Fermat's Last Theorem - Not Enough Margin!

Philippe Michaud-Rodgers

University of Warwick

Young Researchers in Mathematics, 10th edition
08.06.2021

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[Slides available on my webpage.]

Fermat's Little Margin

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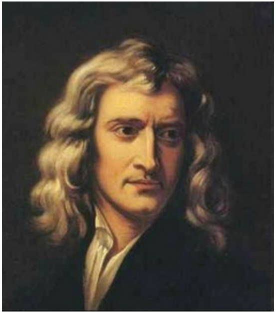

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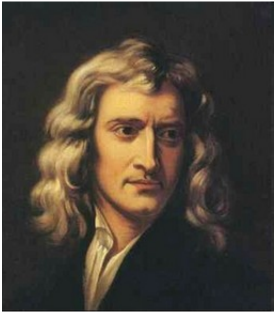

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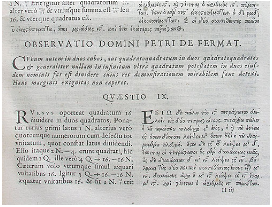

"I have discovered a truly remarkable proof of this result which this margin is too small to contain."

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The equation
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with $p \geq 5$, prime, has no non-trivial solutions for integers x, y, z.

The Frey Curve

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More on Elliptic Curves

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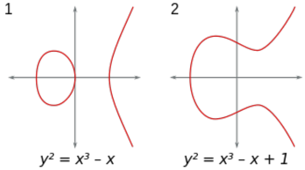

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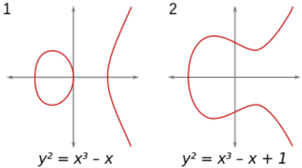

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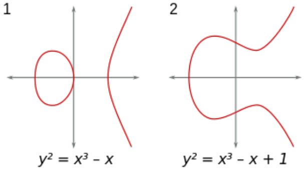

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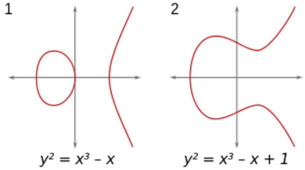

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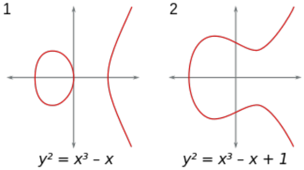

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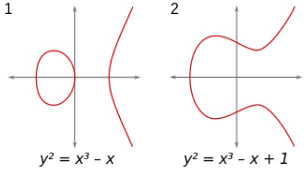

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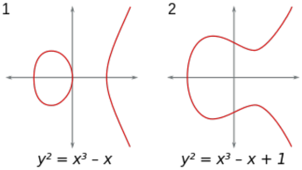

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- $a_\ell(E) := \ell + 1 - \#\widetilde{E}(\mathbb{F}_\ell)$, for ℓ prime with $\ell \nmid N$.

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- There are **no** newforms at level 2.

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$$a_\ell(E) \equiv a_\ell(f) \pmod{p}$$

for all primes $\ell \nmid pNN_p$.

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So $E_{x,y,z,p}$ is modular.

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Conclusion: we can apply the level-lowering theorem to $E_{x,y,z,p}$.

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- **Conclusion:** Fermat's Last Theorem is true!

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Theorem (Jarvis and Meekin, $d = 2$, 2013. Freitas and Siksek, $d > 2$, 2014)

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Thank you for listening! :)