Sieving for quadratic points on bielliptic curves

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Is this true? **Yes!**

- Ozman and Siksek. Quadratic Points on Modular Curves, 2018.
- Box. Quadratic points on modular curves with infinite Mordell–Weil group, 2019.
- Najman and Trbović. Splitting of primes in number fields generated by points on some modular curves, 2020.
- Banwait. Explicit isogenies of prime degree over quadratic fields, 2021.
- M. Fermat's Last Theorem and modular curves over real quadratic fields, 2021.
- Najman and Vukorepa. Quadratic points on bielliptic modular curves, 2021.
- M. On elliptic curves with p-isogenies over quadratic fields, 2022.
- Banwait and Derickx. Explicit isogenies of prime degree over number fields, 2022.
- Vukorepa. Isogenies over quadratic fields of elliptic curves with rational j-invariant, 2022.
- Banwait, Najman, Padurariu. Cyclic isogenies of elliptic curves over fixed quadratic fields, 2022.
- Adžaga, Keller, Najman, M., Ozman, and Vukorepa. Computing quadratic points on modular curves $X_0(N)$, 2023.

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- Mazur and Kenku looked after the case of rational points on $X_0(N)$ a long time ago and quadratic points are the next best thing. ☹
- Studying quadratic points on $X_0(N)$ is hard enough to be interesting, but not too hard.

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- Studying quadratic points on $X_0(N)$ is hard enough to be interesting, but not too hard.

- Deepen our understanding of modular curves.
- Develop techniques for studying low-degree points on curves.
- Deepen our understanding of the arithmetic of elliptic curves and their Galois representations.
- Concrete applications to the modular method for solving Diophantine equations.

Computing quadratic points

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What does **"computing the quadratic points on $X_0(N)$"** actually mean?

1. If $X_0(N)$ has **finitely** many quadratic points (as we range over all quadratic fields) then this means **writing them all down on an explicit model**.
2. If $X_0(N)$ has **infinitely** many quadratic points as we range over all quadratic fields then this means **writing down all the points that do not come from a 'geometric family'**.

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Quadratic points have been computed on all $X_0(N)$ with $2 \leq g(X_0(N)) \leq 8$.

Bielliptic curves $X_0(N)$

Let $N \in \mathcal{N} = \{53, 61, 79, 83, 89, 101, 131\}$.

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Theorem (Box, Najman–Vukorepa, 2021)

Let $N \in \mathcal{N}$ and let P be a quadratic point on $X_0(N)$. Then $\psi(P) = \psi(P^\sigma) \in X_0^+(N)(\mathbb{Q})$.

This result is **great**, but it does not determine $X_0(N)(\mathbb{Q}(\sqrt{d}))$ for a fixed quadratic field $\mathbb{Q}(\sqrt{d})$.

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Theorem (M., 2023)

Let $N \in \mathcal{N} = \{53, 61, 79, 83, 89, 101, 131\}$. Let $d \in \mathbb{Z}$ such that $|d| < 100$. Then

$$\exists P \in X_0(N)(\mathbb{Q}(\sqrt{d}))\backslash X_0(N)(\mathbb{Q}) \iff d \in \mathcal{D}_N,$$

where

$$\mathcal{D}_{53} = \{-43, -11, -7, -1\}, \qquad \mathcal{D}_{61} = \{-19, -3, -1, 61\},$$
$$\mathcal{D}_{79} = \{-43, -7, -3\}, \qquad \mathcal{D}_{83} = \{-67, -43, -19, -2\},$$
$$\mathcal{D}_{89} = \{-67, -11, -2, -1, 89\}, \qquad \mathcal{D}_{101} = \{-43, -19, -1\},$$
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- Write G_ℓ for the order of $\widetilde{R}$ in $E(\mathbb{F}_\ell)$.
- $m \equiv m_0 \pmod{G_\ell}$ for some $m_0 \in \{0, 1, 2, \ldots, G_\ell - 1\}$.

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- $\widetilde{\psi}^{-1}(m_0 \cdot \widetilde{R})$ will either be:
  1. A pair of (distinct) points in $\widetilde{X}(\mathbb{F}_\ell)$.
  2. A pair of (distinct) points in $\widetilde{X}(\mathbb{F}_{\ell^2})$ (not in $\widetilde{X}(\mathbb{F}_\ell)$).
  3. A single point in $\widetilde{X}(\mathbb{F}_\ell)$.

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   If ℓ ramifies or is inert in $\mathbb{Q}(\sqrt{d})$ then $\{\widetilde{P}, \widetilde{P^\sigma}\}$ is a single $\mathbb{F}_\ell$-point or a pair of $\mathbb{F}_{\ell^2}$-points. Contradiction.

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We try and rule out each $m_0 \in \{0, 1, 2, \ldots, G_\ell - 1\}$ to come up with a list of possibilities for $m \pmod{G_\ell}$.

So far: list of possibilities for $m \pmod{G_\ell}$.

- Repeat with several primes $\ell_1, \ell_2, \ldots, \ell_s$.
- No solution to systems of congruences $\Rightarrow$ **Contradiction**.

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Conclusion: $X_0(53)(\mathbb{Q}(\sqrt{-47})) = X_0(53)(\mathbb{Q})$.

In fact, $X_0(53)(\mathbb{Q}(\sqrt{d})) = X_0(53)(\mathbb{Q})$ for any quadratic field $\mathbb{Q}(\sqrt{d})$ in which 5 and 11 are inert, and 7 splits.

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- Apply the sieve using primes $\ell < 1000$:

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- Expected, since $\psi^{-1}(1 \cdot R) \subset X(\mathbb{Q}(\sqrt{-11}))\backslash X(\mathbb{Q})$.

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Thank you!