

On some Generalized Fermat Equations of the form

$$x^2 + y^{2n} = z^p$$

Philippe Michaud-Jacobs

Abstract

The primary aim of this paper is to study the generalized Fermat equation

$$x^2 + y^{2n} = z^{3p}$$

in coprime integers x , y , and z , where $n \geq 2$ and p is a fixed prime. Using modularity results over totally real fields and the explicit computation of Hilbert cuspidal eigenforms, we provide a complete resolution of this equation in the case $p = 7$, and obtain an asymptotic result for fixed p . Additionally, using similar techniques, we solve a second equation, namely $x^{2\ell} + y^{2m} = z^{17}$, for primes $\ell, m \neq 5$.

1 Introduction

The Diophantine equation

$$x^p + y^q = z^r \tag{1}$$

for integers $p, q, r \geq 2$ is known as the *generalized Fermat equation*. Since Wiles' proof of Fermat's Last Theorem [22] some 25 years ago, it has been the subject of intense study, and has been resolved for many infinite families of integer triples (p, q, r) . The *generalized Fermat conjecture*, also known as the *Fermat–Catalan conjecture*, states that there are only finitely many triples of non-zero coprime integer powers (x^p, y^q, z^r) satisfying (1) with $1/p + 1/q + 1/r < 1$. We refer the reader to [5] for an excellent survey on the generalized Fermat equation, which assumes very little background knowledge. We also refer to [2] for all (unconditional) results on this equation prior to 2016, as well as [19, Theorem 8.7], [1, Theorem 1] and [7, Corollary 8.2] for (unconditional) results on this equation since 2016.

Date: September 2, 2021.

Keywords: Elliptic curves, Frey curve, level-lowering, generalized Fermat, Galois representations, irreducibility, Hilbert modular forms.

MSC2010: 11D41, 11F80, 11G05, 11F41.

The author is supported by an EPSRC studentship.

The author has previously used the name Philippe Michaud-Rodgers.

The primary aim of this paper is to study the equation

$$x^2 + y^{2n} = z^{3p}, \quad (2)$$

for $n \geq 2$ and p a fixed prime. By [2, Theorem 1], this equation has no solutions in non-zero coprime integers for $p = 2, 3$, or 5 . Using results on the modularity of elliptic curves over totally real fields, irreducibility of Galois representations, and the explicit computation of Hilbert cuspidal eigenforms, we obtain a complete resolution of this equation in the case $p = 7$.

Theorem 1. *Let $n \geq 2$. The equation*

$$x^2 + y^{2n} = z^{21}$$

has no solutions in non-zero coprime integers x, y , and z .

We also obtain the following asymptotic result.

Theorem 2. *There exists an effectively computable constant $C(p)$, depending only on the prime p , such that for all primes $\ell > C(p)$, the equation*

$$x^2 + y^{2\ell} = z^{3p}$$

has no solutions in non-zero coprime integers x, y , and z .

For small values of $p > 7$, it is possible to compute such a constant $C(p)$. For example, in Proposition 5.2 we find that we can take $C(11) = 10^{2930}$.

We start, in Section 2, by stating some known results on equation (2) and introducing two technical lemmas. Then, in Section 3, we carry out a descent argument. This is initiated by a factorisation of the left-hand side of (2) over the field $\mathbb{Q}(i)$, which leads to new ternary equations over the maximal real subfield of the p th cyclotomic field. In Section 4, we associate a family of Frey elliptic curves to these equations, and use standard level-lowering results to relate these curves (or more precisely their Galois representations) to Hilbert cuspidal eigenforms. Crucially, these Frey curves will have multiplicative reduction at the primes above 3, and this will allow us to circumvent the issues posed by the trivial solutions (those solutions satisfying $xyz = 0$). These arguments allow us to prove Theorems 1 and 2 in Sections 5 and 6 respectively.

Finally, in Section 7 we consider a different, although similar equation, namely

$$x^{2\ell} + y^{2m} = z^p, \quad (3)$$

for primes ℓ and m , and p a fixed odd prime. This equation has no solutions in non-zero coprime integers x, y , and z , for $p \in \{3, 5, 7, 11\}$, and for $p = 13$ when $\ell, m \neq 7$ (see [1, Theorem 1.1], [2, Theorem 1], [4, Theorem 1], and

[3, Theorem 1]). The case $p = 13$ was then completed in [7, Corollary 8.2]. We partially extend these results to the case $p = 17$. The main difficulty in the case $p = 17$ is the impossibility of computing the full Hilbert cusp form data at the required levels. We overcome this by working directly with Hecke operators to prove the following theorem.

Theorem 3. *Let $\ell, m \neq 5$ be primes. The equation*

$$x^{2\ell} + y^{2m} = z^{17}$$

has no solutions in non-zero coprime integers x, y , and z .

By [19, Theorem 8.7], the equation $x^5 + y^5 = z^{17}$ has no solutions in non-zero coprime integers x, y , and z . Using this we obtain the following corollary to Theorem 3.

Corollary 4. *Let $n \geq 2$. The equation*

$$x^{2n} + y^{2n} = z^{17}$$

has no solutions in non-zero coprime integers x, y , and z .

The Magma [8] code used to support the computations in this paper can be found at:

<https://warwick.ac.uk/fac/sci/math/people/staff/michaud/c/>

I am extremely grateful to my supervisors Samir Siksek and Damiano Testa for their support in writing this paper.

2 Known Results and Preliminaries

If a triple of integers (x, y, z) satisfies (1), then we shall say that the solution is *non-trivial* if $xyz \neq 0$, and *primitive* if x, y , and z are coprime.

We start by stating what we can deduce about solutions to (2) from other results on generalized Fermat equations.

Theorem 2.1 ([4], [11], [2]). *Let $n \geq 2$ and let p be prime. Suppose there exist non-zero coprime integers x, y , and z satisfying*

$$x^2 + y^{2n} = z^{3p}.$$

Then $n > 10^7$, $p > 5$, $y \equiv 3 \pmod{6}$, x is even, and z is odd.

Proof. If $n = 2$, then there are no non-trivial primitive solutions to (2) by [4, Theorem 1], so we will suppose $n > 2$. If $p = 2, 3$, or 5 , then there are no non-trivial primitive solutions by [2, Theorem 1]. Next, by the work of Dahmen on the equation $x^2 + y^{2n} = z^3$ (see [11, pp. 3–5]), which builds on work of Chen [9], we obtain that $n > 10^7$ and $y \equiv 3 \pmod{6}$. Finally, since y is odd, we see that x is even and z is odd by considering the equation modulo 4. \square

We note that the equation $x^2 + y^{2n} = z^3$ admits the trivial solutions $(\pm 1, 0, 1)$ and $(0, \pm 1, 1)$. The trivial solution $(0, \pm 1, 1)$ would usually render unfeasible a successful application of the modular method. The reason that this trivial solution can be ruled out in this case is because the corresponding Frey curve (which is defined over \mathbb{Q}) at this solution has complex multiplication (see [11, p. 1306] and [6]).

By Theorem 2.1, we can restrict to the case $n = \ell$, prime, with $\ell > 10^7$.

