Magic Squares of Squares

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What is a Magic Square?

Magic squares have been studied by Chinese mathematicians as far back as 190 BCE.

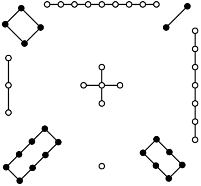


Figure: The Lo Shu magic square.

4	9	2
3	5	7
8	1	6

Magic Squares of Squares: An Open Problem

Open Problem

Does there exist a magic square with nine distinct square entries?

History and Progress

- Euler solved the 4×4 case affirmatively in 1770. In 1876 Edouard Lucas studied the 3×3 case.
- It is known that if a magic square of squares (with nine distinct entries) does exist, then its central element has size $> 25 \times 10^{24}$, so probably not! This is not a very convincing argument...
- The problem has been studied for a long time using elementary number theory techniques and traditional magic-square constructions. **My project looks at the problem using algebraic geometry.**

Euler's 4×4 magic square of squares

The entries in each row, column, and main diagonal sum to 8515.

68^2	29^2	41^2	37^2
17^2	31^2	79^2	32^2
59^2	28^2	23^2	61^2
11^2	77^2	8^2	49^2

"Permettez-moi, Monsieur, que je vous parle encore d'un problème qui me paraît fort curieux et digne de toute attention."

Leonhard Euler in a letter to Joseph Lagrange (1770).

Algebra + Geometry = Algebraic Geometry!

Algebraic Geometry

Algebraic Geometry is, very broadly, the study of solutions of sets of polynomial equations in many variables, using techniques from (commutative) algebra and geometry.

The Algebra-Geometry Correspondence

Algebra: The polynomial ring $k[x_1, \ldots, x_n]$
Geometry: Affine Space $k^n (\approx \mathbb{A}^n)$

- The variety of a subset $\{f_1, \ldots, f_m\} \subseteq k[x_1, \ldots, x_n]$ is $\{(x_1, \ldots, x_n) \in k^n \mid f_i(x_1, \ldots, x_n) = 0, i = 1, \ldots, n\}$. We write this as $V(f_1, \ldots, f_m)$.
- The ideal of a subset $X \subseteq k^n$ is $\{f \in k[x_1, \ldots, x_n] \mid f(x) = 0 \text{ for any } x \in X\}$. We write this as $I(X)$.

This correspondence is almost a bijection!

Magic Squares of Squares using Algebraic Geometry

Consider the magic square:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

We can write out the equations of a magic square of squares as follows:

$$\begin{aligned} a^2+b^2+c^2-d^2-e^2-f^2 &= 0 \\ a^2+b^2+c^2-g^2-h^2-i^2 &= 0 \\ a^2+b^2+c^2-b^2-e^2-h^2 &= 0 \\ a^2+b^2+c^2-a^2-d^2-g^2 &= 0 \\ a^2+b^2+c^2-c^2-f^2-i^2 &= 0 \\ a^2+b^2+c^2-a^2-e^2-i^2 &= 0 \\ a^2+b^2+c^2-c^2-e^2-g^2 &= 0. \end{aligned}$$

Projective Space

Most of algebraic geometry is carried out in projective space, but don't worry! Projective space is a variant on the usual k^n which makes things work nicely.

Defining Projective Space

Projective space is defined as $\mathbb{P}^n = (k^{n+1} \backslash \{0\}) / \sim$, where $(x_1, \ldots, x_n) \sim (y_1, \ldots, y_n)$ if there is a $\lambda \in k \backslash \{0\}$ such that $\lambda(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$ (so when the two points lie on the same line through the origin). We denote a point in projective space by $(x_0 : \cdots : x_n)$.

Polynomials and points in projective space

It makes sense to say that $(x_0 : \cdots : x_n)$ is the zero of a *homogeneous* polynomial $F(X_0, \ldots, X_n)$.

Since our equations for a magic square of squares are all homogeneous, and moreover, a rescaling of a magic square is still a magic square, it makes sense to work in projective space.

