A Unique Perfect Power Decagonal Number

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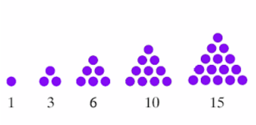
Polygonal Numbers

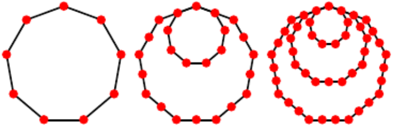
The *nth s-gonal number* is

$$\mathcal{P}_s(n) = \frac{(s-2)n^2 - (s-4)n}{2}.$$

- Triangular numbers: $\mathcal{P}_3(n) = \frac{n^2+n}{2}$.



- Square numbers: $\mathcal{P}_4(n) = n^2$.
- Decagonal numbers: $\mathcal{P}_{10}(n) = D(n) = n(4n-3)$.



Decagonal numbers D_2, D_3 and D_4.

Perfect Powers in Sequences

Perfect Power

$N > 1$ is a perfect power if $N = y^m$ for some $y \in \mathbb{Z}$ and $m > 1$.

- Perfect powers in Fibonacci and Lucas sequences.
- Perfect powers in arithmetic progressions.
- Perfect power polygonal numbers (Kim, Park, Pintér, 2013): All solutions to $\mathcal{P}_s(n) = y^m$ when $m > 2$, $\mathcal{P}_s(n) > 1$, and $s \in \{3, 5, 6, 8, 20\}$ are

$$\mathcal{P}_8(2) = 2^3 \text{ and } \mathcal{P}_{20}(8) = 2^9 = 8^3.$$

A unique perfect power decagonal number (M. 2021)

The only solution to $\mathcal{P}_{10}(n) = D(n) = y^m$ when $m > 1$ and $D(n) > 1$ is $D(3) = 3^3$.

Descent

We have $D(n) = n(4n-3) = y^p$, with p prime.

Case 1: $3 \nmid n$. Then n and $4n-3$ are coprime. So

$$n = a^p \quad \text{and} \quad 4n - 3 = b^p,$$

with $(a, b) = 1$. So

$$4a^p - b^p = 3. \tag{1}$$

Case 2: $3 \parallel n$. Then

$$n = 3t^p \quad \text{and} \quad 4n - 3 = 3^{p-1}u^p,$$

with $(t, u) = 1$ and $3 \nmid t$. So

$$4t^p - 3^{p-2}u^p = 1. \tag{2}$$

Case 3: $3^2 \mid n$.

The Modular Method

These are examples of binomial Thue equations.
We can solve them for $p = 2, 3, 5, 7$ easily. From now on: $p \geq 11$.

We use the modular method:

- Suppose we have a non-zero solution.
- Associate an elliptic curve E (the Frey curve) to this solution.
- Show that E is linked to a **newform** f in some way.
- Obtain a contradiction.

A **newform** at level N is a normalised cusp form in $S_2(\Gamma_0(N))$ that is 'new' at its level.

- **Eichler–Shimura**: $f \longrightarrow A_f$, an abelian variety $/\mathbb{Q}$.

Level-Lowering

Level-Lowering Theorem (Ribet)

Let E be a (modular) elliptic curve over $\mathbb{Q}$ of conductor N and let $p \geq 5$ be prime. Suppose $\overline{\rho}_{E,p}$ is irreducible. Then E arises mod p from a newform f, written $E \sim_p f$, at level N_p, where

$$N_p = \frac{N}{\displaystyle\prod_{q \| N, p \mid \mathrm{ord}_q(\Delta_{\min})} q}.$$

FLT and Case 2

FLT: $x^p + y^p = z^p$.

$E : Y^2 = X(X - x^p)(X + y^p)$.

$E \sim_p f$, f a newform at level 2.

But! No newforms at level 2.

Case 2: $4t^p - 3^{p-2}u^p = 1$.

$E : Y^2 + 3XY - 3^{p-2}u^p Y = X^3$.

$E \sim_p f$, f a newform at level 6.

But! No newforms at level 6.

We were very lucky here! Usually there are newforms at the predicted level.

Frey curves for Case 1

Case 1: $3 \nmid n$:

$$4a^p - b^p = 3.$$

Frey curve depends on $a \pmod 4$.

- If $a \equiv 1 \pmod 4$ then $E_1 : Y^2 = X^3 - 3X^2 + 3a^p X$.
  $E_1 \sim_p f_1$, the unique newform at level 36.
- If $a \equiv 3 \pmod 4$ then $E_2 : Y^2 = X^3 + 3X^2 + 3a^p X$.
  $E_2 \sim_p f_2$, the unique newform at level 72.
- If a is even, we set $E_3 : Y^2 + XY = X^3 - X^2 + \frac{3a^p}{16} X$.
  $E_2 \sim_p f$ at level 18, a contradiction.

We want $E_1 \not\sim_p f_1$ and $E_2 \not\sim_p f_2$.

- Focus on $E_1 \not\sim_p f_1$.

The Trivial Solution

Suppose $E_1 \sim_p f_1$. Now

$$f_1 \longrightarrow W_1 : Y^2 = X^3 + 1.$$

Here, $E_1 \sim_p f_1$ means $\overline{\rho}_{E_1,p} \sim \overline{\rho}_{W_1,p}$.

All methods of elimination will fail.

Why? Because we have a non-zero trivial solution for all p:

$$4(1^p) - 1(1^p) = 3.$$

Comes from $D(1) = 1 = 1^p$.
When $a = 1$, we have $E_1 \cong W_1$ for all p.

Complex Multiplication

The curve W_1 has complex multiplication by $K = \mathbb{Q}(\sqrt{-3})$.

- So $\overline{\rho}_{W_1,p}(G_\mathbb{Q}) \subseteq C^+ \subset \mathrm{GL}_2(\mathbb{F}_p)$. C is a Cartan subgroup.
- Since $\overline{\rho}_{E_1,p} \sim \overline{\rho}_{W_1,p}$, we have $\overline{\rho}_{E_1,p}(G_\mathbb{Q}) \subseteq C'^+$.

Either $E_1 \rightsquigarrow Q \in X^+_{\mathrm{split}}(p)(\mathbb{Q})$ (when p splits in K);
or $E_1 \rightsquigarrow Q \in X^+_{\mathrm{nonsplit}}(p)(\mathbb{Q})$ (when p is inert in K).

Also, E_1 has a 2-torsion point $/\mathbb{Q}$. Forces $j(E_1) \in \mathbb{Z}[\frac{1}{p}]$. So $a = b = 1$.

- Same idea works for $s = 3, 6, 8, 20$ to deal with $\mathcal{P}_s(1) = 1^p$.
- Does not work for any other s.

For more details, see: M. A unique perfect power decagonal number, *Bulletin of the Australian Mathematical Society*, 1–5, 2021.

Thank you!