Proposition 2.2. *Let $\ell, p \geq 5$ be primes. Suppose there exist non-zero coprime integers x, y , and z satisfying*

$$x^2 + y^{2\ell} = z^3.$$

If $p \mid y$, then $\ell < (\sqrt{p} + 1)^2$.

Proof. By [11, pp. 1306–1307], there exist coprime integers u and v , with $uv \neq 0$, $3 \mid v$, u even, and v odd, such that

$$y^\ell = v(3u^2 - v^2). \tag{4}$$

We associate to (4) the Frey elliptic curve

$$W : Y^2 = X^3 + 2uX^2 + v^2,$$

which has minimal discriminant and conductor

$$\Delta_{\min} = 2^6 \cdot 3^{-3} \cdot v^4(3u^2 - v^2), \quad N = 2^5 \cdot 3 \cdot \text{Rad}_{2,3}(\Delta_{\min}).$$

Here, $\text{Rad}_{2,3}(\Delta_{\min})$ denotes the product of all primes other than 2 or 3 dividing Δ_{\min} .

Still following [11, pp. 1306–1307], we level-lower the curve W , and find that $\bar{\rho}_{W,\ell} \sim \bar{\rho}_{W_0,\ell}$, for W_0 an elliptic curve of conductor $96 = 2^5 \cdot 3$. Now, if $p \mid y$, then $p \mid y^\ell = v(3u^2 - v^2)$, so $p \mid \Delta_{\min}$ and W has multiplicative reduction at p . Also $p \nmid 96$ as $p \geq 5$, so

$$\ell \mid p + 1 + a_p(W_0) \quad \text{or} \quad \ell \mid p + 1 - a_p(W_0).$$

Then $|a_p(W_0)| \leq 2\sqrt{p}$, so

$$\ell < p + 1 + 2\sqrt{p} = (\sqrt{p} + 1)^2,$$

as required. □

In order to prove Theorems 1 and 2, we will start (in Section 3) by carrying out a descent argument over the maximal real subfield of the p th cyclotomic field. In this section, we introduce some notation as well as two lemmas that will be useful in the sequel.

Let p be an odd prime. We write ζ_p for a primitive p th root of unity, so that $\mathbb{Q}(\zeta_p)$ is the p th cyclotomic field which has degree $p - 1$. We write $K = \mathbb{Q}(\zeta_p + \zeta_p^{-1})$ for the maximal real subfield of $\mathbb{Q}(\zeta_p)$. The field K is a totally real abelian Galois field of degree $(p - 1)/2$. We write \mathcal{O}_K for the ring of integers of the field K . Then $\mathcal{O}_K = \mathbb{Z}[\zeta_p + \zeta_p^{-1}]$. The prime p is totally ramified in K , and we write \mathfrak{p} for the unique prime ideal of \mathcal{O}_K above p . We have

$$p\mathcal{O}_K = \mathfrak{p}^{(p-1)/2}.$$

More generally we will denote prime ideals of \mathcal{O}_K by \mathfrak{q} , or sometimes by \mathfrak{q}_m for a prime above the rational prime $m \in \mathbb{Z}$. We also introduce the notation

$$\theta_j := \zeta_p^j + \zeta_p^{-j}, \quad \text{for } j = 1, \dots, (p - 1)/2.$$

For further background on cyclotomic fields and their subfields, we refer to [21, pp. 1–19].

Lemma 2.3 ([1, Lemma 3.1]). *For $1 \leq j \leq (p - 1)/2$ we have*

$$\theta_j, \theta_j + 2 \in \mathcal{O}_K^\times \quad \text{and} \quad (\theta_j - 2)\mathcal{O}_K = \mathfrak{p}.$$

For $1 \leq j < k \leq (p - 1)/2$ we have

$$(\theta_j - \theta_k)\mathcal{O}_K = \mathfrak{p}.$$

Lemma 2.4. *For any $m \geq 1$ and $1 \leq j \leq (p - 1)/2$ we have*

$$\theta_j^{2^{(p-1)m}} \equiv \theta_j + 2 \pmod{4\mathcal{O}_K}.$$

Proof. Write $r = (p - 1)m \geq 2$. Then

$$\theta_j^{2^r} = (\zeta_p^j + \zeta_p^{-j})^{2^r} = \sum_{i=0}^{2^r} \binom{2^r}{i} \zeta_p^{ji} \zeta_p^{-j(2^r-i)}.$$

Using Legendre's formula for the prime decomposition of a factorial, we have that $v_2 \binom{2^t}{2^{t-1}} = 1$ for any $t \geq 1$. From this, and the identity

$$\binom{2^r}{i} = \sum_{t=0}^i \binom{2^{r-1}}{t} \binom{2^{r-1}}{2^{r-1}-t},$$

it is straightforward to show by induction on r that

$$v_2 \binom{2^r}{i} \geq 2, \quad \text{for } 0 < i < 2^r, i \neq 2^{r-1}.$$

Then

$$\sum_{i=0}^{2^r} \binom{2^r}{i} \zeta_p^{ji} \zeta_p^{-j(2^r-i)} \equiv (\zeta_p^j)^{2^{m(p-1)}} + 2 + (\zeta_p^{-j})^{2^{m(p-1)}} \equiv \theta_j + 2 \pmod{4\mathcal{O}_K},$$

with the last equivalence coming from the fact that $2^{m(p-1)} \equiv 1 \pmod{p}$. \square

3 Descent

Suppose there exist coprime integers x , y , and z , satisfying

$$x^2 + y^{2\ell} = z^{3p} \quad (5)$$

for primes $\ell > 3$ and $p > 5$. We wish to obtain a factorisation for y^l over the field K . We follow the descent argument of [1, pp. 1154–1155]. We start by considering the following factorisation over $\mathbb{Q}(i)$:

$$(y^\ell + xi)(y^\ell - xi) = (z^3)^p.$$

Since x and y are coprime, there exist $a, b \in \mathbb{Z}$ such that

$$y^\ell + xi = (a + bi)^p \quad \text{and} \quad z^3 = a^2 + b^2.$$

Comparing real and imaginary parts, we obtain

$$y^\ell = \frac{(a + bi)^p + (a - bi)^p}{2}. \quad (6)$$

Since y and z are coprime, we see that a and b are also coprime.

We recall the standard factorisation, for $u, v \in \mathbb{C}$,

$$u^p + v^p = \prod_{j=0}^{p-1} (u + v\zeta_p^j) = (u + v) \prod_{j=1}^{(p-1)/2} (u + v\zeta_p^j)(u + v\zeta_p^{-j}).$$

Applying this to (6), we obtain

$$\begin{aligned} y^\ell &= a \cdot \prod_{j=1}^{(p-1)/2} \left((a + bi) + (a - bi)\zeta_p^j \right) \cdot \left((a + bi) + (a - bi)\zeta_p^{-j} \right) \\ &= a \cdot \prod_{j=1}^{(p-1)/2} \left((\theta_j + 2)a^2 + (\theta_j - 2)b^2 \right). \end{aligned}$$

So

$$y^\ell = a \cdot \prod_{j=1}^{(p-1)/2} \beta_j, \quad (7)$$

where

$$\beta_j := (\theta_j + 2)a^2 + (\theta_j - 2)b^2, \quad \text{for } j = 1, \dots, (p-1)/2.$$

From (7), we see that a is odd (since y is odd), and so b is even.

