Fermat's Last Theorem - Not Enough Margin!

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Statement of Fermat's Last Theorem

Theorem (Wiles + many others! 1995)

The equation

$$x^n + y^n = z^n,$$

with $n \geq 3$, has no non-trivial solutions for integers x, y, z.

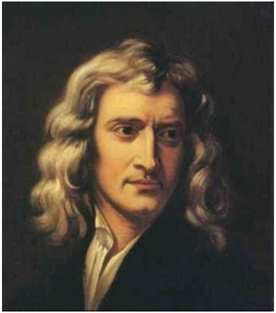
By a non-trivial solution, we mean that $xyz \neq 0$.

Aim for today: Overview the proof.

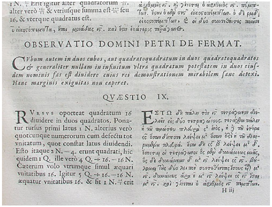
[Slides available on my webpage.]

Fermat's Little Margin

- Pierre de Fermat:



- His margin:



1 N. [illegible] Erit igitur alter quadratorum [illegible] alter verò [illegible] & vtriusque summa est [illegible] seu 16. & vterque quadratus est.

OBSERVATIO DOMINI PETRI DE FERMAT.

Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos & generaliter nullam in infinitum vltra quadratum potestatem in duos eiusdem nominis fas est diuidere cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet.

QVÆSTIO IX.

Rursus oporteat quadratum 16 diuidere in duos quadratos. Ponatur rursus primi latus 1 N. alterius verò quotcunque numerorum cum defectu tot vnitatum, quot constat latus diuidendi. Esto itaque 2 N. − 4. erunt quadrati, hic quidem 1 Q. ille verò 4 Q. + 16. − 16 N. Cæterum volo vtrumque simul æquari vnitatibus 16. Igitur 5 Q. + 16. − 16 N. æquatur vnitatibus 16. & fit 1 N. [illegible] erit

H iij

"I have discovered a truly remarkable proof of this result which this margin is too small to contain."

First Observations

- If $n = p \cdot m$ and (x, y, z) satisfies $x^n + y^n = z^n$, then
$$(x^m)^p + (y^m)^p = (z^m)^p.$$

- $n = 3$ (Euler, 1770) and $n = 4$ (Fermat, 1670): elementary.

So enough to prove:

FLT

The equation
$$x^p + y^p = z^p,$$
with $p \geq 5$, prime, has no non-trivial solutions for integers x, y, z.

The Frey Curve

Suppose (x, y, z) is a (non-trivial) solution and define the Frey curve

$$E_{x,y,z,p} : Y^2 = X(X - x^p)(X + y^p).$$

This is an elliptic curve over $\mathbb{Q}$.

Definition

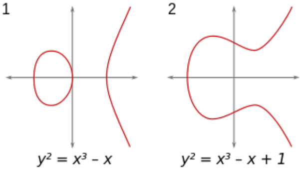
An elliptic curve over $\mathbb{Q}$ is a curve given by an equation

$$E : Y^2 = X^3 + AX^2 + BX + C,$$

where $A, B, C \in \mathbb{Q}$. It is smooth.

More on Elliptic Curves

They look like this:



- **Fact:** $E(\mathbb{Q})$ is a group.
- E has a minimal discriminant, $\Delta_{\min}$.

$$\Delta_{\min}(E_{x,y,z,p}) = 2^{-8}(xyz)^{2p}.$$

- E has a conductor, N.

$$N(E_{x,y,z,p}) = 2 \prod_{p|xyz,\text{odd}} p, \qquad \text{(squarefree)}.$$

- $a_\ell(E) := \ell + 1 - \#\widetilde{E}(\mathbb{F}_\ell)$, for ℓ prime with $\ell \nmid N$.

Newforms

Let $N' > 0$. There are finitely many newforms at level N'.

