



1. INTRODUCTION

If you grabbed a random person off the street and told them that the total mass of the universe is non-negative, they would probably look at you with a mix of confusion and fear. Perhaps this is because you just grabbed a random person off the street, or perhaps it is because the total mass of the universe is *obviously* non-negative. Except, this *isn't* obvious, and proving that this is true for a certain class of Riemannian manifolds was one of the great breakthroughs of geometric analysis in the 1970s.

2. MASS IN GENERAL RELATIVITY

In classical mechanics, one has two distinct notions of mass: *inertial mass*, which quantifies an object's resistance to acceleration, and *gravitational mass*, which quantifies the gravitational force that two bodies exert on each other. Despite being defined in very different ways, all the empirical evidence available to us suggests that these do in fact agree. Einstein's equivalence principle formally bridges this gap, providing the foundation for his theory.

General relativity allows for spacetime to be curved, with mass now being associated directly to this curvature, rather than to a force exerted between objects. However, there are several distinct notions of mass, with each one proving most useful in different situations. For our purposes, the relevant quantity will be the so-called *ADM mass*, which we define below.

Definition. Given an asymptotically flat manifold (M^n, g) with one end, we say that the ADM mass of this end is

$$m_{ADM}(g) := \frac{1}{2(n-1)\omega_{n-1}} \lim_{\sigma \rightarrow \infty} \int_{|x|=\sigma} \sum_{i,j=1}^n (\partial_i g_{ij} - \partial_j g_{ii}) \nu^j$$

An asymptotically flat manifold is, roughly speaking, one whose metric approximates the standard Euclidean metric outside of a compact set. See Figure 1 for an illustration.