By Theorem 2.1, we know that $3 \mid y$. We now claim that $3 \mid a$. Suppose not. If $3 \nmid b$, then $z^3 = a^2 + b^2 \equiv -1 \pmod{3}$, so $z \equiv -1 \pmod{3}$, a contradiction by reducing (5) mod 3. So $3 \mid b$. Write \mathfrak{q}_3 for a prime of

K above 3. Then since $3 \mid y$ but $3 \nmid a$, we have that $\mathfrak{q}_3 \mid \beta_j$ for some $j \in \{1, \dots, (p-1)/2\}$. So

$$\mathfrak{q}_3 \mid \beta_j - (\theta_j - 2)b^2 = (\theta_j + 2)a^2.$$

So $\mathfrak{q} \mid (\theta_j + 2) \in \mathcal{O}_K^\times$, a contradiction. We conclude that $3 \mid a$.

Lemma 3.1. *Suppose $p \nmid y$. Then*

$$a = \alpha^\ell, \quad \beta_j \mathcal{O}_K = \mathfrak{b}_j^\ell,$$

where $\alpha \in \mathbb{Z}$ with $\alpha \equiv 3 \pmod{6}$, and $\alpha \mathcal{O}_K, \mathfrak{b}_1, \dots, \mathfrak{b}_{(p-1)/2}$ are pairwise coprime ideals of \mathcal{O}_K , all coprime to $2p$.

Proof. We follow the first part of the proof of [1, Lemma 4.1]. Since $2 \mid b$ and $2 \nmid a$, we see that the β_j are coprime to $2\mathcal{O}_K$. Let \mathfrak{q} be a prime of K and suppose \mathfrak{q} divides a and β_j . Then it also divides $(\theta_j - 2)b^2$, and since a and b are coprime, it divides $(\theta_j - 2)\mathcal{O}_K = \mathfrak{p}$. So $\mathfrak{q} = \mathfrak{p}$, a contradiction, since $p \nmid y$.

Next, suppose \mathfrak{q} is a prime of K with $\mathfrak{q} \mid \beta_j, \beta_k$ for $j \neq k$. Then

$$\begin{aligned} \mathfrak{q} \mid (\theta_k - 2)\beta_j - (\theta_j - 2)\beta_k &= ((\theta_j + 2)(\theta_k - 2) - (\theta_k + 2)(\theta_j - 2))a^2, \\ \mathfrak{q} \mid (\theta_j + 2)\beta_k - (\theta_k + 2)\beta_j &= ((\theta_j + 2)(\theta_k - 2) - (\theta_k + 2)(\theta_j - 2))b^2. \end{aligned}$$

Since a and b are coprime, we see that

$$\mathfrak{q} \mid (\theta_j + 2)(\theta_k - 2) - (\theta_k + 2)(\theta_j - 2) = 4(\theta_k - \theta_j).$$

Since $(\theta_k - \theta_j)\mathcal{O}_K = \mathfrak{p}$ and β_j is odd, we have $\mathfrak{q} = \mathfrak{p}$, another contradiction. So the ideals $a\mathcal{O}_K$ and $\beta_j\mathcal{O}_K$ are pairwise coprime, and also all coprime to $2p$. The lemma follows. \square

4 Frey Curves

We will now associate a Frey curve (in fact a family of Frey curves) to (7) when $p \nmid y$. The key difference between the Frey curve we define compared to the one defined in [1, p. 1156] is its behaviour at the primes of K above 2. The Frey curve we define will have additive, rather than multiplicative, reduction at the primes above 2, and is therefore not semistable. The main consequences of this are that we will need to apply different modularity and irreducibility results in Sections 5 and 6, and it will also limit our ability to compute Hilbert cusp forms.

Suppose $p \nmid y$. We now fix j and k such that $1 \leq j < k \leq (p-1)/2$. Let

$$u = \beta_j, \quad v = -\frac{(\theta_j - 2)}{(\theta_k - 2)} \cdot \beta_k, \quad w = \frac{4(\theta_j - \theta_k)}{(\theta_k - 2)} \cdot a^2.$$

Then $u + v + w = 0$, and by Lemma 3.1 we have

$$u\mathcal{O}_K = \mathfrak{b}_j^\ell, \quad v\mathcal{O}_K = \mathfrak{b}_k^\ell, \quad w\mathcal{O}_K = 4 \cdot \alpha^{2\ell} \cdot \mathcal{O}_K.$$

We define the Frey elliptic curve

$$E = E_{j,k} : Y^2 = X(X - v)(X + w).$$

We note that u, v , and w , are defined as in [1, p. 1156], but the Frey curve we have chosen differs. We discuss this choice in Remark 4.2.

Write $\text{Rad}(\mathfrak{c})$ to denote the product of prime ideals dividing a non-zero ideal \mathfrak{c} of \mathcal{O}_K .

Lemma 4.1. *The curve E has good reduction at \mathfrak{p} and multiplicative reduction at all primes of K above 3. It has minimal discriminant and conductor*

$$\mathcal{D} = 2^8 \alpha^{4\ell} \mathfrak{b}_j^{2\ell} \mathfrak{b}_k^{2\ell}, \quad \mathcal{N} = 2^3 \cdot \text{Rad}(\alpha \mathfrak{b}_j \mathfrak{b}_k).$$

Proof of Lemma 4.1. We have $\Delta = 16u^2v^2w^2$ and $c_4 = 16(w^2 - uv)$. We see that $\mathfrak{p} \nmid \Delta$, so E has good reduction at \mathfrak{p} . Since c_4 and Δ are coprime away from 2, the Frey curve is semistable away from 2. The curve E has multiplicative reduction at all primes above 3 because $3 \mid \alpha$ by Lemma 3.1.

Let \mathfrak{q} be a prime of K above 2. We note that the model is minimal at \mathfrak{q} since $v_{\mathfrak{q}}(\Delta) = 8 < 12$. So $v_{\mathfrak{q}}(\mathcal{D}) = 8$, and it remains to show that we have $v_{\mathfrak{q}}(\mathcal{N}) = 3$. We do this using Tate's algorithm [20]. We follow the exposition of Tate's algorithm in [18, pp. 364–368] and outline the main steps.

Since $v_{\mathfrak{q}}(2) = 1$, we can take 2 as a uniformiser for the local field $K_{\mathfrak{q}}$. Write k for the residue field $\mathcal{O}_K/\mathfrak{q}$. Now, $\mathfrak{q}^2 \mid w$, so the point $(\tilde{0}, \tilde{0})$ is a singular point of E/k , so $\mathfrak{q} \mid a_3, a_4, a_6$. We then find that $\mathfrak{q} \mid b_2$, $\mathfrak{q}^2 \mid a_6$, and $\mathfrak{q}^3 \mid b_8$, so we proceed directly to Step 6.

