

(Fontaine's Rings and BHT)

give working.

## Fontaine Talk

- Introduce Yourself
- Set out Plan.

Want to Study

- Basic Set-up
- BHT
- Main Result
- complete Defns  
(if time)

$$\varphi : G_{\mathbb{Q}_p} \longrightarrow GL_n(\mathbb{Q}_p)$$

Gal(\mathbb{Q}\_p/\mathbb{Q}\_p)

(local Galois representations)

( $\neq p$ ) ✓ ~ Weeks 1, 2.

( $= p$ ) : More tools : Fontaine's rings.

"Formalism": axiomatise this situation, using more generality, like  
"easier later"

- $F$  field ( $\mathbb{Q}_p$ )
  - $G$  group ( $G_{\mathbb{Q}_p}$ )
  - $B$  on  $F$ -algebra domain.
  - $G$  acts on  $B$  as on  $F$ -algebra.
  - $E = B^G$  a field.
  - $C = \text{Frac}(B)$ ,  $G$  acts on  $C$  too.
- (P-adic field = field  
 $\text{char } B = 0$ , complete w.r.t.  
a given discrete valuation  
that has a perfect  
residue field of char  $p > 0$ )
- ( $B_p$  complete w.r.t. P-adic norm)

Defn:  $B$  is  $(F, G)$ -regular if

$$(i) C^G = B^G$$

(ii) For  $b \in B \setminus \{0\}$ ,

$$Fb \text{ } G\text{-stable} \Rightarrow b \in B^\times$$

(assume set-up above)

[Note:  $\rightarrow E = B^G$  is  
a field]

Example:  $B$  a field. Then  $C = B$ , so  $C^G = B^G$ .

- If  $b \in B \setminus \{0\}$  then  $b \in B^\times$ .

Cases of interest will be BHT, Bars, BDR,  
"B = Borsotti". These are "Period  
Rings".

- Defining  $B_{HT}$ : Hodge-Tate ring of  $\mathbb{Q}_p$

$$B_{HT} := \bigoplus_{q \in \mathbb{Z}} \mathbb{C}_p(q).$$

$$\mathbb{C}_p(R) = \mathbb{C}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[R]]$$

$\left\{ \begin{array}{l} \mathbb{C}_p = \mathbb{C}_p(0) = \text{Completion of } \overline{\mathbb{Q}_p} \\ (\mathbb{C}_p \text{ is alg. closed}). \\ \text{"just a label"} \end{array} \right.$

$$\mathbb{Z}_p(R) = \mathbb{Z}_p[[R]]$$

$\left\{ \begin{array}{l} \mathbb{C}_p(q) = 1\text{-dim } \mathbb{C}_p \text{ rep of } G_{\mathbb{Q}_p} \text{ on which} \\ G_{\mathbb{Q}_p} \text{ acts via } \chi^q \\ g \cdot a_n = g(a_n) \chi(g)^n \end{array} \right.$

$$= \varprojlim_N \mathbb{Z}_p^n \cong \mathbb{Z}_p$$

By choosing a basis,  $T$  of  $\mathbb{Z}_p(1)$  we have a non-canonical isomorphism

$$B_{HT} = \mathbb{C}_p[T, \frac{1}{T}]$$

which is  
a  $\mathbb{Z}_p$ -module

$$g \cdot \sum a_n T^n = \sum g \cdot a_n \chi(g)^n T^n.$$

$$\chi: \mathbb{C}_p \cdot G_{\mathbb{Q}_p} \rightarrow \mathbb{Z}_p^\times$$

Claim:  $B_{HT} = B$  is  $(\mathbb{Q}_p, G_{\mathbb{Q}_p})$ -regular.

Proof:  $B$  is a  $\mathbb{Q}_p$ -algebra ✓

•  $B^G = E$ , a field

•  $B^G = C^G$

• Stability

working with this  
directly is tricky.

Power series,  
a field or  
 $\mathbb{C}_p$  is a field

generated on  
left.

