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EQUIVARIANT FORMAL GROUP LAWS

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CONTENTS

ABSTRACT

In this project, we discuss some aspects of the theory of formal group laws, and we see how the theory changes when we pass to equivariant formal group laws. In particular, we show that it is not true that commutative equivariant formal group laws over the rationals have a logarithm. However, we are able to prove interesting decomposition theorems when the group acting is simple enough. In the last part of the project, we show that any equivariant formal group law on the group with two elements is rationally isomorphic to a suitable multiplicative one.

1

INTRODUCTION

The goal of this project is to introduce the theory of equivariant formal group laws. In the non-equivariant case, covered by the first part of the project, we can think of a formal group law as a power series encoding the behaviour of the multiplication map in a one-dimensional formal group, that is, a group object in the category of formal schemes with some additive properties. We develop the group scheme point of view by following [12] as the main reference. By a theorem of Quillen, giving a formal group law with coefficients in a $\mathbb{Q}$-algebra R, is the same as giving a homomorphism from the rationalized cobordism ring to R. Following [7], [8], and [9] we show that these homomorphisms, which we call Genera, are in turn closely related to characteristic numbers of vector bundles.

These correspondences create an extremely interesting picture in the non-equivariant case that has not a clear analogue when we try to equip things with a group action. The reason is that, in the non-equivariant case, every commutative formal group law is rationally isomorphic to a very simple one, that is the additive formal group law, whereas in the equivariant context this is not true.

We refer to [**greenlees2001equivariant**] and [4] to formalize the notion of equivariant formal group law. The idea is to think about the ring of functions on a formal group that receives a map from the dual of a certain abelian compact Lie group.

To understand what happens in the equivariant case, we explore some examples coming from algebraic topology, indeed any equivariant complex oriented cohomology theory witnesses, in a natural way, an equivariant formal group law. We analyse in detail the case of equivariant K-theory following [2] and [**segal1966equivariant**]. In the last chapter of the project, we show that any commutative equivariant formal group law on the group with two elements is rationally isomorphic to a multiplicative one, that is still a very simple one.

We refer to [6] and [3] for facts regarding G-spectra and equivariant cohomology theories that we will use, to [1] for a complete development of the non-equivariant case, and to [10] for general facts about cohomology and homotopy.

2

CHARACTERISTIC NUMBERS

In this chapter, we introduce characteristic classes of complex vector bundles and we use them to construct the associated characteristic numbers. We start by defining Chern classes as cohomology classes satisfying the following four axioms.

Axiom 1. To each complex vector bundle E over a manifold X there corresponds a sequence of cohomology classes

$$c_i(E) \in H^{2i}(X;\mathbb{Z}),\ i = 0, 1, 2, \ldots$$

called Chern classes. Furthermore $c_0(E) = 1$ and $c_i(E) = 0$ for $i > rk(E)$.

Axiom 2. If $f : X \to Y$, denote with f^*E the pullback bundle with respect to f, then

$$c_i(f^*E) = f^*c_i(E).$$

Axiom 3. Define the total Chern class of E as $c(E) = \sum_{i=0}^{\infty} c_i(E) \in H^*(X;\mathbb{Z})$. If E and F are complex vector bundles over the same base manifold X, then

$$c(E \oplus F) = c(E) \cdot c(F).$$

Axiom 4. Let $g \in H^2(\mathbb{C}P^n;\mathbb{Z})$ denote the generating element of the cohomology ring of $\mathbb{C}P^n$, Poincaré dual to the homology class of the hyperplane $\mathbb{C}P^{n-1} \subset \mathbb{C}P^n$. Let γ_n be the tautological vector bundle over $\mathbb{C}P^n$, having as fiber over each point of $\mathbb{C}P^n$ the line in $\mathbb{C}^{n+1}$ represented by it. Then

$$c(\gamma_n) = 1 - g.$$

As a first calculation, we want to understand who are the Chern classes of the tangent bundle of $\mathbb{C}P^n$. First observe that, since γ_n is contained in the trivial bundle $\mathbb{C}^{n+1}$, we can consider its orthogonal complement $\gamma^\perp$.

Lemma 1. *The tangent bundle* τ *of* $\mathbb{C}P^n$ *is isomorphic to* $\mathrm{Hom}(\gamma_n, \gamma^\perp)$.

Proof. Viewing $\mathbb{C}P^n$ as a quotient of the sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ by S^1, the tangent space of the quotient can be seen as the set of the orbits $S^1 \cdot (x, v)$, where (x, v) is a point of the tangent bundle of S^{2n+1}, that is $x \cdot x = 1$ and $x \cdot v = 0$. Each of these orbits determines and is determined by a linear map

$$l : \mathrm{span}_{\mathbb{C}}(x) \to \mathrm{span}_{\mathbb{C}}(x)^\perp$$

where

$$l(x) = v.$$

It follows that τ is canonically isomorphic to $\mathrm{Hom}(\gamma_n, \gamma^\perp)$. ■

In the following, when X is an almost complex manifold, we will indicate with $c(X)$ the total Chern class of the tangent bundle of X.