Note that $a_2 \equiv v \equiv \theta_j \pmod{\mathfrak{q}}$. We would like to choose γ such that $\gamma^2 \equiv \theta_j \pmod{\mathfrak{q}}$. Write f for the inertia degree of \mathfrak{q} . We choose

$$\gamma = \theta_j^{2^{(p-1)f-1}}.$$

Then $\gamma^2 = \theta_j^{2^{f(p-1)}} \equiv \theta_j \pmod{\mathfrak{q}}$. We then apply the transformation $Y \mapsto Y - \gamma X$ to obtain

$$E' : Y^2 - 2\gamma XY = X(X - v)(X + w) - \gamma^2 X^2.$$

Continuing with Step 6, we consider the polynomial

$$P(T) := T^3 + \frac{a_2}{2}T^2 + \frac{a_4}{2^2}T + \frac{a_6}{2^3} = T \left(T^2 + \frac{a_2}{2}T + \frac{a_4}{2^2} \right).$$

Here, $a_4 = -vw \not\equiv 0 \pmod{\mathfrak{q}^3}$, so P does not have a triple root in \bar{k} , and we continue to Step 7. We claim that $\mathfrak{q}^2 \mid a_2 = -v + w - \gamma^2$, so that P has

a double root in \bar{k} . Since a is odd and b is even, $a^2 \equiv 1 \pmod{\mathfrak{q}}$ and $b^2 \equiv 0 \pmod{\mathfrak{q}^2}$. So

$$\begin{aligned} -v + w - \theta_j^{2^{(p-1)f}} &\equiv \frac{(\theta_j - 2)(\theta_k + 2)}{(\theta_k - 2)} - \theta_j^{2^{(p-1)f}} \pmod{\mathfrak{q}^2} \\ &\equiv \theta_j + 2 - \theta_j^{2^{(p-1)f}} \pmod{\mathfrak{q}^2} \\ &\equiv 0 \pmod{\mathfrak{q}^2}, \end{aligned}$$

where we have applied Lemma 2.4 in the final step.

We now start the subprocedure of Step 7 by choosing $\varphi \in \mathcal{O}_K$ such that $\varphi^2 \equiv -vw/2^2 \pmod{\mathfrak{q}}$. We note that $\mathfrak{q} \nmid \varphi$. We apply the transformation $X \mapsto X + 2\varphi$. We verify that our new polynomial $P(T)$ (defined as above) now has a double root at $\bar{0}$. Our new a_3 value is $-4\gamma\varphi$. We have $v_{\mathfrak{q}}(-4\gamma\varphi) = 2$, so the polynomial

$$Y^2 + \frac{a_3}{2^2}Y + \frac{a_6}{2^4}$$

has distinct roots in \bar{k} , concluding our application of Tate's algorithm. We read off that $v_{\mathfrak{q}}(\mathcal{N}) = v_{\mathfrak{q}}(\Delta) - 5 = 3$, with the reduction type at \mathfrak{q} given by the Kodaira symbol I_1^* . \square

We note that for a fixed value of p , it is possible to verify whether E has split or non-split multiplicative reduction at the primes above 3. This is because $a^2 \equiv 0 \pmod{3}$ and $b^2 \equiv 1 \pmod{3}$, so if \mathfrak{q}_3 denotes a prime of K above 3, we find that

$$c_6 = -32(v-w)(w-u)(u-v) \equiv (\theta_j + 1)^3 \pmod{\mathfrak{q}_3}.$$

The curve E has split multiplicative reduction at \mathfrak{q}_3 if and only if $-c_6$ is a square $\pmod{\mathfrak{q}_3}$ (see [18, pp. 442–444]). For example, E has split multiplicative reduction (for each choice of j) at the unique prime above 3 when $p = 7$, but non-split multiplicative reduction (for each choice of j) at the unique prime above 3 when $p = 11$.

Remark 4.2. In order to simplify the computations in Sections 5 and 6, we would like the conductor of E to be as small as possible. In particular, if we let \mathfrak{q} be a prime of K above 2, then we would like to minimise $v_{\mathfrak{q}}(\mathcal{N})$. The best we can hope for would be to decrease this valuation from 3 to 2. We cannot decrease this valuation further, as E has potential good reduction at \mathfrak{q} . Unfortunately, we found that by twisting E by units and permuting u, v , and w , that we could only increase $v_{\mathfrak{q}}(\mathcal{N})$ to 4. The curve E we have chosen satisfies $v_{\mathfrak{q}}(\mathcal{N}) = 3$ and allows for the easiest application of Tate's algorithm.

5 Asymptotic Results

We would like to apply a suitable level-lowering result to the Frey curve E , and combine this with Proposition 2.2 in order to conclude that any primitive solution (x, y, z) to (5) is trivial, at least for ℓ large enough. We first fix the following notation. We will write \mathfrak{f} for a Hilbert cuspidal eigenform over K of parallel weight 2, and denote by $\mathbb{Q}_{\mathfrak{f}}$ its Hecke eigenfield (the field generated by its eigenvalues under the action of the Hecke operators). If \mathfrak{f} is new at its level, then we will simply refer to \mathfrak{f} as a *Hilbert newform*.

Lemma 5.1. *Let E be the Frey curve defined in Section 4. Suppose E is modular and that $\bar{\rho}_{E,\ell}$ is irreducible. Then $\bar{\rho}_{E,\ell} \sim \bar{\rho}_{\mathfrak{f},\lambda}$ for a Hilbert newform \mathfrak{f} at level \mathcal{N}_{ℓ} , where*

$$\mathcal{N}_{\ell} = 2^3 \cdot \mathcal{O}_K,$$

and $\lambda \mid \ell$ is a prime of $\mathbb{Q}_{\mathfrak{f}}$.

Proof. We apply [16, Theorem 7] to the curve E , which is the standard level-lowering result for elliptic curves defined over totally real fields. The statement follows from Lemma 4.1. \square

Using this, we can prove Theorem 2.

Proof of Theorem 2. We suppose (x, y, z) is a non-trivial primitive solution to (5). Suppose $p \nmid y$. Let E denote the Frey curve, defined in Section 4, associated to the solution (x, y, z) . Since K is a totally real abelian number field in which 3 is unramified, and E has semistable reduction (in fact multiplicative reduction) at all primes above 3, we know that E is modular by [14, Theorem 1.3].

