

BACKGROUND TO THE UNIVERSAL PROPERTY OF MOTIVIC SPECTRA, AFTER ROBALO

In this note we recall Robalo's proof of the universal property of $\mathcal{SH}(k)$ [Rob15].

1. CATEGORICAL SETUP

1.1. Formal inversion. Recall that a symmetric monoidal ∞ -category \mathcal{C}^\otimes is the data of a co-Cartesian fibration $\mathcal{C}^\otimes \rightarrow N\text{Fin}_*$, such that for each $n \geq 0$ and $0 \leq i \leq n$, the maps

$$\rho_i : \langle n \rangle \rightarrow \langle 1 \rangle, \quad j \mapsto \begin{cases} 0, 1, & j = i, \\ 0, & j \neq i \end{cases}$$

induce an equivalence $\mathcal{C}^\otimes_n \xrightarrow{\sim} \prod_{1 \leq i \leq n} \mathcal{C}^\otimes_1$; this is the same as a commutative algebra object in the symmetric monoidal ∞ -category Cat_∞^\times (with the cartesian monoidal structure), i.e. an object of $\text{CAlg}(\text{Cat}_\infty^\times)$. Write $\mathcal{C} := \mathcal{C}^\otimes_1$ for the underlying category of \mathcal{C}^\otimes . An object $X \in \mathcal{C}$ is said to be \otimes -invertible if there is some $X^* \in \mathcal{C}$ with $X \otimes X^* \simeq \mathbb{1} \simeq X^* \otimes X$, or equivalently if the functor $-\otimes X : \mathcal{C} \rightarrow \mathcal{C}$ is an equivalence.

Given a small symmetric monoidal ∞ -category \mathcal{C}^\otimes , write $\text{Mod}_{\mathcal{C}^\otimes}(\text{Cat}_\infty^\times)$ for the ∞ -category of modules over \mathcal{C}^\otimes . There is a canonical equivalence

$$\text{CAlg}(\text{Mod}_{\mathcal{C}^\otimes}(\text{Cat}_\infty^\times)) \simeq \text{CAlg}(\text{Cat}_\infty^\times)_{\mathcal{C}^\otimes/},$$

and a forgetful functor $\text{CAlg}(\text{Mod}_{\mathcal{C}^\otimes}(\text{Cat}_\infty^\times)) \rightarrow \text{Mod}_{\mathcal{C}^\otimes}(\text{Cat}_\infty^\times)$ (forgetting the algebra structure). This functor preserves limits, so we get by presentability a left adjoint

$$\text{Free}_{\mathcal{C}^\otimes} : \text{Mod}_{\mathcal{C}^\otimes}(\text{Cat}_\infty^\times) \rightarrow \text{CAlg}(\text{Mod}_{\mathcal{C}^\otimes}(\text{Cat}_\infty^\times)),$$

associating to a \mathcal{C}^\otimes -module \mathcal{D}^\otimes the free \mathcal{C}^\otimes -module generated by \mathcal{D} . Given now an object $X \in \mathcal{C}$, write

$$S_X := \{ \text{Free}_{\mathcal{C}^\otimes}(\mathcal{C}) \xrightarrow{\text{Free}_{\mathcal{C}^\otimes}(-\otimes X)} \text{Free}_{\mathcal{C}^\otimes}(\mathcal{C}) \}.$$

The full subcategory of S_X -local objects in $\text{CAlg}(\text{Cat}_\infty^\times)_{\mathcal{C}^\otimes/}$ identifies with the full subcategory $\text{CAlg}(\text{Cat}_\infty^\times)_{\mathcal{C}^\otimes/}^X$ on objects for which the structure map $\mathcal{C}^\otimes \rightarrow \mathcal{D}^\otimes$ sends X to a \otimes -invertible object. Again by presentability, there is an adjunction

$$\text{CAlg}(\text{Cat}_\infty^\times)_{\mathcal{C}^\otimes/} \xrightleftharpoons{\mathcal{L}_{(\mathcal{C}^\otimes, X)}^\otimes} \text{CAlg}(\text{Cat}_\infty^\times)_{\mathcal{C}^\otimes/}^X.$$