Theorem 1. $c(\mathbb{C}P^n) = (1+g)^{n+1} = 1 + (n+1)\cdot g + \binom{n+1}{2}\cdot g^2 + ... + (n+1)\cdot g^n$.

Note that we have omitted the summand g^{n+1} because $H^{2n+2}(\mathbb{C}P^n) = 0$.

Proof. Note that the bundle $\mathrm{Hom}(\gamma_n, \gamma_n)$ is trivial since it is a line bundle with a canonical nowhere zero cross-section. Therefore,

$$\tau \oplus \mathbb{C} \cong \mathrm{Hom}(\gamma_n, \gamma^{\perp}) \oplus \mathrm{Hom}(\gamma_n, \gamma_n) \cong \mathrm{Hom}(\gamma_n, \mathbb{C}^{n+1}).$$

This is clearly isomorphic to

$$\mathrm{Hom}(\gamma_n, \mathbb{C} \oplus ... \oplus \mathbb{C}) \cong \mathrm{Hom}(\gamma_n, \mathbb{C}) \oplus ... \oplus \mathrm{Hom}(\gamma_n, \mathbb{C}).$$

Therefore,

$$\tau \oplus \mathbb{C} \cong (\gamma_n)^* \oplus ... \oplus (\gamma_n)^* .$$

Now by [9] lemma 14.9,

$$c_k\left((\gamma_n)^*\right) = (-1)^k c_k\left(\gamma_n\right)$$

so the result follows by applying axiom 3. ■

Let E be a real vector bundle over X. we define the Pontryagin classes $p_i(E)$ by

$$p_i(E) := (-1)^i c_{2i}(E \otimes \mathbb{C}) \in H^{4i}(X; \mathbb{Z}).$$

In case E is the real vector bundle underlying a complex vector bundle, we have that $E \otimes \mathbb{C} \cong E \oplus \overline{E}$. This observation allows us to prove the following result.

Theorem 2. *The equality*

$$\sum_{i=0}^{\infty}(-1)^i p_i(E) = c(E) \cdot \sum_{i=0}^{\infty}(-1)^i c_i(E)$$

holds in $H^*(X; \mathbb{Z})$.

Proof.

$$\sum_{i=0}^{\infty}(-1)^i p_i(E) = \sum_{i=0}^{\infty} c_{2i}(E \otimes \mathbb{C}) = \sum_{i=0}^{\infty}\sum_{j=0}^{2i} c_j(E)c_{2i-j}(\overline{E}).$$

Again by [9] lemma 14.9, $c_i\left(\overline{E}\right) = (-1)^i c_i(E)$, therefore the last term is equal to

$$\sum_{i=0}^{\infty}\sum_{j=0}^{2i}(-1)^{2i-j} c_j(E)c_{2i-j}(E) = \sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^{i-j} c_j(E)c_{i-j}(E).$$

Indeed, in the last sum the terms in which i is odd vanish. Now the last term is simply $c(E \oplus \overline{E})$ and the result follows. ■

By applying the theorem to the tangent bundle of $\mathbb{C}P^n$ we get

$$\sum_{i=0}^{\infty}(-1)^i p_i(E) = (1+g)^{n+1} \cdot (1-g)^{n+1} = (1-g^2)^{n+1}.$$

Therefore

$$p(\mathbb{C}P^n) = \sum_{i=0}^{\infty} p_i(E) = (1+g^2)^{n+1}.$$

Having these characteristic classes we can define their characteristic numbers as follows. Let X be a compact, oriented, almost complex manifold of dimension $2n$ and let $(i_1, ..., i_r)$ be a partition of n (i.e. $\sum_j i_j = n$). Then, the Chern number corresponding to this partition is defined as

$$\left(\prod_{j=1}^{r} c_{i_j}(X)\right)[X]$$

where $[X]$ denotes the fundamental cycle of the oriented manifold X.
In the differentiable case, the Pontryagin numbers of a manifold X of dimension $4n$ are defined as

$$\left(\prod_{j=1}^{r} p_{i_j}(X)\right)[X]$$

where $(i_1, ..., i_r)$ is again a partition of n.

3

GENERA

In this Chapter, we introduce the notion of cobordism and we study the ring homomorphisms from the rationalized cobordism ring to any commutative $\mathbb{Q}$-algebra.
For an oriented manifold *W* we indicate with ∂W its boundary with the induced orientation.