Next, as K is a totally real Galois field and E is semistable away from 2, we can apply Theorem 2 of [15]. Write B_p for the non-zero constant, which depends only on p , defined in Theorem 1 of [15]. Then if $\ell \nmid p \cdot B_p$ (we include a factor of p , as p is the only prime that ramifies in K), then $\bar{\rho}_{E,\ell}$ is irreducible for $\ell > (1 + 3^{3h(p-1)})^2$, where h denotes the class number of K . Since $(1 + 3^{3h(p-1)})^2 > p$, it follows that $\bar{\rho}_{E,\ell}$ is irreducible for $\ell > C'(p)$, where

$$C'(p) := B_p \cdot (1 + 3^{3h(p-1)})^2.$$

Suppose $\ell > C'(p)$. Then applying Lemma 5.1, we have $\bar{\rho}_{E,\ell} \sim \bar{\rho}_{\mathfrak{f},\lambda}$, for a Hilbert newform \mathfrak{f} at level \mathcal{N}_{ℓ} , and $\lambda \mid \ell$ a prime of $\mathbb{Q}_{\mathfrak{f}}$. We write d for the dimension of the space of Hilbert cusp forms that are new at level \mathcal{N}_{ℓ} . Let \mathfrak{q}_3 denote a prime of K above 3. Then E has multiplicative reduction at \mathfrak{q}_3 by Lemma 4.1. Write $a_{\mathfrak{q}_3}$ for the trace of Frobenius of $\bar{\rho}_{\mathfrak{f},\lambda}$ at \mathfrak{q}_3 . Then

$$\lambda \mid \text{Norm}(\mathfrak{q}_3) + 1 + a_{\mathfrak{q}_3}(\mathfrak{f}) \quad \text{or} \quad \lambda \mid \text{Norm}(\mathfrak{q}_3) + 1 - a_{\mathfrak{q}_3}(\mathfrak{f}).$$

It follows that

$$\begin{aligned} \ell &| \text{Norm}_{\mathbb{Q}_f/\mathbb{Q}}(\text{Norm}(\mathfrak{q}_3) + 1 + a_{\mathfrak{q}_3}(f)) \text{ or} \\ \ell &| \text{Norm}_{\mathbb{Q}_f/\mathbb{Q}}(\text{Norm}(\mathfrak{q}_3) + 1 - a_{\mathfrak{q}_3}(f)). \end{aligned}$$

The size of $a_{\mathfrak{q}_3}(f)$ is bounded by $2\sqrt{\text{Norm}(\mathfrak{q}_3)}$, and since $[\mathbb{Q}_f : \mathbb{Q}] < d$, we have

$$\ell \leq \left(\text{Norm}(\mathfrak{q}_3) + 1 + 2\sqrt{\text{Norm}(\mathfrak{q}_3)} \right)^d = (\sqrt{\text{Norm}(\mathfrak{q}_3)} + 1)^{2d}.$$

We set $C(p) = \max\{C'(p), (\sqrt{\text{Norm}(\mathfrak{q}_3)} + 1)^{2d}\}$. If, instead, $p \mid y$, then $\ell < (\sqrt{p} + 1)^2 < C'(p)$ by Proposition 2.2.

We conclude that if $\ell > C(p)$ then we have a contradiction, so no such non-trivial primitive solution exists. \square

Although the constant $C(p)$ in Theorem 2 is effectively computable, actually computing it is another matter. In the case $p = 7$, we are able to compute the Hilbert newforms (using Magma's Hilbert modular form package) at the level \mathcal{N}_ℓ and this allows us to compute a (relatively) small value for $C(7)$. Combining this with the fact that we have no solutions for $\ell < 10^7$ will allow us to prove Theorem 1 in the next section. Unfortunately, we were unable to compute the Hilbert newforms at level \mathcal{N}_ℓ for $p > 7$.

When $p \equiv 1 \pmod{4}$, it is possible to choose j and k appropriately and twist the Frey curve E (as in [1, p. 1157–1158]) so that E is defined over a subfield of K , but we were still unable to compute the Hilbert newforms (at the new required level) for $p = 13$ (or any larger $p \equiv 1 \pmod{4}$).

Even though we cannot compute the required Hilbert newforms for $p > 7$, we can still (following the proof of Theorem 2) compute a value $C(p)$, provided we can bound the dimensions of the spaces of Hilbert cusp forms that are new at the level \mathcal{N}_ℓ . We consider the cases $p = 11, 13$, and 17 .

Proposition 5.2. *Let $p = 11, 13$, or 17 . Suppose $\ell > C(p)$, with ℓ prime, where*

$$C(11) = 10^{2930}, \quad C(13) = 10^{90946}, \quad C(17) = 10^{160315410}.$$

Then the equation

$$x^2 + y^{2\ell} = z^{3p}$$

has no solutions in non-zero coprime integers x, y , and z .

Proof. We follow the proof of Theorem 2, computing explicit constants. We first compute the quantity B_p . We find that

$$B_{11} = 1, \quad B_{13} = 2^{18} \cdot 3^{12} \cdot 5^6 \cdot 13^3, \quad B_{17} = 2^{32} \cdot 5^8 \cdot 13^8 \cdot 17^4 \cdot 67^8.$$

Since $\ell > 10^7$, we can safely ignore the contribution from $\ell \mid p \cdot B_p$. Since K has class number 1 in each case, we set $C'(p) = (1 + 3^{3(p-1)})^2$. Next, 3 is inert in K in each case, and $\text{Norm}(3 \cdot \mathcal{O}_K) = 3^{(p-1)/2}$. The dimensions d of the spaces of Hilbert cusp forms that are new at level \mathcal{N}_ℓ can be computed directly with `Magma`, and are 1201, 31422, and 41883752, for $p = 11, 13$, and 17 respectively. We set

$$C(p) = \max \left\{ (1 + 3^{3(p-1)})^2, \left(\sqrt{3^{(p-1)/2}} + 1 \right)^{2d} \right\} = \left(\sqrt{3^{(p-1)/2}} + 1 \right)^{2d},$$

and the proposition follows. \square

As discussed after the proof of Theorem 2, for $p = 13$ and $p = 17$, we could work over a subfield of K and obtain smaller (although still very large) constants in the above proposition.

It would be interesting to see if it is possible to find a bound on the dimension of the space of Hilbert cusp forms that are new at level \mathcal{N}_ℓ in terms of p , or quantities associated to p . In this way, it would be possible to obtain a value for the constant $C(p)$ without the need for calculating the dimension explicitly.

6 The Equation $x^2 + y^{2n} = z^{21}$

We now set $p = 7$. The field $K = \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$ has degree 3. We would first like to prove the irreducibility of $\bar{\rho}_{E,\ell}$ for $\ell > 10^7$. As the curve E is not semistable, we cannot use the same techniques as in [1, pp. 1160–1166].

Lemma 6.1. *Let $p = 7$. Let E be the Frey curve defined in Section 4. Then $\bar{\rho}_{E,\ell}$ is irreducible for $\ell > 65 \cdot 6^6$.*

Proof. The prime 3 is inert in K , and by Lemma 4.1, E has multiplicative reduction at $3\mathcal{O}_K$. Since $3 > \deg(K) - 1 = 2$, we can apply Theorem 1.3 of [17] to deduce that the representation $\bar{\rho}_{E,\ell}$ is irreducible for $\ell > 65 \cdot 6^6$. \square

We note that $65 \cdot 6^6 < 10^7$, so $\bar{\rho}_{E,\ell}$ is irreducible for $\ell > 10^7$.

Proof of Theorem 1. By Theorem 2.1, we may restrict to the case of $n = \ell$ prime, with $\ell > 10^7$. We suppose (x, y, z) is a non-trivial primitive solution to (2). If $p \mid y$, then $\ell \leq 13$ by Proposition 2.2, so we will assume that $p \nmid y$, and associate the Frey curve E to this solution, as in Section 4.