Formal Laurent  
series

•  $B_G = \mathbb{Q}_p$  by Ax-Sen-Tate theorem

•  $C = \mathbb{C}_p(T) \hookrightarrow \text{Frac}(\mathbb{C}_p[[T]]) = \mathbb{C}_p((T))$

Enough to show  $\mathbb{C}_p((T))^G = \mathbb{Q}_p$ .

$$g \cdot \sum_{n \in \mathbb{Z}} a_n T^n = \sum_{n \in \mathbb{Z}} g(a_n) \chi(g)^n T^n$$

$$g(a_n) \in \mathbb{Q}_p$$

so we require  $g(a_n) = \chi(g)^n a_n$ ,

so  $a_n \in \mathbb{C}_p((T))^G$ , so  $a_0 \in \mathbb{Q}_p$ ,  $a_n = 0$  if  $n \neq 0$

by Tate-Sen. So  $C^G = \mathbb{Q}_p$ .

(Stability is similar). □

• Definition:  $B$  on  $(F, G)$  regular domain,  
 $E = B^G = C^G$ .

Let  $V \in \text{Rep}_F(G) = \{\text{finite dimensional } F\text{-linear representations of } G\}$

Then  $D_B(V) := (B \otimes_F V)^G$

$$e \circ g \cdot (e(\lambda \otimes x)) = g \cdot e \lambda \otimes g x = e \cdot (g \lambda \otimes g x) = e(g \cdot (\lambda \otimes x)) = e(\lambda \otimes x)$$

This is an  $E$ -vector space, equipped with a canonical map

$$\begin{aligned} \alpha_V : B \otimes_E D_B(V) &\longrightarrow B \otimes_F V \\ \lambda \otimes x &\mapsto \lambda x \\ (\text{then extend}). \end{aligned}$$

$\alpha_V$  is  $B$ -linear,  $G$  equivariant

Theorem:  $\alpha_V$  is injective and  $\dim_E(D_B(V)) \leq \dim_F(V)$  with equality if and only if  $\alpha_V$  is an isomorphism.

Consequence:  $D_B(V) \in \text{Rep}_F(G)$

Interesting remark:  
 dimension of the graded piece is the multiplicity of  $g$  as a HT weight of  $V$ .

Definition: If equality holds in the above, then we say that  $V$  is  $B$ -admissible.

$B_{HT}$  admissible  $\iff$  Hodge-Tate Rep

$B_{dR}$  admissible  $\iff$  de Rham Rep

$B_{cris}$  admissible  $\iff$  Crystalline Rep

Idea: Introduce subset of all  $P$ -adic rep's of  $G_{\overline{Q}_p}$ .

Stable under direct sums, tensor products etc.

If  $B$  has more structure  $\rightsquigarrow$  descends to  $D_B(V)$ , then obtain non-trivial invariants of HT rep's, which we can use to classify them.

Example :  $V = F \in \text{Rep}_F(G)$  with trivial  $G$ -action.

$$\text{Then } D_B(V) = (B \otimes_F F)^G = B^G = E,$$

$$\alpha_v : B \otimes_E E \rightarrow B \otimes_F F, \quad \alpha_v = \text{id}.$$

$$\begin{array}{ccc} \parallel & & \parallel \\ B & & B \end{array}$$

So  $\alpha_v$  is an isomorphism, and we note that

$$\dim_E(D_B(V)) = \dim_E E = 1 = \dim_F(F) = \dim_F(V)$$

as expected.

In particular,  $V = F$  is always  $B$ -admissible.

Proof ( $\alpha_v$  injective)

Consider  $D_C(V) = (C \otimes_F V)^G$ , which comes with its map  $C \otimes_F D_C(V) \rightarrow C \otimes_F V$

$$\lambda \otimes y \mapsto \lambda y.$$

We have

$$B \otimes_E D_B(V) \xrightarrow{\alpha_v} B \otimes_F V$$

$$\downarrow$$

$$\downarrow$$

$$C \otimes_E D_C(V) \xrightarrow{\quad} C \otimes_F V \quad \text{commutes.}$$

To show  $\alpha_v$  injective, enough to show bottom is.

so by replacing  $B$  with  $C$ , can assume  $B$  is a field.