Definition 1. *Let* V *be a compact, oriented, differentiable,* n*-dimensional manifold without boundary. We say that* V *bounds if there exists a compact, oriented, differentiable,* $(n+1)$*-dimensional manifold* W *such that* $\partial W = V$*.*

Theorem 3. $\mathbb{C}P^{2k+1}$ *bounds.*

Idea of the Proof. Observe that the bijection $\sigma : \mathbb{C}^{2k+2} \rightarrow \mathbb{H}^{k+1}$

$$\sigma(z_1, ..., z_{2k+2}) = (z_1 + z_2 \cdot j, ..., z_{2k+1} + z_{2k+2} \cdot j)$$

induces a fibration

$$S^2 \rightarrow \mathbb{C}P^{2k+1} \rightarrow \mathbb{H}P^k.$$

This fibration yields a locally trivial fiber bundle which can be given the structure group $SO(3)$. One can extend this operation to the associated disk bundle

$$D^3 \rightarrow E \rightarrow \mathbb{H}P^k$$

with fiber the 3-ball obtaining a $4k+3$-dimensional manifold E whose boundary is $\mathbb{C}P^{2k+1}$.

■

Definition 2. *Two compact, oriented, differentiable,* n*-dimensional manifold without boundary* V *and* W *are said to be cobordant* $(V \sim W)$ *if the manifold* $V + (-W)$ *bounds. Here "+" denotes the disjoint union and* $-W$ *is the manifold* W *with reversed orientation.*

It's easy to check that the relation $\sim$ is an equivalence relation, for example for reflexivity we have that $V + (-V) = \partial(V \times [0, 1])$ and for transitivity, we can glue two manifolds along diffeomorphic boundary components. We call Ω^n the set of equivalence classes of compact, oriented, differentiable, n-dimensional manifolds with respect to $\sim$.
$(\Omega^n, +)$ is a finitely generated abelian group and the Cartesian product of manifolds induces a map $\Omega^n \times \Omega^m \rightarrow \Omega^{n+m}$ making $\Omega := \sum_{n=0}^{\infty} \Omega^n$ into a graded commutative unital ring called cobordism ring.

Theorem 4. *$\Omega^n \otimes \mathbb{Q} = 0$ when n is not a multiple of 4 and Ω^{4k} is a finitely generated abelian group with rank equal to the number of partitions of k. Furthermore,*

$$\Omega \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{C}P^2, \mathbb{C}P^4, \mathbb{C}P^6, ...]$$

as a graded polynomial ring.

See [7] Chapter II for a proof. For example, $\Omega^4 \otimes \mathbb{Q} = \langle \mathbb{C}P^2 \times \mathbb{C}P^2, \mathbb{C}P^4 \rangle$.

Definition 3. *Let R be a commutative unital $\mathbb{Q}$-algebra. A genus is a ring homomorphism $\varphi : \Omega \otimes \mathbb{Q} \to R$.*

Consider an even power series $Q(x)$ with constant term 1 and with coefficients in R. If $x_1, x_2, ..., x_n$ are indeterminates of weight two, then the product $Q(x_1) \cdots Q(x_n)$ is symmetric in the x_i^2 and hence can be written in terms of the elementary symmetric functions p_j of the x_i^2 in the form

$$Q(x_1) \cdots Q(x_n) = 1 + K_1(p_1) + K_2(p_1, p_2) + ... + K_n(p_1, ..., p_n) + K_{n+1}(p_1, ..., p_n, 0) + ...$$

for homogeneous polynomials K_r of weight $4r$. For example, if $Q(x) = 1 + a_2x^2 + a_4x^4 + ...$ then $Q(x_1) \cdots Q(x_n) = 1 + a_2 \sum_{i=1}^n x_i^2 + ...$ so $K_1(p_1) = a_2p_1$.
Given a power series Q as described above, we can associate to it a genus φ_Q. Namely, if M is a compact, oriented, differentiable, $4n$-dimensional manifold, then we can define $\varphi_Q(M) := K_n(p_1, ..., p_n)[M] \in R$, with $p_i = p_i(M) \in H^{4i}(M; \mathbb{Z})$. In addition, if $4 \nmid \dim(M)$ we put $\varphi_Q(M) = 0$. Furthermore, we define $K(TM) := K(p_1, ..., p_n) := 1 + K_1(p_1) + K_2(p_1, p_2) + ...$, so that $\varphi_Q(M) = K(TM)[M]$.