In particular, there is a universal functor

$$\iota : \mathcal{C}^\otimes \rightarrow \mathcal{L}_{(\mathcal{C}^\otimes, X)}^\otimes(\mathcal{C}) =: \mathcal{C}^\otimes[X^{-1}]$$

with ιX invertible in $\mathcal{C}^\otimes[X^{-1}]$, and such that restriction along ι induces an equivalence

$$\text{Fun}_{\mathcal{C}^\otimes}(\mathcal{C}^\otimes[X^{-1}], \mathcal{D}^\otimes) \rightarrow \text{Fun}_{\mathcal{C}^\otimes}(\mathcal{C}^\otimes, \mathcal{D}^\otimes)$$

for any \mathcal{C}^\otimes -algebra $\mathcal{C}^\otimes \rightarrow \mathcal{D}^\otimes$ in $\text{CAlg}(\text{Cat}_\infty^\times)_{\mathcal{C}^\otimes/}^X$. We can upgrade this to the presentable setting with the following observation: the forgetful functor $\text{CAlg}(\text{Cat}_\infty^\times) \rightarrow \text{Cat}_\infty^\times$ admits a left adjoint $\text{free}^\otimes : \text{Cat}_\infty^\times \rightarrow \text{CAlg}(\text{Cat}_\infty^\times)$, and we write $*^\otimes := \text{free}^\otimes(\Delta^0)$. An object of \mathcal{C}^\otimes is the data of a functor

$$*^\otimes \rightarrow \mathcal{C}^\otimes,$$

and a \mathcal{C}^\otimes -algebra $f : \mathcal{C}^\otimes \rightarrow \mathcal{D}^\otimes$ has fX \otimes -invertible if and only if there is a factorisation, for $\ast \in \Delta^0$ the unique object,

$$(1) \quad \begin{array}{ccc} \ast^\otimes & \longrightarrow & \mathcal{L}_{(\ast^\otimes, \ast^\otimes)}^\otimes(\ast^\otimes) \\ \downarrow X & & \vdots \\ \mathcal{C}^\otimes & \xrightarrow{f} & \mathcal{D}^\otimes. \end{array}$$

The pushout (in Cat_∞^\times)

$$\mathcal{C}^\otimes \coprod_{\ast^\otimes} \mathcal{L}_{(\ast^\otimes, \ast^\otimes)}^\otimes(\ast^\otimes),$$

is such that a monoidal functor $\mathcal{C}^\otimes \rightarrow \mathcal{D}^\otimes$ sends X to a \otimes -invertible object if and only if it factors through the map

$$\mathcal{C}^\otimes \rightarrow \mathcal{C}^\otimes \coprod_{\ast^\otimes} \mathcal{L}_{(\ast^\otimes, \ast^\otimes)}^\otimes(\ast^\otimes),$$

and this factorisation is unique up to contractible choice. Accordingly, there is a canonical equivalence

$$\mathcal{C}^\otimes \coprod_{\ast^\otimes} \mathcal{L}_{(\ast^\otimes, \ast^\otimes)}^\otimes(\ast^\otimes) \simeq \mathcal{C}^\otimes[X^{-1}].$$

This diagram (1) factors by the universal monoidal property of presheaves as

$$\begin{array}{ccc} \ast^\otimes & \longrightarrow & \mathcal{L}_{(\ast^\otimes, \ast^\otimes)}^\otimes(\ast^\otimes) \\ \downarrow j & & \downarrow j \\ \mathcal{P}^\otimes(\ast^\otimes) & \longrightarrow & \mathcal{P}^\otimes(\mathcal{L}_{(\ast^\otimes, \ast^\otimes)}^\otimes(\ast^\otimes)) \\ \vdots & & \vdots \\ \mathcal{C}^\otimes & \xrightarrow{f} & \mathcal{D}^\otimes, \end{array}$$