The Magic Square Variety

We define the magic square variety, which we denote X, to be the projective variety in $\mathbb{P}^8$ given as

$$X = \left\{ (a : b : c : d : e : f : g : h : i) \in \mathbb{P}^8 \;\middle|\; \text{the equations hold} \right\}.$$

This is a geometric object. Our aim is to understand the geometry of this object as much as possible to gain insight into whether a magic square of squares may exist.

- What is its dimension?
- What are its singular points?
- Does it contain lines?
- Does it contain curves?

Hilbert Polynomials and Dimension (I)

Intuitively, the dimension of a variety (the zero set of polynomial equations) is what it looks like close up.

The dimension of a variety can be defined in many different ways, all of which are complicated!

A natural way is using Hilbert Polynomials. There are two versions: affine and projective. They work in more or less the same way.

Coordinate Ring

Let $X \in \mathbb{A}^n$ be an affine variety. We define the coordinate ring of X, which we denote $k[X]$, as the quotient ring $k[x_1, \ldots, x_n]/I(X)$. This is also a vector space over k.

Rather than think of $k[X]$ as a quotient ring, we think of it as polynomial functions restricted to X. The coordinate ring encodes almost all the information of X.

Hilbert Polynomials and Dimension (II)

let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal. The (affine) Hilbert function of I, denoted by H_I, is defined as

$$H_I(s) = \dim \left(\frac{k[x_1, \ldots, x_n]_{\leq s}}{I_{\leq s}} \right) = \dim \left(k[x_1, \ldots, x_n]_{\leq s} \right) - \dim \left(I_{\leq s} \right).$$

For s large enough, this is a polynomial, known as the (affine) Hilbert polynomial.

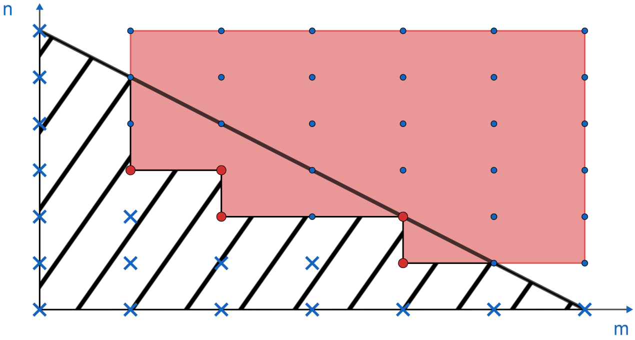
We define the dimension of an affine variety $X \in \mathbb{A}^n$ to be $\deg H_{I(X)}(s)$ for s large enough.

As the monomials form a basis for these vector spaces, this is the **number of monomials of degree $\leq s$ in the complement of I.**

It turns out, that using Gröbner bases, it is enough to understand how to calculate the Hilbert Polynomial of a monomial ideal.

Hilbert Polynomials and Dimension (III)

We want to understand the number of monomials of degree $\leq s$ in the complement of I. We can visualise this in $k[x, y]$.



To find the dimension of the magic square variety, we find a Gröbner basis for $I(X)$ and find the Hilbert polynomial of $LT(I(X))$, which has degree 2, so the magic square variety is a surface!

Singular Points

Intuitively, a singular point is a 'non-smooth' point. More formally, it is a point where the dimension of the tangent space at that point is not equal to the dimension of the variety.

The singular points of the magic square variety are given by

$$\begin{pmatrix} \pm 1 & \pm\sqrt{2} & 0 \\ 0 & \pm 1 & \pm\sqrt{2} \\ \pm\sqrt{2} & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & \pm 1 & \pm\sqrt{-1} \\ \pm\sqrt{-1} & 0 & \pm 1 \\ 1 & \pm\sqrt{-1} & 0 \end{pmatrix}, \quad \begin{pmatrix} \pm\sqrt{-1} & 0 & \pm 1 \\ \pm\sqrt{2} & 0 & \pm\sqrt{-2} \\ \pm\sqrt{-1} & 0 & 1 \end{pmatrix},$$

as well as their transposes, and the matrices obtained by reflecting in the central column. All the $\pm$ signs are independent from one another.

There are precisely 256 singular points (note that we work over $\mathbb{C}$ here). Each singular point has precisely three zero entries.