Remark 1. *φ_Q is a well-defined genus, that is*

- *φ_Q vanishes on boundaries*
- *It is compatible with the disjoint union*
- *It is compatible with the Cartesian product.*

Indeed, for the first assertion, suppose $M = \partial W$ then the tangent bundle of W restricted to M is the tangent bundle of M plus the normal bundle of M in W, which is trivial. Therefore, the Pontryagin classes of M are those of W restricted to the boundary, hence

$$P(M) \cap [M] = i^*P(W) \cap [\partial W] = \delta \circ i^*P(W) \cap [W] = 0$$

where $P(M)$ is a cohomology class of M of maximum dimension given by any product of Pontryagin classes. The second assertion is clear, and the last follows from the following general fact (see [7] Lemma 1.2.2 for a proof): if $p_i, p_i^{'}, p_i^{''}$ are indeterminates for which

$$1 + p_1 + p_2 + ... = (1 + p_1^{'} + p_2^{'} + ...) \cdot (1 + p_1^{''} + p_2^{''} + ...)$$

then

$$\sum_{n\geqslant 0} K_n(p_1,...,p_n) = \sum_{n\geqslant 0} K_n(p_1^{'},...,p_n^{'}) \cdot \sum_{n\geqslant 0} K_n(p_1^{''},...,p_n^{''}) \tag{3.1}$$

where the K_n are the polynomials defined by Q as before. This is exactly the multiplicative property satisfied by the Cartesian product of two manifolds.

Now we discuss an important lemma relating characteristic numbers and genera.

Start with an even power series Q with constant tern 1 and with coefficients in a $\mathbb{Q}$-algebra R as before, and define f as the odd power series $f(x) = x/Q(x)$.

Since $Q(x)$ begin with 1, $f(x)$ begins with x and has coefficients in R. Now define g as the formal inverse function of f.

Lemma 2. $g^{'}(y) = \sum\limits_{n=0}^{\infty} \varphi_Q(\mathbb{C}P^n) \cdot y^n.$

Proof.

$$c(\mathbb{C}P^n) = (1+x)^{n+1} \Rightarrow p(\mathbb{C}P^n) = (1+x^2)^{n+1}.$$

Then by the multiplicative property 3.1 we have

$$\sum_{n\geqslant 0} K_n(p_1,...,p_n) = \left(\sum_{n\geqslant 0} K_n(x^2,0,...,0)\right)^{n+1} = Q(x)^{n+1}$$

where the p_i are the Pontryagin classes of $\mathbb{C}P^n$ and $x \in H^2(\mathbb{C}P^n;\mathbb{Z})$ is the generator Poincaré-dual to $\mathbb{C}P^{n-1} \subset \mathbb{C}P^n$.

$$\begin{aligned}
\varphi_Q(\mathbb{C}P^n) &= \left(\frac{x}{f(x)}\right)^{n+1} [\mathbb{C}P^n] \\
&= \text{coefficient of } x^n \text{ in } \left(\frac{x}{f(x)}\right)^{n+1} = \text{res}_0 \left(\frac{1}{f(x)}\right)^{n+1} dx \\
&= \frac{1}{2\pi i} \int_{\gamma} \left(\frac{1}{f(x)}\right)^{n+1} dx = \frac{1}{2\pi i} \int_{f(\gamma)} \left(\frac{1}{y^{n+1}}\right)^{n+1} g^{'}(y)dy \\
&= \text{res}_0 \left(\frac{g^{'}(y)dy}{y^{n+1}}\right) = \text{coefficient of } y^n \text{ in } g^{'}(y).
\end{aligned}$$

Note that f is a power series starting with x, so when it converges, $f(\gamma)$ is also a closed path with winding number 1. When f does not converge, the substitution formula works anyway for formal power series. ■

The power series g is called the logarithm of the genus φ_Q and the lemma shows that the genus is determined by its logarithm.

On the other hand, if we fix the values of a genus on the complex projective spaces, then we can form the power series $g^{'}$ and reverse the process ending up with $Q(x)$. This determines a one-to-one correspondence between genera and even power series starting with 1. Roughly speaking, this shows in particular that genera are nothing but infinite linear combinations of characteristic numbers.

As an example, if we start with $Q(x) = x/\tanh(x)$ we get $f(x) = \tanh(x)$. Then $f^{'}(x) = 1 - f(x)^2$ and $g^{'}(y) = 1/(1-y^2) = 1 + y^2 + y^4 + ...$ So we obtain the genus whose value on all $\mathbb{C}P^{2n}$ is 1, this is called the L-genus.

Recall that a stably almost-complex manifold M is a differentiable manifold for which there exists a trivial vector bundle $\underline{\mathbb{R}^n}$ such that $TM \oplus \underline{\mathbb{R}^n}$ admits a complex structure.
If instead of considering cobordism classes of compact oriented differentiable manifolds we consider cobordism classes of stably almost-complex manifolds, we obtain a variant of the cobordism ring, called the complex cobordism ring Ω_U.
In this case we have $\Omega_U \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{C}P^1, \mathbb{C}P^2, \mathbb{C}P^3, ...]$ and we can consider U-genera, as ring homomorphisms from the rationalized complex cobordism ring to a $\mathbb{Q}$-algebra R.
If we apply the same constructions that we have seen in this chapter, replacing the Pontryagin numbers with the Chern numbers, we see that these U-genera are in one-to-one correspondence with the power series $Q(x)$ with constant term 1 (not necessarly even).