The curve E is modular by [14, Theorem 1.3], or alternatively by applying the more general result that any elliptic curve defined over a totally real cubic field is modular [13, Theorem 1]. By Lemma 6.1, $\bar{\rho}_{E,\ell}$ is irreducible, and we can therefore apply Lemma 5.1 and level-lower. We have $\bar{\rho}_{E,\ell} \sim \bar{\rho}_{\mathfrak{f},\lambda}$, for a Hilbert newform \mathfrak{f} at level \mathcal{N}_ℓ , and $\lambda \mid \ell$ a prime of $\mathbb{Q}_{\mathfrak{f}}$. The prime 3 is inert in K , and we write $\mathfrak{q}_3 = 3 \cdot \mathcal{O}_K$, which has norm 27.

The dimension of the space of cusp forms that are new a level \mathcal{N}_ℓ is 5. We note that using this information alone is not enough to obtain a contradiction, as the bound obtained following the proof of Theorem 2 is $(\sqrt{27} + 1)^{10} > 10^7$. Instead, we compute the newform decomposition using `Magma`, and find there are five newforms at level \mathcal{N}_ℓ (each with $\mathbb{Q}_f = \mathbb{Q}$ necessarily). We can now mimic the proof of Theorem 2 to obtain the bound $\ell < (\sqrt{27} + 1)^2 < 39$, giving the desired contradiction. \square

We note that explicitly computing the values $a_{q_3}(f)$ for each of the five newforms at level \mathcal{N}_ℓ would allow us to obtain a sharper bound than $\ell < 39$ in the final step of the above proof, but since we are assuming $\ell > 10^7$ anyway, this is not necessary.

It is in fact possible to avoid the newform computation in the proof of Theorem 1. Setting $a = 1$ and $b = 0$ (which corresponds to the trivial solution $(0, 1, 1)$), the Frey curve E is an elliptic curve with conductor \mathcal{N}_ℓ . By modularity, we obtain a Hilbert newform f at level \mathcal{N}_ℓ with $\mathbb{Q}_f = \mathbb{Q}$. In particular, following the proof of Theorem 2, we obtain the improved inequality $\ell < (\sqrt{27} + 1)^8 < 10^7$.

7 The Equation $x^{2\ell} + y^{2m} = z^{17}$

We now consider the equation

$$x^{2\ell} + y^{2m} = z^{17} \tag{8}$$

for primes ℓ and m , and p a fixed odd prime. Our aim is to prove Theorem 3. We directly extend the work carried out in [1], and so we do not provide a very detailed exposition when the same ideas are present. We continue using the same notation as in the previous sections.

We suppose that (x, y, z) is a primitive solution to (8). We can interchange x and y to ensure that x is even. This is a key step, as it means that the only trivial solutions are $(0, \pm 1, 1)$. When the values corresponding to these trivial solutions are substituted into the Frey curve F_1 we define below, we will obtain a singular elliptic curve, and this will not endanger the success of the modular method. If $\ell = 2$ then there are no non-trivial primitive solutions by [4, Theorem 1]. If $\ell = 3$ then there are no non-trivial primitive solutions by [3, Theorem 1]. If $\ell = 17$ then there are no non-trivial primitive solutions by [12, Main Theorem]. We therefore suppose $\ell \geq 5$ and $\ell \neq 17$.

As in Section 3, there exist coprime integers a and b such that

$$x^\ell + y^m i = (a + bi)^p \quad \text{and} \quad z = a^2 + b^2.$$

Since x is even, a is even and b is odd.

We fix $j = 1$ and $k = 4$ so that θ_j and θ_k are interchanged by the unique involution in $\text{Gal}(K/\mathbb{Q})$. We write K' for the unique degree 2 subfield of K , and \mathfrak{B} for the unique prime of K' over p .

Case 1: $p \nmid x$

Let

$$u' = (\theta_k - 2)\beta_j, \quad v' = -(\theta_j - 2)\beta_k, \quad w' = 4(\theta_j - \theta_k) \cdot a^2.$$

The Frey elliptic curve we define is

$$F_1 : Y^2 = X(X - u')(X + v').$$

The curve F_1 is defined over K , but not over K' .

Case 2: $p \mid x$

Let

$$u' = \frac{\beta_j}{(\theta_j - 2)}, \quad v' = -\frac{\beta_k}{(\theta_k - 2)}, \quad w' = \frac{4(\theta_j - \theta_k)}{(\theta_j - 2)(\theta_k - 2)} \cdot a^2.$$

The Frey elliptic curve we define is

$$F_2 : Y^2 = X(X - u')(X + v').$$

By our choice of j and k , the curve F_2 is defined over K' , and we view it as a curve defined over K' .

Lemma 7.1 ([1, Lemma 6.1]). *Let $i = 1$ or 2 , so that F_i is one of the Frey curves defined above. Suppose $\bar{\rho}_{F_i, \ell}$ is irreducible and F_i is modular. Then $\bar{\rho}_{F_i, \ell} \sim \bar{\rho}_{\mathfrak{f}_i, \lambda_i}$ for a Hilbert newform \mathfrak{f}_i at level $\mathcal{N}_{\ell, i}$, where*

$$\mathcal{N}_{\ell, 1} = 2 \cdot \mathcal{O}_K, \quad \mathcal{N}_{\ell, 2} = 2 \cdot \mathfrak{B},$$

and $\lambda_i \mid \ell$ is a prime of $\mathbb{Q}_{\mathfrak{f}_i}$.

The curves F_1 and F_2 are modular by [14, Theorem 1.3] (or by using the modularity results in [1]). In order to apply this lemma, we must first prove the irreducibility of $\bar{\rho}_{F_i, \ell}$ for $i = 1$ and 2 . Although we need only prove this for $\ell > 5$, we prove irreducibility for $\ell = 5$ too, in the hope that our subsequent results may, in the future, be extended to include the case $\ell = 5$.

Lemma 7.2. *Let $i = 1$ or 2 , so that F_i is one of the Frey curves defined above. Then $\bar{\rho}_{F_i, \ell}$ is irreducible for $\ell \geq 5$.*

The main difficulty will be the case $\ell = 5$. We first prove the following lemma.

