Isogenies of elliptic curves and Diophantine equations

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Section 1

Motivation

Fermat's Last Theorem

The equation

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with $n \geq 3$, has no solutions for $x, y, z \in \mathbb{Z}$ with $xyz \neq 0$.

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Mazur's isogeny theorem, 1978

Let p be a prime such that there exists an elliptic curve $E/\mathbb{Q}$ that admits a rational p-isogeny. Then

$$p \in \{2, 3, 5, 7, 11, 13, 17, 19, 37, 43, 67, 163\}.$$

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Why is this an important theorem?

- Isogenies are the basic building blocks of maps between elliptic curves.
- It's proof introduced many important concepts and techniques.
- Leads to a deeper understanding of **modular curves** and **Galois representations**.
- Plays a crucial role in the **modular method**.

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Key question: Does this theorem generalise to number fields?

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Example.

$$E_1 : Y^2 = X^3 + X^2 - X, \quad E_2 : Y^2 = X^3 - 2X^2 + 5X.$$

$$\varphi : (x, y) \mapsto \left(\frac{y^2}{x^2}, \frac{y(x^2+1)}{x^2} \right).$$

$\ker(\varphi) = \{0_{E_1}, (0,0)\}$, it is a ($\mathbb{Q}$-)rational 2-isogeny.

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- Isogenies are the basic building blocks of maps between elliptic curves.
- An answer would lead to a deeper understanding of **modular curves** and **Galois representations**.
- An answer would lead to a simpler application of the **modular method over number fields**.

The modular method over number fields

- Start with an equation:

$$x^p + y^p = z^p, \quad \text{for } x, y, z \in K.$$

$$x^{2p} + y^{2p} = z^7, \quad \text{for } x, y, z \in \mathbb{Z}.$$

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No set method for proving that E does not admit a K-rational p-isogeny.

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- This is not a very restrictive assumption.
- It is *already* an assumption in the modular method for the level-lowering theorem.

Section 2

Results

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Theorem (M., 2022)

Let $K = \mathbb{Q}(\sqrt{5})$ and let p be a prime. There exists an elliptic curve E/K which admits a K-rational p-isogeny and is semistable at all primes of K above p if and only if $p \in \{2, 3, 5, 7, 13, 17, 37\}$.

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Theorem (M., 2023)

Let K be a real quadratic field with $h(K) \leq 7$ and let ϵ be a fundamental unit of K. Let p be a prime such that there exists an elliptic curve E/K which admits a K-rational p-isogeny and is semistable at all primes of K above p. Then either

(i) *p ramifies in K; or*

(ii) *$p \in \{2, 3, 5, 7, 11, 13, 17, 19, 37\}$; or*

(iii) *p splits in K and $p \mid \mathrm{Norm}_{K/\mathbb{Q}}(\epsilon^{12} - 1)$.*

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(iii) *p splits in K and $p \mid \mathrm{Norm}_{K/\mathbb{Q}}(\epsilon^{12} - 1)$.*

- This gives a simple criterion for ruling out primes.
- Any leftover primes can normally be dealt with separately.

Section 3

Proofs

Modular curves and Galois representations

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The modular curve $X_0(p)$

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- The cusps $0, \infty \in X_0(p)(\mathbb{Q})$ are the poles of the j-map.

A prime of potentially multiplicative reduction

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- Argue that $x = \infty$ or 0, a contradiction (think of **Hensel's lemma**!).

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Then $\overline{\rho}_{E,p}(\sigma) = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$.

A key equivalence

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Proof.

(i) $\implies$ (ii) $\ker(\varphi)$ is a nontrivial proper G_K-submodule of $E[p]$.

(ii) $\implies$ (i) For some basis $\{R_1, R_2\} \subset E[p]$,

$$\overline{\rho}_{E,p} \sim \begin{pmatrix} \lambda & * \\ 0 & \lambda' \end{pmatrix}.$$

Then $\langle R_1 \rangle \subset E[p]$ is a K-rational subgroup of order p. Quotienting out by this subgroup gives rise to a K-rational p-isogeny. $\square$

The isogeny character

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We study λ as it encodes key information about E and φ.

The group G_K and Frobenius elements

We want to study $\lambda : G_K \to \mathbb{F}_p^\times$. The group G_K is complicated and we want to work with concrete elements.

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We study $\lambda(\sigma_\mathfrak{q}) \in \mathbb{F}_p^\times$.

A prime of potentially good reduction

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Can prove: $\lambda(\sigma_\mathfrak{q})$ is a root of the following polynomials (after reducing mod p):

(I) $X^{12} - \alpha^t$ for some $t \in \{0, 12\}$; **and**

(II) $X^2 - aX + \mathrm{Norm}(\mathfrak{q})$ for some $|a| \leq 2\sqrt{\mathrm{Norm}(\mathfrak{q})}$.

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The fact that E is semistable at the primes of K above p means that $t \in \{0, 12\}$. Otherwise, $t \in \{0, 4, 6, 8, 12\}$.

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- For each $\mathfrak{q}_i$, obtain a finite list of primes $\mathcal{P}_i$ such that if there exists an elliptic curve E/K that admits a K-rational p-isogeny and is semistable at the primes of K above p, then $p \in \mathcal{P}_i$.

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- Intersect the sets $\mathcal{P}_i$ to obtain a finite list of primes.
- Use additional techniques to try and remove even more primes.

Section 4

Examples

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Conclusion: $p \leq 19$ or $p = 37$.

The Fermat equation over $K = \mathbb{Q}(\sqrt{2})$

Theorem (Jarvis–Meekin, 2004)

The equation

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- 'Classical' for $n \in \{4, 5, 6, 7, 9, 11, 13\}$.
- Let $n = p \geq 17$ be prime and suppose $x^p + y^p = z^p$ with $xyz \neq 0$.
- Define the Frey elliptic curve $E : Y^2 = X(X - x^p)(X + y^p)$.
- E does not admit a K-rational p-isogeny.
- E is modular.
- E 'corresponds' to a newform at level $\sqrt{2} \cdot \mathcal{O}_K \rightsquigarrow$ contradiction.

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Hvala!