Sieving for quadratic points on bielliptic curves

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Is this true?



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Is this true? Yes!

- Ozman and Siksek. Quadratic Points on Modular Curves, 2018.
- Box. Quadratic points on modular curves with infinite Mordell–Weil group, 2019.
- Najman and Trbović. Splitting of primes in number fields generated by points on some modular curves, 2020.
- Banwait. Explicit isogenies of prime degree over quadratic fields, 2021.
- M. Fermat's Last Theorem and modular curves over real quadratic fields, 2021.
- Najman and Vukorepa. Quadratic points on bielliptic modular curves, 2021.
- M. On elliptic curves with p-isogenies over quadratic fields, 2022.
- Banwait and Derickx. Explicit isogenies of prime degree over number fields, 2022.
- Vukorepa. Isogenies over quadratic fields of elliptic curves with rational *j*-invariant, 2022.
- Banwait, Najman, Padurariu. Cyclic isogenies of elliptic curves over fixed quadratic fields, 2022.
- Adžaga, Keller, Najman, M., Ozman, and Vukorepa. Computing quadratic points on modular curves $X_0(N)$, 2023.

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Why?

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Why?

- Mazur and Kenku looked after the case of rational points on X₀(N) a long time ago and quadratic points are the next best thing. ☺
- Studying quadratic points on $X_0(N)$ is hard enough to be interesting, but not too hard.

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Why?

- Mazur and Kenku looked after the case of rational points on $X_0(N)$ a long time ago and quadratic points are the next best thing. \odot
- Studying quadratic points on $X_0(N)$ is hard enough to be interesting, but not too hard.
- Deepen our understanding of modular curves.
- Develop techniques for studying low-degree points on curves.
- Deepen our understanding of the arithmetic of elliptic curves and their Galois representations.
- Concrete applications to the modular method for solving Diophantine equations.

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Computing quadratic points

A quadratic point on a curve X/\mathbb{Q} is a point

$P \in X(\mathbb{Q}(\sqrt{d})) \backslash X(\mathbb{Q}).$

We think of *P* with its Galois conjugate, P^{σ} .

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What does **"computing the quadratic points on** $X_0(N)$ " actually mean?

- 1. If $X_0(N)$ has **finitely** many quadratic points (as we range over all quadratic fields) then this means **writing them all down** on an explicit model.
- 2. If $X_0(N)$ has infinitely many quadratic points as we range over all quadratic fields then this means writing down all the points that do not come from a 'geometric family'.

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Quadratic points have been computed on all $X_0(N)$ with $2 \le g(X_0(N)) \le 8$.



Bielliptic curves $X_0(N)$

Let $N \in \mathcal{N} = \{53, 61, 79, 83, 89, 101, 131\}.$

• $X_0(N)$ is bielliptic, with a degree 2 map defined over \mathbb{Q} :

 $\psi: X_0(N) \longrightarrow X_0^+(N) = X_0(N)/w_N.$



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Theorem (Box, Najman–Vukorepa, 2021)

Let $N \in \mathcal{N}$ and let P be a quadratic point on $X_0(N)$. Then $\psi(P) = \psi(P^{\sigma}) \in X_0^+(N)(\mathbb{Q})$.

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This result is **great**, but it does not determine $X_0(N)(\mathbb{Q}(\sqrt{d}))$ for a fixed quadratic field $\mathbb{Q}(\sqrt{d})$.

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Theorem (M., 2023)

Let $N \in \mathcal{N} = \{53, 61, 79, 83, 89, 101, 131\}$. Let $d \in \mathbb{Z}$ such that |d| < 100. Then

$$\exists P \in X_0(N)(\mathbb{Q}(\sqrt{d})) \setminus X_0(N)(\mathbb{Q}) \Longleftrightarrow d \in \mathcal{D}_N,$$

where

$$\begin{split} \mathcal{D}_{53} &= \{-43, -11, -7, -1\}, & \mathcal{D}_{61} &= \{-19, -3, -1, 61\}, \\ \mathcal{D}_{79} &= \{-43, -7, -3\}, & \mathcal{D}_{83} &= \{-67, -43, -19, -2\}, \\ \mathcal{D}_{89} &= \{-67, -11, -2, -1, 89\}, & \mathcal{D}_{101} &= \{-43, -19, -1\}, \\ \mathcal{D}_{131} &= \{-67, -19, -2\}. \end{split}$$

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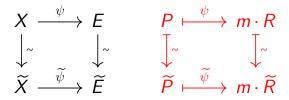
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- Know: $P \in X(\mathbb{Q}(\sqrt{d}))$ and $\psi(P) = \psi(P^{\sigma}) = m \cdot R$.
- Want: information about m by investigating matters mod ℓ .

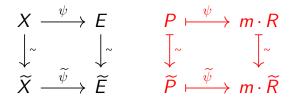
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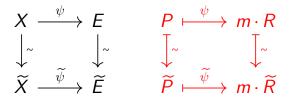


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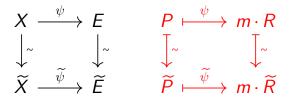
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$$\widetilde{\psi}(\widetilde{P}) = \widetilde{\psi}(\widetilde{P^{\sigma}}) = m \cdot \widetilde{R}.$$

• Write G_{ℓ} for the order of \widetilde{R} in $E(\mathbb{F}_{\ell})$.