where the dashed arrows are given by left Kan extension. For \mathcal{C}^\otimes a presentably symmetric monoidal category (a commutative algebra object in $\text{Pr}^{\text{L}, \otimes}$) and $X \in \mathcal{C}$, we define $\mathcal{C}^\otimes[X^{-1}]$ as the pushout in $\text{CAlg}(\text{Pr}^{\text{L}})$

$$\mathcal{C}^\otimes[X^{-1}] := \mathcal{C}^\otimes \coprod_{\mathcal{P}^\otimes(\ast^\otimes)} \mathcal{P}^\otimes(\mathcal{L}_{(\ast^\otimes, \ast^\otimes)}^\otimes(\ast^\otimes)).$$

This can again be identified as the image of \mathcal{C}^\otimes under the left adjoint $\mathcal{L}_{(\mathcal{C}^\otimes, X)}^{\text{Pr}}$ to the fully faithful restriction functor

$$\text{CAlg}(\text{Pr}^{\text{L}, \otimes})_{\mathcal{C}^\otimes[X^{-1}]} \rightarrow \text{CAlg}(\text{Pr}^{\text{L}, \otimes})_{\mathcal{C}^\otimes/}.$$

1.2. Stabilisation. Suppose \mathcal{C} is an ∞ -category, and (G, U) an adjoint pair $\mathcal{C} \xrightleftharpoons[U]{G} \mathcal{C}$. The stabilisation of \mathcal{C} with respect to (G, U) is the limit in CAT_∞

$$\text{Stab}_{(G, U)}(\mathcal{C}) := \lim(\dots \xrightarrow{G} \mathcal{C} \xrightarrow{U} \mathcal{C} \xrightarrow{G} \mathcal{C}),$$

and for free we get a functor $\Omega_{(G, U)}^\infty : \text{Stab}_{(G, U)}(\mathcal{C}) \rightarrow \mathcal{C}$. In the case \mathcal{C} is finitely (co)complete with final object \ast , the stabilisation $\text{Stab}(\mathcal{C})$ of $\mathcal{C}_{\ast/}$ with respect to the pair

$$\mathcal{C}_{\ast/} \xrightleftharpoons[\Omega]{\Sigma} \mathcal{C}_{\ast/}$$

is a stable ∞ -category, and restriction along Ω^∞ induces an equivalence, for any stable ∞ -category \mathcal{D} ,

$$(\Omega^\infty)^\ast : \text{Fun}^{\text{ex}}(\text{Stab}(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \text{Fun}^{\text{lex}}(\mathcal{C}, \mathcal{D}).$$

In the case \mathcal{C} is presentable, the limit can equivalently be taken in Pr^{R} , and we get for free an adjoint $\Sigma^\infty : \mathcal{C} \rightarrow \text{Stab}(\mathcal{C})$, with the universal property that for \mathcal{D} a stable presentable ∞ -category, the restriction

$$(\Sigma^\infty)^* : \text{Fun}^L(\text{Stab}(\mathcal{C}), \mathcal{D}) \rightarrow \text{Fun}^L(\mathcal{C}, \mathcal{D})$$

is an equivalence.

There is an analogous construction in the symmetric monoidal setting: for $\mathcal{C}^\otimes \in \text{CAlg}(\text{Cat}_\infty^\times)$, $X \in \mathcal{C}$, and $M \in \text{Mod}_{\mathcal{C}^\otimes}(\text{Cat}_\infty^\times)$, we have an endofunctor $- \otimes X : M \rightarrow M$ induced by the functor

$$\mathcal{C}^\otimes \rightarrow \text{End}(M)^\otimes$$

classifying the \mathcal{C}^\otimes -action. We wish to find some universal approximation to M on which X acts as an equivalence; if M is \mathcal{C} itself, the first obstruction to this comes from the observation that the automorphism group of a \otimes -invertible object is abelian: for such a U , we can write $U \simeq U \otimes U^* \otimes U$, and an endomorphism $U \rightarrow U$ is equivalent to both $f \otimes 1 \otimes 1$ or $1 \otimes 1 \otimes f$. Accordingly, for $f, g : U \rightarrow U$,

$$f \circ g \simeq (1 \otimes f \otimes 1) \circ (g \otimes 1 \otimes 1) \simeq (g \otimes 1 \otimes 1) \circ (1 \otimes f \otimes 1) \simeq g \circ f.$$