Lines on varieties (I): The Grassmannian

Grassmannians

The Grassmannian is the set of two-dimensional subspaces of k^{n+1}. We denote it $Gr(2, n+1)$. Equivalently, it is the set of lines in $\mathbb{P}^n$.

So $Gr(2, 4)$ is the set of lines in $\mathbb{P}^3$.

Our aim is to find an injective map, ψ, called the Plücker embedding:

$$\psi : Gr(2, n+1) \longrightarrow \mathbb{P}^m,$$

for some m.

So, we want to embed $Gr(2, n+1)$ into projective space. If we can do this, then any point in the image of ψ will correspond to a unique line in $\mathbb{P}^n$, and vice-versa.

Lines on varieties (II): The Plücker embedding

We will try and understand the Plücker embedding when $n = 4$ (so we are looking at lines in $\mathbb{P}^3$).

Let $W \subseteq k^4$ be a two-dimensional subspace. Let

$$v_1 = (u_{11}, u_{12}, u_{13}, u_{14}) \text{ and } v_2 = (u_{21}, u_{22}, u_{23}, u_{24})$$

be a basis for W and form the matrix

$$M_W = \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ u_{21} & u_{22} & u_{23} & u_{24} \end{pmatrix}$$

We define

$$\psi(W) = (\omega_{12} : \omega_{13} : \omega_{14} : \omega_{23} : \omega_{24} : \omega_{34}),$$

where ω_{ij} is the (i, j)th minor of M_W. So $\omega_{24} = u_{12}u_{24} - u_{14}u_{22}$, for example. This map is well defined (independent of the choice of basis).

Lines on Varieties (III): The Grassmannian (again)

Key Facts

- The Plücker embbedding, ψ is an embedding (i.e. it is injective).
- The image of ψ is a projective variety (i.e. we can find defining equations).

We also call the image of the Plücker embedding the Grassmannian.

The image of the Plücker embedding, $\psi(Gr(2,4)) \in \mathbb{P}^5$, is known as the *Plücker quadric* and is defined by the single equation

$$\omega_{12}\omega_{34} - \omega_{13}\omega_{24} + \omega_{14}\omega_{23} = 0.$$

So, any point in $\mathbb{P}^5$ satisfying this equation corresponds to a unique line in $\mathbb{P}^3$, and any line in $\mathbb{P}^3$ corresponds to a unique point in $\mathbb{P}^5$ satisfying this equation. We have parametrised the lines in $\mathbb{P}^3$ using a projective variety in $\mathbb{P}^5$.

Lines on Varieties (IV): Open subsets of the Grassmannian

Suppose $\omega_{12} \neq 0$. Then the first 2×2 minor of the corresponding matrix, M_W, is non-zero, so by row reducing M_W we can put it into the following form:

$$\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}.$$

The row space of this matrix still gives us the subspace W, because row reducing does not change the rowspace.

So, a general two-dimensional subspace of k^4 **corresponding to a point with $\omega_{12} \neq 0$ under the Plücker embedding**, can be expressed parametrically as

$$\{(\lambda, \mu, \lambda a + \mu c, \lambda b + \mu d) \mid \lambda, \mu \in k\}.$$

Lines on Varieties (V): Grassmannians in action!

Consider the variety $\Omega \in \mathbb{P}^3$ defined by the equation

$$X^3 + Y^3 + Z^3 + W^3 = 0.$$

This is known as the Fermat cubic. It is a surface and it is smooth (no singular points). We would like to find the lines contained in this surface (if there are any).

We work on the subset $\{\omega_{12} \neq 0\}$ of the Grassmannian. If a line

$$l = \{(\lambda, \mu, \lambda a + \mu c, \lambda b + \mu d) \mid \lambda, \mu \in k\}$$

is contained in Ω, then by substituting into the equation for Ω, we get that

$$1 + a^3 + b^3 = 0, \; 1 + c^3 + d^3 = 0, \; a^2c + b^2d = 0, \; ac^2 + bd^2 = 0.$$

Lines on Varieties (VI): Grassmannians in action! (Cont.)