4
FORMAL GROUP LAWS

Using the notation of [12] we think of a scheme as a represented functor from $\mathbf{Ring}$ to $\mathbf{Set}$. We define the category of formal schemes as the category whose objects are small filtered colimits of schemes and whose morphisms are natural transformations.

As an example, we have the functor $\widehat{\mathbb{A}}^1$ whose value at a ring R is the set of its nilpotent elements $\widehat{\mathbb{A}}^1(R) = \mathrm{Nil}(R)$. This is a formal scheme because it is the colimit of the represented functors $\mathbf{Ring}(\mathbb{Z}[t]/t^n, -)$.

We will indicate the category of formal schemes with $\widehat{\mathcal{X}}$ and, if X is a scheme, we will indicate the slice category over X as $\widehat{\mathcal{X}}_X$.

When not differently specified, we will assume that our formal schemes are one-dimensional over a scheme X, this means formal schemes isomorphic to $\widehat{\mathbb{A}}^1 \times X$ in $\widehat{\mathcal{X}}_X$ (in general an n-dimensional formal scheme over X is one isomorphic to $\widehat{\mathbb{A}}^n \times X$ in $\widehat{\mathcal{X}}_X$).

Given $S \in \widehat{\mathcal{X}}_X$, a coordinate on S is a map $S \to \widehat{\mathbb{A}}^1 \times X$ giving rise to an isomorphism.

Associated to a formal scheme $S \in \widehat{\mathcal{X}}$ we have its ring of functions $\mathcal{O}_S$ that is the set of maps from S to the forgetful functor $\mathbb{A}^1 : \mathbf{Ring} \to \mathbf{Set}$ with the ring structure given by pointwise operations. That is, given two natural transformations $\phi, \psi : S \to \mathbb{A}^1$, their product is the natural transformation that, on a ring R, is given by $m_R \circ (\phi \times \psi) \circ \Delta : S(R) \to R$, where m_R is the multiplication in the ring R and Δ is the diagonal inclusion. Note that the forgetful functor is a scheme, indeed it is represented by the ring $\mathbb{Z}[t]$.

By [12] chapter II, the functor that associates to any scheme its ring of functions $X \mapsto \mathcal{O}_X$ is an isomorphism between the category of schemes and the opposite category of rings, in particular, for a scheme X it holds that $X = \mathbf{Ring}(\mathcal{O}_X, -)$.

If $S \in \widehat{\mathcal{X}}_X$ is a one-dimensional formal scheme over a scheme X, then

$$\begin{aligned}
\mathcal{O}_S &\cong \mathrm{Nat}(X \times \widehat{\mathbb{A}}^1, \mathbb{A}^1) \\
&\cong \mathrm{Nat}(X \times \operatorname{colim} \mathbf{Ring}(\mathbb{Z}[t]/t^n, -), \mathbb{A}^1) \\
&\cong \lim \mathrm{Nat}(\mathbf{Ring}(\mathcal{O}_X \otimes \mathbb{Z}[t]/t^n, -), \mathbb{A}^1) \\
&\cong \lim \mathbb{A}^1(\mathcal{O}_X[t]/t^n) \cong \mathcal{O}_X[[t]].
\end{aligned}$$

For the second isomorphism we have used that in any presheaf category, filtered colimits are stable under pullback. So, in our cases of interest, the rings of functions will always be power series rings, at least in this non-equivariant setting.

A formal group over a scheme X is a group object in the category of formal schemes over X, note that if $\mathbb{G} \in \widehat{\mathcal{X}}_X$, specifying a multiplication map $\mathbb{G} \times_X \mathbb{G} \to \mathbb{G}$ is the same as specifying an element of $\widehat{\mathcal{X}}_X(X \times \widehat{\mathbb{A}}^2, X \times \widehat{\mathbb{A}}^1)$ where $\widehat{\mathbb{A}}^2 = \widehat{\mathbb{A}}^1 \times \widehat{\mathbb{A}}^1$.