If X is \otimes -invertible in \mathcal{C} , so is $X^{\otimes 3}$, and we see that the cyclic permutation $(1\ 2\ 3)$ on $X^{\otimes 3}$ generating the commutator $A_3 \subset \Sigma_3$ must be homotopic to the identity. Any cyclic permutation of $n \geq 3$ objects is a product of 3-bloc permutations, and accordingly this necessary condition turns out to be sufficient. Call an object X with the property that $(1\ 2\ 3)$ on $X^{\otimes 3}$ is homotopic to the identity symmetric.

Proposition 1.2.1. *Let \mathcal{C}^\otimes be a small symmetric monoidal ∞ -category, and $X \in \mathcal{C}$ symmetric; then for any \mathcal{C}^\otimes -module M , the colimit*

$$\text{Stab}_{(\mathcal{C}^\otimes, X)}(M) := \text{colim}(M \xrightarrow{- \otimes X} M \xrightarrow{- \otimes X} M \xrightarrow{- \otimes X} \dots)$$

taken in $\text{Mod}_{\mathcal{C}^\otimes}(\text{Cat}_\infty^\times)$ is a \mathcal{C}^\otimes -module on which X acts an equivalence.

The same holds in the presentable setting:

Proposition 1.2.2. *Let \mathcal{C}^\otimes be a presentably symmetric monoidal ∞ -category and $X \in \mathcal{C}$ symmetric. Then for a \mathcal{C}^\otimes -module M , $\text{Stab}_{(\mathcal{C}^\otimes, X)}(M)$ is a \mathcal{C}^\otimes -module on which X acts as an equivalence, and the functor*

$$\mathcal{L}_{(\mathcal{C}^\otimes, X)}^{\text{Pr}}(M) \rightarrow \text{Stab}_{(\mathcal{C}^\otimes, X)}(M)$$

induced by adjunction is an equivalence. In particular, there is an equivalence of underlying ∞ -categories

$$\mathcal{C}^\otimes[X^{-1}] \simeq \text{Stab}_{(\mathcal{C}^\otimes, X)}(\mathcal{C}^\otimes).$$

Moreover, if \mathcal{C}^\otimes is additionally stable, $\mathcal{C}^\otimes[X^{-1}]$ is again stably presentably symmetric monoidal.

Idea of proof. The functor $\mathcal{L}_{(\mathcal{C}^\otimes, X)}^{\text{Pr}}(M) \rightarrow \text{Stab}_{(\mathcal{C}^\otimes, X)}(M)$ factors as

$$\mathcal{L}_{(\mathcal{C}^\otimes, X)}^{\text{Pr}}(M) \rightarrow \mathcal{L}_{(\mathcal{C}^\otimes, X)}^{\text{Pr}}(\text{Stab}_{(\mathcal{C}^\otimes, X)}(M)) \rightarrow \text{Stab}_{(\mathcal{C}^\otimes, X)}(M),$$

and since X acts invertibly on $\text{Stab}_{(\mathcal{C}^\otimes, X)}(M)$, the second map is an equivalence (adjoint to the identity on $\text{Stab}_{(\mathcal{C}^\otimes, X)}(M)$). Since $\text{Stab}_{(\mathcal{C}^\otimes, X)}$ is a colimit and $\mathcal{L}_{(\mathcal{C}^\otimes, X)}^{\text{Pr}}$ a left adjoint, we have a canonical equivalence

$$\mathcal{L}_{(\mathcal{C}^\otimes, X)}^{\text{Pr}}(\text{Stab}_{(\mathcal{C}^\otimes, X)}(M)) \simeq \text{Stab}_{(\mathcal{C}^\otimes, X)}(\mathcal{L}_{(\mathcal{C}^\otimes, X)}^{\text{Pr}}(M))$$