Lemma 7.3. *Let X be the modular curve $X_0(20)$, and let $K = \mathbb{Q}(\zeta_{17} + \zeta_{17}^{-1})$. Then $X(K) = X(\mathbb{Q}(\sqrt{17}))$.*

Proof. The curve X is an elliptic curve, given by Cremona label 20a1, and admits the following model over \mathbb{Q} :

$$X : y^2 = x^3 + x^2 + 4x + 4.$$

The minimal polynomial of θ_1 over K' is a quadratic polynomial and we set d to be its discriminant, so that $K = K'(\sqrt{d})$. We denote by X_d the quadratic twist of X by d . Then X and X_d are isomorphic over K , with an isomorphism given by

$$\varphi : X(K) \longrightarrow X_d(K), \quad (x, y) \longmapsto \left(\frac{x}{d}, \frac{y}{d\sqrt{d}} \right).$$

Using **Magma** we compute the following:

$$\begin{aligned} X(K') &= X(\mathbb{Q}(\sqrt{17})) = \mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} = \langle R \rangle \oplus \langle Q \rangle, \\ X_d(K') &= \mathbb{Z}/2\mathbb{Z}, \\ X(K)_{\text{tors}} &= X(\mathbb{Q})_{\text{tors}} = \mathbb{Z}/6\mathbb{Z} = \langle Q \rangle, \end{aligned}$$

where $R = \left((3\sqrt{17} + 5)/8, (9\sqrt{17} + 47)/16 \right)$ and $Q = (4, 10)$. We were unable to directly compute $X(K)$ with **Magma**. However, we can start by noting that

$$\text{Rank}(X(K)) = \text{Rank}(X(K')) + \text{Rank}(X_d(K')) = 1.$$

Next, let $P \in X(K)$ and let $\sigma \in \text{Gal}(K/K')$. Then

$$P + P^\sigma \in X(K') \quad \text{and} \quad \varphi(P - P^\sigma) \in X_d(K') = X_d(K')_{\text{tors}}.$$

Applying φ^{-1} we have $P - P^\sigma \in X(K)_{\text{tors}} = X(\mathbb{Q})_{\text{tors}}$. It follows that $2P = (P + P^\sigma) + (P - P^\sigma) \in X(K')$.

Now choose $P \in X(K)$ such that $X(K) = \langle P \rangle \oplus \langle Q \rangle$, and write $R = rP + sQ$ for $r, s \in \mathbb{Z}$ with $0 \leq s \leq 5$. If $r = 2r' + 1$ is odd, then

$$P = R - sQ - r'(2P) \in X(K').$$

If $r = 2r'$ is even, then $R - sQ = 2(r'P)$. To obtain a contradiction, it will suffice to show that R and $R + Q$ are not 2-divisible. The prime 137 is totally split in K . Let \mathfrak{q} denote a prime of K above 137, and let $k = \mathcal{O}_K/\mathfrak{q}$. We find that $X(k) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/60\mathbb{Z}$, and that the points \tilde{R} and $\tilde{R} + \tilde{Q}$ both have order 60; a contradiction in each case since there are no points of order 120 in $X(k)$. We conclude that $P \in X(K')$, and thus $X(K) = X(K') = X(\mathbb{Q}(\sqrt{17}))$. \square

Proof of Lemma 7.2. Suppose $\bar{\rho}_{F_i, \ell}$ is reducible. In Case 1, arguing as in [1, p. 1165], we find that there exists an elliptic curve defined over K with good reduction at \mathfrak{p} , full 2-torsion over K , and a torsion point of order 2ℓ over K . By the Hasse–Weil bounds, we have

$$\ell \leq \frac{(\sqrt{17} + 1)^2}{4} < 7,$$

so $\ell = 5$. In Case 2, again arguing as in [1, p. 1165], we deduce the existence of an elliptic curve defined over K' with a torsion point of order 2ℓ over K' . The field K' is a cyclic quartic extension of \mathbb{Q} , and so we can apply [10, Theorem 1.3] (which classifies the possible torsion subgroups of elliptic curves defined over such extensions) to conclude that $\ell = 5$.

It remains to deal with the case $\ell = 5$. In both cases we will obtain a non-cuspidal K -point on the modular curve $X = X_0(20)$, with j -value $j(F_i)$ for $i = 1$ or 2 . Using Lemma 7.3 we have that $X(K) = X(\mathbb{Q}(\sqrt{17}))$. It follows that $j(F_i) \in \mathbb{Q}(\sqrt{17})$ for $i = 1$ and 2 . Let $\hat{\mathfrak{q}}$ denote one of the two primes of $\mathbb{Q}(\sqrt{17})$ above 2, and let $\mathfrak{q} = \hat{\mathfrak{q}}\mathcal{O}_K$, which is a prime of K above 2. For $i = 1$ and 2 , we have $v_{\mathfrak{q}}(j(F_i)) = -(20v_2(a) - 4)$, and we find that

$$2^{20v_2(a)-4}j(F_i) \equiv \frac{\theta_j^2\theta_k^2}{(\theta_j - \theta_k)^2} \pmod{\mathfrak{q}}.$$

We verify that $\frac{\theta_j^2\theta_k^2}{(\theta_j - \theta_k)^2} \pmod{\mathfrak{q}} \notin \mathbb{F}_2$ for $j = 1$ and $k = 4$ (in fact this holds for any choice of $1 \leq j < k \leq 8$). However, $\mathcal{O}_{\mathbb{Q}(\sqrt{17})}/\hat{\mathfrak{q}} = \mathbb{F}_2$, contradicting $j(F_i) \in \mathbb{Q}(\sqrt{17})$. \square

Having proven the necessary modularity and irreducibility statements, we can proceed to apply Lemma 7.1 to the curves F_1 and F_2 .

Proof of Theorem 3. We suppose (x, y, z) is a non-trivial primitive solution to (5). Let $i = 1$ or 2 according to whether $p \nmid x$ or $p \mid x$, and let F_i denote the Frey curve associated to this solution. We apply Lemma 7.1 to conclude that $\bar{\rho}_{F_i, \ell} \sim \bar{\rho}_{\mathfrak{f}_i, \lambda_i}$ for a Hilbert newform \mathfrak{f}_i at level $\mathcal{N}_{\ell, i}$, where $\lambda_i \mid \ell$ is a prime of $\mathbb{Q}_{\mathfrak{f}_i}$.

The spaces of Hilbert cusp forms that are new at levels $\mathcal{N}_{\ell, 1}$ and $\mathcal{N}_{\ell, 2}$ respectively have dimensions 647 and 49, and we can compute their newform decompositions using **Magma**. We use the same notation as in [1, pp. 1166–1168], and follow the same method, when possible, to eliminate the newforms at these levels. In Case 2, we have $\mathcal{N}_{\ell, 2} = 2 \cdot \mathfrak{B}$ and using the set of primes $S = \{3, 67, 101\}$ in the sieve, we are able to eliminate all the newforms for primes $\ell > 5$ with $\ell \neq 17$. In Case 1, there are 35 newforms we would like to eliminate. Using the set of primes $S = \{3, 67, 101\}$ again, we eliminate 31 of these newforms for all primes $\ell > 3$ with $\ell \neq 17$.