From the matrix $\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}$, we see that

$$a = -\omega_{23}, \; b = -\omega_{24}, \; c = \omega_{13}, \; d = \omega_{14}.$$

So, substituting in for a, b, c, and d, we obtain the four equations

$$1-\omega_{23}^3-\omega_{24}^3 = 0, \; 1+\omega_{13}^3+\omega_{14}^3 = 0, \; \omega_{23}^2\omega_{13}+\omega_{24}^2\omega_{14} = 0, \; -\omega_{23}\omega_{13}^2-\omega_{24}\omega_{14}^2 = 0.$$

We combine these with the dehomogenised Plücker quadric (where $\omega_{12} = 1$),

$$\omega_{34} - \omega_{13}\omega_{24} + \omega_{14}\omega_{23} = 0.$$

Then any point in $\mathbb{P}^5$ satisfying these five equations corresponds to a unique line in $\mathbb{P}^3$ lying on Ω. We find that there are 18 such points by solving the equations.

Lines on Varieties (VII): Grassmannians in action! (Cont.)

One such point is $(\xi : 0 : -\xi : 1 : 0 : 1)$, where $\xi = e^{\frac{2\pi i}{3}}$. We find that this has corresponding matrix

$$\begin{pmatrix} -\xi & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix},$$

and we read off the line in $\mathbb{P}^3$ given by the two equations $X + \xi Z = 0$ and $Y + W = 0$.

We then repeat for the other 17 points.

We then repeat the whole process on each other set $\{\omega_{ij} \neq 0\}$. We end up with a total of 27 lines:

$$\begin{aligned} X + \xi^r Z &= 0 \quad \text{and} \quad Y + \xi^s W = 0, \qquad 0 \le r, s \le 2 \\ X + \xi^r W &= 0 \quad \text{and} \quad Y + \xi^s Z = 0, \qquad 0 \le r, s \le 2 \\ X + \xi^r Y &= 0 \quad \text{and} \quad W + \xi^s Z = 0, \qquad 0 \le r, s \le 2. \end{aligned}$$

Theorem: Any smooth cubic surface in $\mathbb{P}^3$ over an algebraically closed field contains 27 distinct lines.

Lines on Varieties (VII): Magic Squares and Variants

- The **magic square variety** contains no lines.
- The **magic hourglass variety** contains no lines.

a^2	b^2	c^2
	d^2	
e^2	f^2	g^2

- The **near magic square of squares variety** contains infinitely many lines (in fact a two-dimensional set of lines).

58^2	46^2	127^2
94^2	113^2	2^2
97^2	82^2	74^2

Lines on Varieties (VII): Curves on Varieties

By eliminating variables from sets of equations we can find lines on the corresponding subsets, and then 'lift' these up to curves on the original variety.

Carrying out this process on the magic square variety, we find a whole class of degree 8 curves lying on the variety. Unfortunately, none of these curves go through any rational points. This is perhaps further evidence on the non-existence of solutions to our original problem.

One such curve is given by the following equations:

$$c = a + b, g = \alpha(a - b), h = \alpha(2a + b), i = \alpha(a + 2b),$$
$$\frac{1}{3}(2a^2 - 4ab - b^2) - f^2 = 0, \frac{2}{3}(a^2 + ab + b^2) - e^2 = 0,$$
$$\frac{2}{3}(a^2 + 4ab + b^2) - d^2 = 0,$$

where $\alpha = \frac{1}{\sqrt{-3}}$.

The Fourth-Year Project (I): Timeline

Disclaimer: The process may differ in future years. Moreover, any of the advice I give is personal and should be taken as such.

- Two options: Research Project and Maths In Action.

  - **Term 3 Year 3**: Speak to potential supervisors, agree roughly on a project.

  - **Term 1 Year 4**: Meetings with supervisor, mainly background reading. Write up some pages of project.

  - **Christmas holidays**: Write up project and progress report.

  - **Term 2 Year 4**: Less background reading, more research. Continue meetings with supervisor.

  - **Easter Holidays**: Submit project, usually third week of holidays.

  - **Term 3 Year 4**: Presentation.

The Fourth-Year Project (II): Some advice

- Speak to members of staff early!
- Try and find a good project-module balance.
- Writing up takes a long time.
- Don't be afraid of research!

Thank you for listening! :)

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