With a calculation similar to the previous one we obtain that

$$\begin{aligned}\widehat{\mathcal{X}}_X(X\times\widehat{\mathbb{A}}^2, X\times\widehat{\mathbb{A}}^1) &\cong \widehat{\mathcal{X}}(X\times\widehat{\mathbb{A}}^2,\widehat{\mathbb{A}}^1)\\ &\cong \widehat{\mathcal{X}}(\mathbf{Ring}(\mathcal{O}_X,-)\times \operatorname{colim}\mathbf{Ring}(\mathbb{Z}[x]/x^n\otimes\mathbb{Z}[y]/y^m,-),\widehat{\mathbb{A}}^1)\\ &\cong \lim \widehat{\mathcal{X}}(\mathbf{Ring}(\mathcal{O}_X\otimes\mathbb{Z}[x]/x^n\otimes\mathbb{Z}[y]/y^m,-),\widehat{\mathbb{A}}^1)\\ &\cong \lim \widehat{\mathbb{A}}^1(\mathcal{O}_X\otimes\mathbb{Z}[x]/x^n\otimes\mathbb{Z}[y]/y^m)\subseteq \mathcal{O}_X[[x,y]].\end{aligned}$$

It means that we can encode the behaviour of the multiplication map in a power series in two variables with coefficients in the ring $\mathcal{O}_X$. As a consequence, we can also encode the group axioms in the coefficients of this power series, for example, we unravel the right-unit axiom, which says that the following diagram commutes

$$\begin{tikzcd}\mathbb{G} \arrow[r, "\cong"] \arrow[rrr, bend right, "\mathrm{id}_{\mathbb{G}}"'] & \mathbb{G}\times_X X \arrow[r, "\mathbb{G}\times e"] & \mathbb{G}\times_X\mathbb{G} \arrow[r, "m"] & \mathbb{G}\end{tikzcd}$$

We start by observing that $\mathrm{Id}_{\mathbb{G}}:\mathbb{G}\to\mathbb{G}$ corresponds to $x\in\mathcal{O}_X[[x]]$. This can be seen just by unrevealing the isomorphisms, indeed $\mathrm{Id}_{X\times\widehat{\mathbb{A}}^1}\in\widehat{\mathcal{X}}_X(X\times\widehat{\mathbb{A}}^1, X\times\widehat{\mathbb{A}}^1)$ corresponds to the natural transformation $\operatorname{colim}\mathbf{Ring}(\mathcal{O}_X\otimes\mathbb{Z}[x]/x^n,-)\to\operatorname{colim}\mathbf{Ring}(\mathbb{Z}[x]/x^n,-)$ given by precomposition with the ring homomorphism $x\mapsto 1\otimes x$. As a consequence, the corresponding element in $\widehat{\mathcal{X}}(\mathbf{Ring}(\mathcal{O}_X\otimes\mathbb{Z}[x]/x^n,-),\widehat{\mathbb{A}}^1)$ is the one that corresponds to $x\in\widehat{\mathbb{A}}^1(\mathcal{O}_X\otimes\mathbb{Z}[x]/x^n)$ through the Yoneda lemma for each n, so by taking the image in the limit we get that the corresponding power series is $x\in\mathcal{O}_X[[x]]$.

Now if our multiplication m is represented by the power series $F(x,y)=\sum_{j,k}a_{jk}x^jy^k\in\mathcal{O}_X[[x,y]]$, and the unit map by $0\in\mathcal{O}_X$, then we must have that $x\mapsto(x,0)\mapsto F(x,0)$ coincides with the identity $x\mapsto x$. In other words

$$a_{i0}=\begin{cases}1, & \text{if } i=1.\\ 0, & \text{otherwise.}\end{cases}\tag{4.1}$$

Asking that e is a two-sided unit witness also the symmetrical condition so we conclude that

$$F(x,0)=x=F(0,x).\tag{4.2}$$

The associativity axiom translates into an associativity for F, namely

$$F(x,F(y,z))=F(F(x,y),z)\tag{4.3}$$

and the inverse axiom in

$$F(\iota(x),x)=F(x,\iota(x))=0.\tag{4.4}$$

For some power series $\iota(x)\in\mathcal{O}_X[[x]]$.

We will call formal group law an element of $\mathcal{O}_X[[x,y]]$ satisfying 4.2, 4.3 and 4.4.

When not differently specified we will also assume that our formal group laws are commutative, that is $F(x,y)=F(y,x)$.

It's worth remarking that we started by saying that a power series has to satisfy the formal group law axioms in order to specify a group structure on a formal scheme, but now we can also guarantee that given a one-dimensional formal group $\mathbb{G} \in \hat{\mathcal{X}}_X$ and a coordinate $y : \mathbb{G} \to X \times \hat{\mathbb{A}}^1$, a formal group law over the ring $\mathcal{O}_X$ really specifies a group structure on $\mathbb{G}$ because the first axiom forces the constant term of the formal group law to vanish, therefore any formal group law is really in the image of the inclusion
$\lim \hat{\mathbb{A}}^1(\mathcal{O}_X \otimes \mathbb{Z}[x]/x^n \otimes \mathbb{Z}[y]/y^m) \subseteq \mathcal{O}_X[[x, y]]$.