The four remaining newforms, which we denote $\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3$, and \mathfrak{g}_4 , have Hecke eigenfields of degree 136, 152, 152, and 160 respectively, and we are unable to compute their Hecke eigenvalues using **Magma**. However, for a prime $\mathfrak{q} \mid q$ of K with $q \nmid 2p$, by considering the factorisation of the characteristic polynomial of the Hecke operator at \mathfrak{q} , we can compute the minimal polynomial of $a_{\mathfrak{q}}(\mathfrak{g}_i)$ for each i . In particular, this allows us to compute $\text{Norm}_{\mathbb{Q}_{\mathfrak{q}_i}/\mathbb{Q}}(c - a_{\mathfrak{q}}(\mathfrak{g}_i))$ for any $c \in \mathbb{Z}$. Let

$$A_{\mathfrak{q}} := \{0 \leq \eta, \mu \leq q - 1, (\eta, \mu) \neq (0, 0)\}.$$

Then for $(\eta, \mu) \in A_{\mathfrak{q}}$, we can compute the quantity $\text{Norm}_{\mathbb{Q}_{\mathfrak{q}_i}/\mathbb{Q}}(B_{\mathfrak{q}}(\mathfrak{g}_i, \eta, \mu))$, where $B_{\mathfrak{q}}(\mathfrak{g}_i, \eta, \mu)$ is defined as in [1, pp. 1167]. We then have that $\ell \mid B_{\mathfrak{q}}(\mathfrak{g}_i)$, where

$$B_{\mathfrak{q}}(\mathfrak{g}_i) := q \prod_{(\eta, \mu) \in A_{\mathfrak{q}}} \text{Norm}_{\mathbb{Q}_{\mathfrak{q}_i}/\mathbb{Q}}(B_{\mathfrak{q}}(\mathfrak{g}_i, \eta, \mu)).$$

Since this holds for each prime $\mathfrak{q} \mid q$ with $q \nmid 2p$, we can choose several such primes \mathfrak{q} and compute the greatest common divisor of the values $B_{\mathfrak{q}}(\mathfrak{g}_i)$. We choose one prime of K above each of the rational primes in the set $\{3, 67, 157\}$ and compute the greatest common divisor of the values $B_{\mathfrak{q}}(\mathfrak{g}_i)$ for each i . This greatest common divisor is not divisible by any prime > 3 when $i = 1$, and is not divisible by any prime > 5 for $i = 2, 3$, and 4. These computations complete the proof of the theorem. \square

It would of course be preferable to eliminate the condition $\ell, m \neq 5$ in Theorem 3. There are three obstructing newforms at level $\mathcal{N}_{\ell, 1}$ (the newforms $\mathfrak{g}_2, \mathfrak{g}_3$, and \mathfrak{g}_4) with Hecke eigenfields of degree 152, 152, and 160. There are four obstructing newforms at level $\mathcal{N}_{\ell, 2}$ with Hecke eigenfields of degree 2, 2, 6, and 6. We expect that for each of these newforms \mathfrak{f} the representation $\bar{\rho}_{\mathfrak{f}, \lambda}$ is reducible, for $\lambda \mid 5$ a prime of $\mathbb{Q}_{\mathfrak{f}}$, and that this is why we are unable to discard them. Proving this would in fact allow us to discard these newforms since $\bar{\rho}_{F_i, 5}$ is irreducible for $i = 1$ and 2. This seems like a difficult task, and would likely require an extension of the ideas present in [7, 5–8]. We note that it is also possible to twist the curve F_1 so that it is defined over the smaller field K' (see [1, p. 1158]) and obtain a different set of Hilbert newforms. We were still unable to eliminate $\ell = 5$ in this case due to the presence of four obstructing newforms.

References

- [1] S. Anni and S. Siksek. Modular elliptic curves over real abelian fields and the generalized Fermat equation $x^{2\ell} + y^{2m} = z^p$. *Algebra Number Theory*, 10(6):1147–1172, 2016.
- [2] M. Bennett, I. Chen, S. Dahmen, and S. Yazdani. Generalized Fermat equations: a miscellany. *Int. J. Number Theory*, 11(1):1–28, 2015.

- [3] M. Bennett and I. Chen. Multi-Frey \mathbb{Q} -curves and the Diophantine equation $a^2 + b^6 = c^n$. *Algebra Number Theory*, 6(4):707–730, 2012.
- [4] M. Bennett, J. Ellenberg, and N. Ng. The Diophantine equation $A^4 + 2^\delta B^2 = C^m$. *Int. J. Number Theory*, 6(2):311–338, 2010.
- [5] M. Bennett, P. Mihăilescu, and S. Siksek. The generalized Fermat equation. In *Open Problems in Mathematics*, 173–205, Springer, Cham, 2016.
- [6] M. Bennett and C. Skinner. Ternary Diophantine equations via Galois representations and modular forms. *Canad. J. Math.*, 56(1):23–54, 2004.
- [7] N. Billerey, I. Chen, L. Dembélé, L. Dieulefait, and N. Freitas. Some extensions of the modular method and Fermat equations of signature $(13, 13, n)$. *arXiv preprint*, arXiv:1802.04330v2, 2018.
- [8] W. Bosma, J. Cannon, and C. Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997.
- [9] I. Chen. On the equation $s^2 + y^{2p} = \alpha^3$. *Math. Comp.*, 77(262):1223–1227, 2008.
- [10] M. Chou. Torsion of rational elliptic curves over quartic Galois number fields. *J. Number Theory*, 160:603–628, 2016.
- [11] S. Dahmen. A refined modular approach to the Diophantine equation $x^2 + y^{2n} = z^3$. *Int. J. Number Theory*, 7(5):1303–1316, 2011.
- [12] H. Darmon and L. Merel. Winding quotients and some variants of Fermat’s last theorem. *J. Reine Angew. Math.*, 490:81–100, 1997.
- [13] M. Derickx, F. Najman, and S. Siksek. Elliptic curves over totally real cubic fields are modular. *Algebra Number Theory*, 14(7):1791–1800, 2020.
- [14] F. Freitas. Recipes to Fermat-type equations of the form $x^r + y^r = Cz^p$. *Math. Z.*, 279(3-4):605–639, 2015.
- [15] F. Freitas and S. Siksek. Criteria for irreducibility of mod p representations of Frey curves. *J. Théor. Nombres Bordeaux*, 27(1):67–76, 2015.
- [16] F. Freitas and S. Siksek. The asymptotic Fermat’s last theorem for five-sixths of real quadratic fields. *Compos. Math.*, 151(8):1395–1415, 2015.

- [17] F. Najman and G. Țurcaș. Irreducibility of mod p Galois representations of elliptic curves with multiplicative reduction over number fields. *Int. J. Number Theory*, 1–10, 2021.
- [18] J. Silverman. *Advanced Topics in the Arithmetic of Elliptic Curves*, vol. 151 of *Graduate Texts in Mathematics*. Springer, New York, NY, 1994.
- [19] M. Stoll. Chabauty without the Mordell–Weil group. In *Algorithmic and Experimental Methods in Algebra, Geometry, and Number Theory*, 623–663, Springer, Cham, 2017.
- [20] J. Tate. Algorithm for determining the type of a singular fiber in an elliptic pencil. In *Modular Functions of One Variable IV*, 33–52, Springer, Berlin, Heidelberg, 1975.
- [21] L. Washington. *Introduction to Cyclotomic Fields*, vol. 83 of *Graduate Texts in Mathematics*. Springer, New York, NY, 1997.
- [22] A. Wiles. Modular elliptic curves and Fermat’s last theorem. *Ann. Math.*, 141(3):443–551, 1995.

MATHEMATICS INSTITUTE, UNIVERSITY OF WARWICK, CV4 7AL, UNITED KINGDOM
E-mail address: p.rodgers@warwick.ac.uk