Enough to show  $\alpha_v$  takes  $E$ -basis of  $D_B(V)$  to a  $B$ -linearly indep set in  $B \otimes_F V$

Why? Suppose  $\alpha_r (\sum_i \lambda_i \otimes y_i) = 0$

$$\text{so } \sum_i \lambda_i y_i = 0,$$

express each  $y_i$  as an  $E$ -linear combination of the  $x_i$ .

Then expand and rearrange. By  $B$ -linear indep  $\lambda_i = 0 \forall i$  ( $B$  a domain).

So enough to show if  $x_1, \dots, x_r \in D_B(V)$  one  $E$ -linearly indep, then they are  $B$ -linearly indep.

Suppose  $x_i$  are not  $B$ -linearly independent, minimal relation

$$x_r = \sum_{i < r} b_i \cdot x_i$$

$$\text{Then } x_r = g \cdot x_r = \sum_{i < r} g(b_i) \cdot g(x_i) = \sum_{i < r} g(b_i) x_i$$

By minimality  $b_i = g(b_i) \forall i < r$ ,

$$\text{so } b_i \in B^G = E \forall i < r,$$

contradicting  $E$ -linear independence.  $\square$

Proof ( $\dim_E D_B(V) \leq \dim_F(V)$ )

We have just showed that

$$B \otimes_E D_B(V) \hookrightarrow B \otimes_F V$$

(tensor up to field of fractions)

$$\text{Con deduce } C \otimes_E D_B(V) \leq_C C \otimes_F V$$

Then compare  $C$  dimensions.  $\square$

How the definitions match up:

- ① Last time:  $V \in \text{Rep}_{\mathbb{Q}_p}(G_{\mathbb{Q}_p})$  is HT if  $V \otimes_{\mathbb{Q}_p} \mathbb{C}_p \in \text{Rep}_{\mathbb{Q}_p}(G_{\mathbb{Q}_p})$  is HT, and this occurs if

$$\bigoplus_q [V \otimes_{\mathbb{Q}_p} \mathbb{C}_p] \{q\} \otimes_{\mathbb{Q}_p} \mathbb{C}_p(-q) \xrightarrow{\sim} V \otimes_{\mathbb{Q}_p} \mathbb{C}_p$$

$$(\text{Now } W\{q\} = W(q)G_{\mathbb{Q}_p})$$

So this becomes

$$\bigoplus_q (V \otimes_{\mathbb{Q}_p} \mathbb{C}_p(q))^{G_{\mathbb{Q}_p}} \otimes_{\mathbb{Q}_p} \mathbb{C}_p(-q) \xrightarrow{\sim} V \otimes_{\mathbb{Q}_p} \mathbb{C}_p$$

- ② This time:  $V \in \text{Rep}_{\mathbb{Q}_p}(G_{\mathbb{Q}_p})$  is  $B_{HT}$  admissible if

$$(V \otimes_{\mathbb{Q}_p} B_{HT}) \otimes_{\mathbb{Q}_p} B_{HT} \xrightarrow{\sim} V \otimes_{\mathbb{Q}_p} B_{HT}$$

[Note: we showed that  $E = \mathbb{Q}_p = F$ ]

So if

$$\bigoplus_q (V \otimes_{\mathbb{Q}_p} \mathbb{C}_p(q))^{G_{\mathbb{Q}_p}} \otimes_{\mathbb{Q}_p} B_{HT} \xrightarrow{\sim} \bigoplus_q V \otimes_{\mathbb{Q}_p} \mathbb{C}_p(q)$$

To go from ① to ②:

- twist by  $q'$  then  $\bigoplus_{q'}$  this gives

$$\bigoplus_{q'} \left( \bigoplus_q (V \otimes_{\mathbb{Q}_p} \mathbb{C}_p(q))^{G_{\mathbb{Q}_p}} \right) \otimes_{\mathbb{Q}_p} \mathbb{C}_p(q') \xrightarrow{\sim} \bigoplus_{q'} V \otimes_{\mathbb{Q}_p} \mathbb{C}_p(q')$$

as expected, and similarly for the converse we more or less "undo" this.