4.1 Topological Hopf algebra structure on the ring of functions

In this section, we make explicit some links between the formal group aspects and the implications they have on its ring of functions, this digression will be very useful in understanding how to generalize these concepts to the equivariant case and will be used a lot in chapter 6.

A formal group law defines a comultiplication on the ring of functions $\mathcal{O}_X[[t]]$ of a formal group $\mathbb{G}$ over X by $\Delta(t) = F(1 \otimes t, t \otimes 1) \in \mathcal{O}_X[[t]] \hat{\otimes}_{\mathcal{O}_X} \mathcal{O}_X[[t]]$ that makes it in a complete Hopf algebra over $\mathcal{O}_X$. Furthermore, we can think of the ring of functions on a one-dimensional formal group $\mathbb{G}$ over X as a complete topological Hopf algebra where the topology is generated under powers, translations, finite intersections and unions by a certain ideal $I \trianglelefteq \mathcal{O}_{\mathbb{G}}$.

In general, for each ring S and for each element $x \in \mathbb{G}(S)$, we have a corresponding natural transformation $\varphi_x : Y = \mathbf{Ring}(S, -) \to \mathbb{G}$ through the Yoneda lemma. We obtain a map on the respective rings of functions by pullback $\varphi_x^* : \mathcal{O}_{\mathbb{G}} \to \mathcal{O}_Y$, call I_x the kernel of φ_x^*. More explicitly, if $x \in \mathbb{G}(S)$, a natural transformation $\eta \in \hat{\mathcal{X}}_x(\mathbb{G}, \mathbb{A}^1)$ belongs to the kernel of φ_x^* if and only if the composition $\eta \circ \varphi_x$ vanishes and, by the Yoneda lemma, this is equivalent to $\eta(S)(x) = 0 \in \mathbb{A}^1(S)$. Now $x \in \mathbb{G}(S) = \operatorname{colim} \mathbf{Ring}(\mathcal{O}_X \otimes \mathbb{Z}[t]/t^n, S)$ because colimits in functors categories are computed pointwise, and the Yoneda lemma tells us that for each n we have

$$\begin{tikzcd}
\mathrm{Id}_{\mathcal{O}_X \otimes \mathbb{Z}[t]/t^n} \arrow[r, mapsto] \arrow[d, phantom, sloped, "\in"] \arrow[ddd, mapsto, bend right=60] & x_n \arrow[d, phantom, sloped, "\in"] \arrow[ddd, mapsto, bend left=60] \\
\mathbf{Ring}(\mathcal{O}_X \otimes \mathbb{Z}[t]/t^n, \mathcal{O}_X \otimes \mathbb{Z}[t]/t^n) \arrow[r] \arrow[d, "\eta(\mathcal{O}_X \otimes \mathbb{Z}[t]/t^n)"'] & \mathbf{Ring}(\mathcal{O}_X \otimes \mathbb{Z}[t]/t^n, S) \arrow[d, "\eta(S)"] \\
\mathbb{A}^1(\mathcal{O}_X \otimes \mathbb{Z}[t]/t^n) \arrow[r] \arrow[d, phantom, sloped, "\ni"] & \mathbb{A}^1(S) \arrow[d, phantom, sloped, "\ni"] \\
\eta(\mathrm{Id}_{\mathcal{O}_X \otimes \mathbb{Z}[t]/t^n}) \arrow[r, mapsto] & x_n(\eta(\mathrm{Id}_{\mathcal{O}_X \otimes \mathbb{Z}[t]/t^n})) = \eta(S)(x_n)
\end{tikzcd}$$

where we have called x_n the preimage of x under the universal map in the colimit, this preimage exists for n sufficiently large. So the natural transformation η belongs to I_x if and only if, for some n, we can evaluate the morphism x_n on the power series corresponding to η truncated at level n and obtain zero. But x_n, by definition, is just a map of rings $\phi : \mathcal{O}_X \to S$ together with an element $s \in S$ such that $s^n = 0$, therefore an element x in the colimit $\mathbb{G}(S)$ is a map of rings $\phi : \mathcal{O}_X \to S$ together with a nilpotent element $s \in S$ and $x(\sum \alpha_i x^i) = \sum \phi(\alpha_i)s^i$.

Define now the ideal I as the kernel of φ_0^* with $0 \in \mathcal{O}_X$, this is just the ideal of the power series that evaluated at 0 give 0. Note that φ_0^* acts on a power series by evaluation at 0 exactly as the counit θ of $\mathcal{O}_{\mathbb{G}}$, obtained by pullback with respect to the unit of the formal group $\mathbb{G}$. With the topology on $\mathcal{O}_{\mathbb{G}}$ defined by I, that is the kernel of θ, it is apparent that the product $\mathcal{O}_{\mathbb{G}} \hat{\otimes}_{\mathcal{O}_X} \mathcal{O}_{\mathbb{G}} \to \mathcal{O}_{\mathbb{G}}$ and the coproduct $\mathcal{O}_{\mathbb{G}} \to \mathcal{O}_{\mathbb{G}} \hat{\otimes}_{\mathcal{O}_X} \mathcal{O}_{\mathbb{G}}$ are continuous maps once the completed tensor product is equipped with the right topology.

