that the automorphism group of a $\otimes$-invertible object is abelian: for such a U, we can write $U \simeq U \otimes U^* \otimes U$, and an endomorphism $U \to U$ is equivalent to both $f \otimes 1 \otimes 1$ or $1 \otimes 1 \otimes f$. Accordingly, for $f, g : U \to U$,

$$f \circ g \simeq (1 \otimes f \otimes 1) \circ (g \otimes 1 \otimes 1) \simeq (g \otimes 1 \otimes 1) \circ (1 \otimes f \otimes 1) \simeq g \circ f.$$

If X is $\otimes$-invertible in $\mathcal{C}$, so is $X^{\otimes 3}$, and we see that the cyclic permutation $(1\ 2\ 3)$ on $X^{\otimes 3}$ generating the commutator $A_3 \subset \Sigma_3$ must be homotopic to the identity. Any cyclic permutation of $n \geq 3$ objects is a product of 3-bloc permutations, and accordingly this necessary condition turns out to be sufficient. Call an object X with the property that $(1\ 2\ 3)$ on $X^{\otimes 3}$ is homotopic to the identity symmetric.

Proposition 1.2.1. *Let $\mathcal{C}^\otimes$ be a small symmetric monoidal ∞-category, and $X \in \mathcal{C}$ symmetric; then for any $\mathcal{C}^\otimes$-module M, the colimit*

$$\mathrm{Stab}_{(\mathcal{C}^\otimes, X)}(M) := \mathrm{colim}(M \xrightarrow{-\otimes X} M \xrightarrow{-\otimes X} M \xrightarrow{-\otimes X} \dots)$$

taken in $\mathrm{Mod}_{\mathcal{C}^\otimes}(\mathcal{C}\mathrm{at}_\infty^\times)$ is a $\mathcal{C}^\otimes$-module on which X acts an equivalence.

The same holds in the presentable setting:

Proposition 1.2.2. *Let $\mathcal{C}^\otimes$ be a presentably symmetric monoidal ∞-category and $X \in \mathcal{C}$ symmetric. Then for a $\mathcal{C}^\otimes$-module M, $\mathrm{Stab}_{(\mathcal{C}^\otimes, X}(M)$ is a $\mathcal{C}^\otimes$-module on which X acts as an equivalence, and the functor*

$$\mathcal{L}^{\mathrm{Pr}}_{(\mathcal{C}^\otimes, X)}(M) \to \mathrm{Stab}_{(\mathcal{C}^\otimes, X)}(M)$$

induced by adjunction is an equivalence. In particular, there is an equivalence of underlying ∞-categories

$$\mathcal{C}^\otimes[X^{-1}] \simeq \mathrm{Stab}_{(\mathcal{C}^\otimes, X)}(\mathcal{C}^\otimes).$$

Moreover, if $\mathcal{C}^\otimes$ is additionally stable, $\mathcal{C}^\otimes[X^{-1}]$ is again stably presentably symmetric monoidal.

Idea of proof. The functor $\mathcal{L}^{\mathrm{Pr}}_{(\mathcal{C}^\otimes, X)}(M) \to \mathrm{Stab}_{(\mathcal{C}^\otimes, X)}(M)$ factors as

$$\mathcal{L}^{\mathrm{Pr}}_{(\mathcal{C}^\otimes, X)}(M) \to \mathcal{L}^{\mathrm{Pr}}_{(\mathcal{C}^\otimes, X)}(\mathrm{Stab}_{(\mathcal{C}^\otimes, X)}(M)) \to \mathrm{Stab}_{(\mathcal{C}^\otimes, X)}(M),$$

and since X acts invertibly on $\mathrm{Stab}_{(\mathcal{C}^\otimes, X)}(M)$, the second map is an equivalence (adjoint to the identity on $\mathrm{Stab}_{(\mathcal{C}^\otimes, X)}(M)$). Since $\mathrm{Stab}_{(\mathcal{C}^\otimes, X)}$ is a colimit and $\mathcal{L}^{\mathrm{Pr}}_{(\mathcal{C}^\otimes, X)}$ a left adjoint, we have a canonical equivalence

$$\mathcal{L}^{\mathrm{Pr}}_{(\mathcal{C}^\otimes, X)}(\mathrm{Stab}_{(\mathcal{C}^\otimes, X)}(M)) \simeq \mathrm{Stab}_{(\mathcal{C}^\otimes, X)}(\mathcal{L}^{\mathrm{Pr}}_{(\mathcal{C}^\otimes, X)}(M))$$

