On some generalized Fermat equations of the form $x^2 + y^{2n} = z^p$

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Modern Breakthroughs in Diophantine Problems

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The Generalized Fermat Conjecture

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$$x^p + y^q + z^r = 0$$
has finitely many (10) solutions (x^p, y^q, z^r) in non-zero coprime integers x, y, and z and $p, q, r \in \mathbb{Z}_{\geq 2}$ satisfying $1/p + 1/q + 1/r < 1$.

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Many 'solved' cases:

- $(2,3,7)$, $(3,4,5)$, $(5,5,7)$, $\ldots$
- $\underbrace{(\ell,\ell,\ell)}_{\text{FLT}}$, $(\ell,\ell,2)$, $(4,2\ell,3)$, $\ldots$

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$$x^2 + y^{2\ell} = z^p,$$

where p is a fixed prime and ℓ varies + highlight the role played by modular curves.

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Eliminate Compare traces of Frobenius	$\mathrm{tr}(\overline{\rho}_{E,\ell}(\sigma_3)) \equiv \mathrm{tr}(\overline{\rho}_{f_i,\ell}(\sigma_3)) \pmod{\ell}$ $\Rightarrow \ell \leq 5$

Over totally real fields

Frey curve - Modularity - Irreducibility - Level-lower - Eliminate

Over a totally real field K, the same strategy works.

- Need to prove **modularity**
- Need to prove **irreducibility**
- Newforms $\rightsquigarrow$ **Hilbert** newforms

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- Suppose $p \nmid y$ and $\ell \neq p$. Each term on the RHS is an ℓth power.

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$$R = 1, \quad S = -\frac{\zeta_p^j - \zeta_p^{-j} - 2}{\zeta_p^k - \zeta_p^{-k} - 2}, \quad T = 4\frac{\zeta_p^j + \zeta_p^{-j} + \zeta_p^k + \zeta_p^{-k}}{\zeta_p^k - \zeta_p^{-k} - 2} \in K$$

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- If $p = 3$, we define a Frey curve over $\mathbb{Q}$:

$$E_{x,y,z,\ell} = E : \quad Y^2 = X^3 + 6b^2X^2 + 3(a^2 - 3b^2)X.$$

Suppose $\overline{\rho}_{E,\ell}$ is modular and irreducible.

$$\overline{\rho}_{E,\ell} \sim \overline{\rho}_{f,\lambda}, \text{ where } \begin{cases} f \text{ is a newform at level } 288 & \text{if } p = 3, \\ f \text{ is a Hilbert newform at level } 2^3 \cdot \mathcal{O}_K & \text{if } p > 3. \end{cases}$$

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Theorem (B, A–S, B–C–D–D–F, M)

Let $\ell \geq 2$ and $p \in \{3, 5, 7, 11, 13, 17\}$. The equation

$$x^{2\ell} + y^{2\ell} = z^p$$

has no solutions in non-zero coprime integers x, y, and z.

Complex multiplication

Solution II: When $p = 3$, the curve $E_{\mathrm{triv}} : Y^2 = X^3 + 3X$ has complex multiplication by $\mathbb{Q}(i)$.

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$$E \rightsquigarrow P \in \begin{cases} X^+_{\mathrm{split}}(\ell)(\mathbb{Q}) & \text{if } p \equiv 1 \pmod 4 \quad ☺ \\ X^+_{\mathrm{nonsplit}}(\ell)(\mathbb{Q}) & \text{if } p \equiv -1 \pmod 4 \quad ☹ \end{cases}$$

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- Same idea works when $\ell = p$ (for any p).

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Theorem (Chen, Dahmen)

Let ℓ be a prime and suppose there exist non-zero coprime integers x, y, and z satisfying

$$x^2 + y^{2\ell} = z^3.$$

Then $3 \mid y$ and $\ell > 10^7$.

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Assume $p > 5$ is fixed and $p \nmid y$.

- Know $\ell > 10^7$.
- Since $3 \mid y$, the Frey curve $E_{x,y,z,\ell}/K$ has multiplicative reduction at all primes $\mathfrak{q} \mid 3$.

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But

$$|a_{\mathfrak{q}_3}(f)| \leq 2\sqrt{\mathrm{Norm}(\mathfrak{q}_3)}.$$

Using this,

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where d is the dimension of the space of Hilbert newforms at level $2^3 \cdot \mathcal{O}_K$.

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Example. Let $p = 17$. Then $d = 41883752$ and $\ell \leq 10^{160315410}$.

Modularity

Theorem (Freitas)

Let K be an abelian totally real number field where 3 is unramified. Let C/K be an elliptic curve semistable at all primes $\mathfrak{q}|3$. Then, C is modular.

Consequence: $\overline{\rho}_{E,\ell}$ is modular for all ℓ.

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- When $p = 7$, we have

$$\ell \mid 15369 \textbf{ or } \ell \leq (1 + 3^9)^2 > 10^7 \ldots$$

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- When $p = 7$, we have

$$\ell \mid 15369 \textbf{ or } \ell \leq (1 + 3^9)^2 > 10^7 \ldots$$

  We have $P \equiv \text{cusp} \pmod{3 \cdot \mathcal{O}_K} \implies \ell < 65 \cdot 6^6 < 10^7$.

An asymptotic result

Theorem (M)

Let p be a prime. There exists a constant $C(p)$ such that for $\ell > C(p)$, the equation

$$x^2 + y^{2\ell} = z^{3p}$$

has no solutions in non-zero coprime integers x, y, and z.

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We could take

$$C(p) = \underbrace{(\sqrt{p}+1)^2}_{\color{red}{p|y}} \cdot \underbrace{\mathrm{Norm}_{K/\mathbb{Q}}(\epsilon^{12}-1)\cdot(1+3^{3(p-1)h_K/2})^2}_{\text{irreducibility}} \cdot \underbrace{(\sqrt{\mathrm{Norm}(\mathfrak{q}_3)}+1)^{2d}}_{\text{eliminating } \overline{\rho}_{E,\ell}\sim\overline{\rho}_{f,\lambda}}$$

The case $p = 7$

Theorem (M)

Let $\ell \geq 2$. The equation

$$x^2 + y^{2\ell} = z^{21}$$

has no solutions in non-zero coprime integers x, y, and z.