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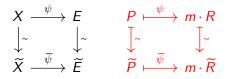
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$$\widetilde{\psi}(\widetilde{P}) = \widetilde{\psi}(\widetilde{P^{\sigma}}) = m \cdot \widetilde{R}.$$

- Write G_{ℓ} for the order of \widetilde{R} in $E(\mathbb{F}_{\ell})$.
- $m \equiv m_0 \pmod{G_\ell}$ for some $m_0 \in \{0, 1, 2, \dots, G_\ell 1\}$.

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Fix an $m_0 \in \{0, 1, 2, \dots, G_{\ell} - 1\}$ and suppose $m \equiv m_0 \pmod{G_{\ell}}$.

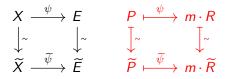




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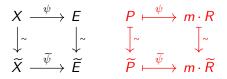
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• Then
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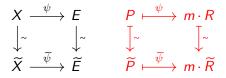
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So {\$\$\tilde{P}\$, \$\$\tilde{P}\$\tilde{\tilde{P}\$}\$}\$} = \$\$\tilde{\psi}^{-1}(m_0 \cdot \tilde{R}\$}\$) ⊂ \$\$\tilde{X}(\mathbb{F}_{\ell^2})\$, a set which we can compute explicitly.

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- So {\$\$\tilde{P}\$, \$\$\tilde{P}\$\tilde{\textsf{\vee}\$}\$}\$} = \$\$\tilde{\psi}^{-1}(m_0 \cdot \tilde{R}\$}\$) ⊂ \$\$\tilde{X}(\textsf{\vee}\$_{\ell^2}\$)\$, a set which we can compute explicitly.
- $\widetilde{\psi}^{-1}(m_0 \cdot \widetilde{R})$ will either be:
 - 1. A pair of (distinct) points in $\widetilde{X}(\mathbb{F}_{\ell})$.
 - 2. A pair of (distinct) points in $\widetilde{X}(\mathbb{F}_{\ell^2})$ (not in $\widetilde{X}(\mathbb{F}_{\ell})$).
 - 3. A single point in $\widetilde{X}(\mathbb{F}_{\ell})$.

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• We are supposing $\{\widetilde{P}, \widetilde{P^{\sigma}}\} = \widetilde{\psi}^{-1}(m_0 \cdot \widetilde{R})$. Is this possible?

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Using a nice explicit (diagonalised) model:

1. Suppose $\widetilde{\psi}^{-1}(m_0 \cdot \widetilde{R})$ is a pair of points in $\widetilde{X}(\mathbb{F}_{\ell})$.

If ℓ ramifies or is inert in $\mathbb{Q}(\sqrt{d})$ then $\{\widetilde{P}, \widetilde{P^{\sigma}}\}$ is a single \mathbb{F}_{ℓ} -point or a pair of \mathbb{F}_{ℓ^2} -points. Contradiction.



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3. Suppose $\widetilde{\psi}^{-1}(m_0 \cdot \widetilde{R})$ is a single point in $\widetilde{X}(\mathbb{F}_{\ell})$.

We can't rule anything out here.

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We can't rule anything out here.

We try and rule out each $m_0 \in \{0, 1, 2, ..., G_{\ell} - 1\}$ to come up with a list of possibilities for $m \pmod{G_{\ell}}$.

So far: list of possibilities for $m \pmod{G_{\ell}}$.

- Repeat with several primes $\ell_1, \ell_2, \ldots, \ell_s$.
- No solution to systems of congruences ⇒ **Contradiction**.

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• $\ell_1 = 5$ is inert in $\mathbb{Q}(\sqrt{-47})$ and $G_5 = 6$.



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- $\ell_1 = 5$ is inert in $\mathbb{Q}(\sqrt{-47})$ and $G_5 = 6$.
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- But when $m_0 \in \{0, 1, 2, 4\}$, the set $\widetilde{\psi}^{-1}(m_0 \cdot \widetilde{R})$ is a pair of \mathbb{F}_5 -points.
- Conclusion: $m \equiv 3 \text{ or } 5 \pmod{6}$.

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- Conclusion: $m \equiv 3 \text{ or } 5 \pmod{6}$.
- $\ell_2 = 7$ splits in $\mathbb{Q}(\sqrt{-47})$, $G_7 = 12$, and we find that $m \equiv 0, 3, 4, 7$, or 11 (mod 12).

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- $\ell_3 = 11$ is inert in $\mathbb{Q}(\sqrt{-47})$, $G_7 = 12$, and we find that $m \equiv 1, 2, 5, 7$, or 10 (mod 12). Contradiction.

Let $X = X_0(53)$ and suppose $P \in X(\mathbb{Q}(\sqrt{-47})) \setminus X(\mathbb{Q})$.

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Conclusion: $X_0(53)(\mathbb{Q}(\sqrt{-47})) = X_0(53)(\mathbb{Q}).$

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- $\ell_3 = 11$ is inert in $\mathbb{Q}(\sqrt{-47})$, $G_7 = 12$, and we find that $m \equiv 1, 2, 5, 7$, or 10 (mod 12). **Contradiction.**

Conclusion: $X_0(53)(\mathbb{Q}(\sqrt{-47})) = X_0(53)(\mathbb{Q}).$

In fact, $X_0(53)(\mathbb{Q}(\sqrt{d})) = X_0(53)(\mathbb{Q})$ for any quadratic field $\mathbb{Q}(\sqrt{d})$ in which 5 and 11 are inert, and 7 splits.

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 - Expected, since $\psi^{-1}(1 \cdot R) \subset X(\mathbb{Q}(\sqrt{-11})) \setminus X(\mathbb{Q})$.

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Examples

Violating the Hasse principle

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The sieve

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Thank you!