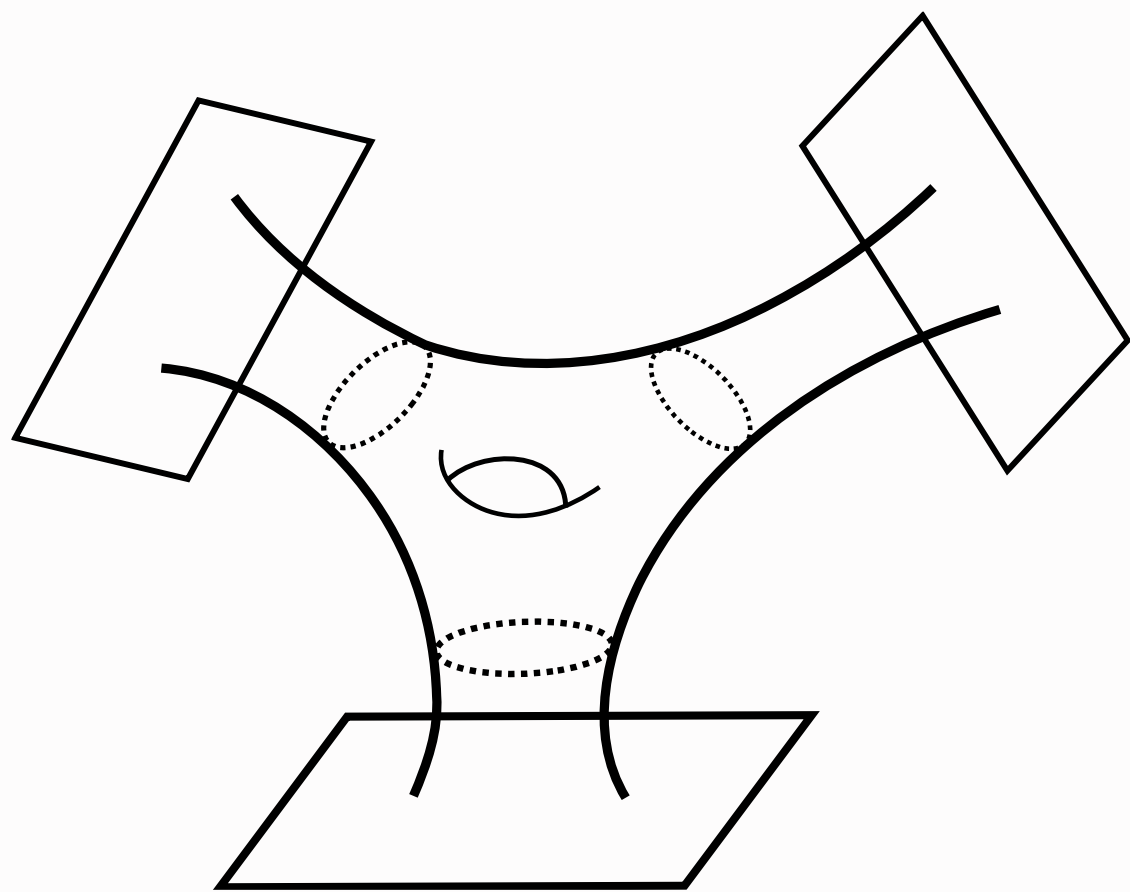


Figure 1. An asymptotically flat manifold with three ends.

Before stating our main result, let us make a few remarks on the above definition. Recall that the ADM mass is closely related to the (scalar) curvature R_g . Since (M^n, g) is 'close' to the standard Euclidean space, we can approximate R_g by its linearisation at the Euclidean metric. Integrating this linearisation over a large ball and applying the divergence theorem yields the ADM mass defined above, up to a normalising constant.

3. THE POSITIVE MASS THEOREM

Schoen and Yau proved in [SY79] that the ADM mass of an asymptotically flat manifold is non-negative whenever its scalar curvature is non-negative. More precisely, we have the following statements.

Theorem. Let (M^n, g) be an oriented and asymptotically flat Riemannian manifold, where $3 \leq n \leq 7$. If the scalar curvature R_g is non-negative, then the ADM mass $m_{ADM}(g)$ is also non-negative. Moreover, the ADM mass is identically zero if and only if (M^n, g) is isometric to the standard Euclidean space.

Before outlining the proof, let us make a few remarks on the dimensional constraint. First, note that the statement also holds in the cases $n = 1, 2$. For the first case, recall that the Riemann curvature tensor of a one-dimensional Riemannian manifold is identically zero. For the second case, the Gauß–Bonnet theorem allows us to avoid much of the heavy machinery required in higher dimensions. Second, the proof by Schoen and Yau only extends up to $n = 7$, since it relies on the construction of an area-minimising hypersurface; such objects are known to develop singularities in higher dimensions. Finally, we note that the recent generic regularity results of Chodosh, Mantoulidis, Schulze, and Wang [CMS23, CMSW25] allow us to extend the positive mass theorem, and other related results, up to dimension $n = 11$.

4. OUTLINE OF THE PROOF

This is a proof by contradiction. In other words, we begin by assuming that the scalar curvature R_g is non-negative, but that the ADM mass is strictly negative. We can then arrive at a contradiction with the following three steps:

1. By performing a suitable conformal transformation of the metric g , we may assume that the scalar curvature is strictly positive outside some compact set. For this we pay the price of making the ADM mass slightly less negative.
2. Using a compactness argument, we construct a complete area-minimizing hypersurface $\Sigma^{n-1} \subset M^n$. This is smoothly embedded because of our dimension restriction.
3. The stability inequality, combined with the positivity of R_g from Step 1 implies that the area-minimizing hypersurface constructed in the previous step cannot exist. This yields the contradiction.

It is well-known that a positive scalar curvature places strong topological restrictions on a Riemannian manifold, so Step 3 is perhaps not too surprising. In fact, it is known more generally that the positive mass theorem is closely related to other results in the study of positive scalar curvature. One such example is the *Geroch conjecture*.

Theorem. The n -torus T^n , where $3 \leq n \leq 7$, does not admit a metric of positive scalar curvature.

5. BEYOND THE PMT

We conclude by mentioning that the positive mass theorem admits a quantitative generalisation known as the *Riemannian Penrose inequality*. This inequality gives a lower bound on the ADM mass in terms of the area of the outermost minimal surface in (M^n, g) . The first proof was due to Huisken and Ilmanen [HI01], who made use of the inverse mean curvature flow. A second proof was given by Bray, and this was later generalized to higher dimensions in [BL09].

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