- A newform is a holomorphic function on the upper half-plane.
- It has a Fourier, or q-expansion:

$$f = \sum_{n=1}^{\infty} a_n q^n, \qquad \text{where } q = e^{\frac{2\pi i}{z}}.$$

- Newforms at level 38:

$$f_1 = q - q^2 + q^3 + q^4 - q^6 - q^7 + \cdots$$
$$f_2 = q + q^2 - q^3 + q^4 - 4q^5 - q^6 + 3q^7 + \cdots$$

- There are **no** newforms at level 2.

The Key Theorem

Level-Lowering Theorem (Ribet)

Let E be a modular elliptic curve over $\mathbb{Q}$ of conductor N and let $p \geq 5$ be prime. Suppose E has no rational subgroups of size p. Then E arises mod p from a newform f at level N_p, where

$$N_p = \frac{N}{\prod_{q \| N, p | \mathrm{ord}_q(\Delta_{\min})} q}.$$

- E arises mod p from a newform $f = \sum_{n=1}^{\infty} a_n q^n$ at level N_p means

$$a_\ell(E) \equiv a_\ell(f) \pmod{p}$$

for all primes $\ell \nmid pNN_p$.

Modularity

Condition in level-lowering theorem: E is modular.

Definition

Let $E/\mathbb{Q}$ be an elliptic curve of conductor N. Then E is modular if there exists a newform $f = \sum_{n=1}^{\infty} a_n q^n$ at level N such that

$$a_\ell(E) = a_\ell(f)$$

for every prime $\ell \nmid N$.

Theorem (Wiles)

If N is squarefree, then E is modular.

So $E_{x,y,z,p}$ is modular.

Mazur's Theorem

Condition in level-lowering theorem: E has no rational subgroups of size p.

Mazur's Theorem

Let $E : Y^2 = g(X)$ be an elliptic curve over $\mathbb{Q}$ of conductor N and let $p \geq 5$. Suppose N is squarefree and that g has 3 rational roots. Then E has no rational subgroups of size p.

This is true for our Frey curve

$$E_{x,y,z,p} : Y^2 = X(X - x^p)(X + y^p).$$

Conclusion: we can apply the level-lowering theorem to $E_{x,y,z,p}$.

QED

Apply the level-lowering theorem to the Frey curve $E_{x,y,z,p}$:

- $E_{x,y,z,p}$ arises mod p from a newform f at level N_p, where

$$N_p = \frac{N}{\displaystyle\prod_{q \| N, p \mid \mathrm{ord}_q(\Delta_{\min})} q} = \frac{2 \displaystyle\prod_{p \mid xyz, \mathrm{odd}} p}{\displaystyle\prod_{p \mid xyz, \mathrm{odd}} p} = 2.$$

- **Note:** N_p no longer depends on x, y, z, or p.
- BUT! There are no newforms at level 2, a contradiction.
- **Conclusion:** Fermat's Last Theorem is true!

FLT over quadratic fields

Theorem (Jarvis and Meekin, $d = 2$, 2013. Freitas and Siksek, $d > 2$, 2014)

The equation

$$x^n + y^n = z^n, \quad x, y, z \in K,$$

has no non-trivial solutions for $n \geq 4$ and $K = \mathbb{Q}(\sqrt{d})$, when $d \in \{2, 3, 6, 7, 10, 11, 13, 14, 15, 19, 21, 22, 23\}$. (No $d = 5, 17$).

Theorem (M. 2021)

The equation

$$x^n + y^n = z^n, \quad x, y, z \in K,$$

has no non-trivial solutions for $n \geq 4$ and $K = \mathbb{Q}(\sqrt{d})$, when $d \in \{26, 29, 30, 31, 35, 37, 38, 42, 43, 46, 47, 51, 53, 58, 59, 61, 62, 65, 66, 67, 69, 71, 73, 74, 77, 79, 82, 83, 85, 86, 87, 91, 93, 94, 97\}$.

Thank you for listening! :)