under $\mathcal{L}^{\mathrm{Pr}}_{(\mathcal{C}^\otimes,X)}(M)$. But X acts as an equivalence on $\mathrm{Stab}_{(\mathcal{C}^\otimes,X)}(M)$, and so $M \to \mathrm{Stab}_{(\mathcal{C}^\otimes,X)}(M)$ is sent to an equivalence under $\mathcal{L}^{\mathrm{Pr}}_{(\mathcal{C}^\otimes,X)}$. Since the diagram

$$\begin{array}{ccc} & \mathcal{L}^{\mathrm{Pr}}_{(\mathcal{C}^\otimes,X)}(M) & \\ \swarrow & & \searrow \\ \mathcal{L}^{\mathrm{Pr}}_{(\mathcal{C}^\otimes,X)}(\mathrm{Stab}_{(\mathcal{C}^\otimes,X)}(M)) & \xrightarrow{\simeq} & \mathrm{Stab}_{(\mathcal{C}^\otimes,X)}(\mathcal{L}^{\mathrm{Pr}}_{(\mathcal{C}^\otimes,X)}(M)) \end{array}$$

commutes, we are done.

For the last statement, if $\mathcal{C}^\otimes$ is stable presentable, $-\otimes X$ is an exact functor, and the diagram defining $\mathrm{Stab}_{(\mathcal{C}^\otimes,X)}(\mathcal{C}^\otimes) \simeq \mathcal{C}^\otimes[X^{-1}]$ lives in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. Since the inclusion $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}} \subset \mathrm{Pr}^{\mathrm{L}}$ preserves colimits, $\mathcal{C}^\otimes[X^{-1}]$ is again stable presentable. □

Example 1.2.3. Write $\mathcal{S}_*^\wedge$ for the presentable symmetric monoidal ∞-category of pointed spaces with the smash product, with unit $S^0 := * \coprod *$. The space S^1 is a symmetric object in $\mathcal{S}_*^\wedge$ since the diagram

$$\begin{array}{ccc} S^1 \wedge S^1 & \xrightarrow{\cong} & S^2 \\ \downarrow T & & \downarrow -1 \\ S^1 \wedge S^1 & \xrightarrow{\cong} & S^2, \end{array}$$

commutes up to homotopy, and hence the cyclic permutation $(1\ 2\ 3)$ on $(S^1)^{\wedge 3}$ is homotopic to 1_{S^1}; accordingly we have that the stabilisation $\mathrm{Stab}_{(\mathcal{S}_*,S^1)}(\mathcal{S}_*)$ with respect to S^1 is equivalently given by the formal inversion $\mathcal{S}_*[S^1)^{-1}]$; note that this stabilisation is a stable ∞-category since this coincides with the stabilisation with respect to the adjunction $\Sigma \dashv \Omega$. In fact, $\mathcal{S}_*^\wedge$ is the initial pointed presentable symmetric monoidal ∞-category. For any pointed presentable symmetric monoidal ∞-category $\mathcal{C}^\otimes$, we have accordingly a unique monoidal pointed colimit-preserving functor $f : \mathcal{S}_*^\wedge \to \mathcal{C}^\otimes$, and by comparing universal properties, we see that there is an equivalence

$$\mathcal{C}^\otimes[f(S^1)^{-1}] \simeq \mathcal{S}\mathrm{p} \coprod_{\mathcal{S}_*^\wedge} \mathcal{C}^\otimes,$$

with the pushout taken in $\mathrm{Pr}^{\mathrm{L},\otimes}$.

2. Motivic stable homotopy theory

We now apply the machinery above to give a characterisation of (stable) motivic homotopy theory.

2.1. Unstable $\mathbb{A}^1$-homotopy theory.

Recall for an ∞-category $\mathcal{C}$ that the Yoneda embedding $\mathcal{C} \to \mathcal{P}(\mathcal{C})$ is a free cocompletion. This cocompletion replaces colimits that existed in $\mathcal{C}$ with formal colimits, and we may reimpose these by (Bousfiedl) localising at an appropriate class of maps. For S a

Construction 2.1.1.

2.2. Motivic spheres.

2.3. Betti realisation.