under $\mathcal{L}_{(\mathcal{C}^\otimes, X)}^{\text{Pr}}(M)$. But X acts as an equivalence on $\text{Stab}_{(\mathcal{C}^\otimes, X)}(M)$, and so $M \rightarrow \text{Stab}_{(\mathcal{C}^\otimes, X)}(M)$ is sent to an equivalence under $\mathcal{L}_{(\mathcal{C}^\otimes, X)}^{\text{Pr}}$. Since the diagram

$$\begin{array}{ccc} & \mathcal{L}_{(\mathcal{C}^\otimes, X)}^{\text{Pr}}(M) & \\ & \swarrow \quad \searrow & \\ \mathcal{L}_{(\mathcal{C}^\otimes, X)}^{\text{Pr}}(\text{Stab}_{(\mathcal{C}^\otimes, X)}(M)) & \xrightarrow{\simeq} & \text{Stab}_{(\mathcal{C}^\otimes, X)}(\mathcal{L}_{(\mathcal{C}^\otimes, X)}^{\text{Pr}}(M)) \end{array}$$

commutes, we are done.

For the last statement, if \mathcal{C}^\otimes is stable presentable, $- \otimes X$ is an exact functor, and the diagram defining $\text{Stab}_{(\mathcal{C}^\otimes, X)}(\mathcal{C}^\otimes) \simeq \mathcal{C}^\otimes[X^{-1}]$ lives in $\text{Pr}_{\text{st}}^{\text{L}}$. Since the inclusion $\text{Pr}_{\text{st}}^{\text{L}} \subset \text{Pr}^{\text{L}}$ preserves colimits, $\mathcal{C}^\otimes[X^{-1}]$ is again stable presentable. \square

Example 1.2.3. Write \mathcal{S}_*^\wedge for the presentable symmetric monoidal ∞ -category of pointed spaces with the smash product, with unit $S^0 := * \amalg *$. The space S^1 is a symmetric object in \mathcal{S}_*^\wedge since the diagram

$$\begin{array}{ccc} S^1 \wedge S^1 & \xrightarrow{\cong} & S^2 \\ \downarrow T & & \downarrow -1 \\ S^1 \wedge S^1 & \xrightarrow{\cong} & S^2, \end{array}$$

commutes up to homotopy, and hence the cyclic permutation $(1\ 2\ 3)$ on $(S^1)^\wedge^3$ is homotopic to 1_{S^1} ; accordingly we have that the stabilisation $\text{Stab}_{(\mathcal{S}_*, S^1)}(\mathcal{S}_*)$ with respect to S^1 is equivalently given by the formal inversion $\mathcal{S}_*[S^1]^{-1}$; note that this stabilisation is a stable ∞ -category since this coincides with the stabilisation with respect to the adjunction $\Sigma \dashv \Omega$. In fact, \mathcal{S}_*^\wedge is the initial pointed presentable symmetric monoidal ∞ -category. For any pointed presentable symmetric monoidal ∞ -category \mathcal{C}^\otimes , we have accordingly a unique monoidal pointed colimit-preserving functor $f : \mathcal{S}_*^\wedge \rightarrow \mathcal{C}^\otimes$, and by comparing universal properties, we see that there is an equivalence

$$\mathcal{C}^\otimes[f(S^1)^{-1}] \simeq \text{Sp} \amalg_{\mathcal{S}_*^\wedge} \mathcal{C}^\otimes,$$

with the pushout taken in $\text{Pr}^{\text{L}, \otimes}$.

REFERENCES

- [Rob15] Marco Robalo. “ K -theory and the bridge from motives to noncommutative motives”. In: *Adv. Math.* 269 (2015), pp. 399–550. ISSN: 0001-8708,1090-2082. DOI: [10.1016/j.aim.2014.10.011](https://doi.org/10.1016/j.aim.2014.10.011). URL: <https://doi.org/10.1016/j.aim.2014.10.011>.