We can verify either by direct computation at the power series level or via the formal group axioms that $(\theta \hat{\otimes} \theta) \circ \Delta = \Delta \circ \theta = \theta$.

The upshot of this discussion is that making a one-dimensional formal scheme S over X into a formal group is the same as specifying a complete topological Hopf algebra structure over on its ring of functions $\mathcal{O}_S$ where the topology is defined by θ. This is the point of view that we will use to generalize the theory to the equivariant case.

4.2 FORMAL GROUP LAWS FROM SPECTRA

In this section we see how formal group laws naturally arise in algebraic topology from cohomology theories.

Let E be a homotopy associative and homotopy commutative ring spectrum provided with a homotopy unit $i : S^0 \to E$. Suppose that E is provided with an orientation, that is an element $x \in \tilde{E}^*(\mathbb{CP}^\infty)$ such that $\tilde{E}^2(\mathbb{C}P^1)$ is a free module over E^* on the generator i^*x, where $i : \mathbb{C}P^1 \to \mathbb{CP}^\infty$ is the inclusion. Note that, since $\mathbb{C}P^1 \approx S^2$ and $\tilde{E}^2(S^2) \cong \tilde{E}^0(S^0) = \pi_0(E)$, by the suspension isomorphism, we have a canonical generator γ of the cyclic module $\tilde{E}^2(\mathbb{C}P^1)$ represented by the unit map $S^0 \to E$. However, we are not requiring i^*x to be exactly that generator, so, in general, we will have a relation of the form $i^*x = u\gamma$ for some invertible $u \in E^*$.

For example, for $E = H$ we can take as orientation the usual generator $x \in H^2(\mathbb{CP}^\infty; \mathbb{Z})$. For $E = K$, the spectrum representing K-theory, we can fix the orientation $\xi = 1 - z \in \tilde{K}^0(\mathbb{CP}^\infty)$, where z is the tautological vector bundle over $\mathbb{CP}^\infty$. If we consider the pullback

$$\begin{tikzcd}
i^*z \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & z \arrow[d] \\
\mathbb{CP}^1 \arrow[r, "i"] & \mathbb{CP}^\infty
\end{tikzcd}$$

we see that i^*z is what we called γ_1 in chapter 2, therefore, by [2], $1 - i^*z$ generates the cyclic module $\tilde{K}^0(S^2)$. In this particular case, we are allowed to choose the orientation in degree zero instead of two because of the Bott periodicity theorem.

Now consider $E^*(\mathbb{CP}^\infty)$, by [1] lemma 2.5 $E^*(\mathbb{CP}^\infty) \cong E^*[[x]]$ and $E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong E^*[[x_1, x_2]]$.

Over $\mathbb{CP}^\infty \times \mathbb{CP}^\infty$ we have the line bundles $\pi_1^* z$ and $\pi_2^* z$ obtained by pullback along the projections, so we can form their tensor product that is a line bundle over $\mathbb{CP}^\infty$. But $z \to \mathbb{CP}^\infty$ is the universal line bundle, it follows that we have a map $m : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ such that

$$\begin{tikzcd}
\pi_1^*(z) \otimes \pi_2^*(z) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & z \arrow[d] \\
\mathbb{CP}^\infty \times \mathbb{CP}^\infty \arrow[r, "m"'] & \mathbb{CP}^\infty
\end{tikzcd}$$

Therefore, $m^* : E^*[[x]] \to E^*[[x_1, x_2]]$ maps the orientation x in a power series $m^*(x) = \mu(x_1, x_2) = \sum\limits_{i,j} a_{ij} x_1^i x_2^j \in E_*[[x_1, x_2]]$ and this power series is a formal group law in the sense discussed before.

For example, the unity axiom is ensured by the commutativity of the diagram

$$\begin{tikzcd}
z \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \pi_1^*(z) \otimes \pi_2^*(z) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & z \arrow[d] \\
\mathbb{CP}^\infty \arrow[r, "Id \times 0"] \arrow[rr, bend right, "Id"'] & \mathbb{CP}^\infty \times \mathbb{CP}^\infty \arrow[r, "m"] & \mathbb{CP}^\infty
\end{tikzcd}$$

and the others are obtainable similarly.

For example, when $E = H$ with our choices for the orientations we have $m^*(x) = m^*(-c_1(z)) = -c_1(m^*(z)) = -c_1(\pi_1^*(z) \otimes \pi_2^*(z)) = -c_1(\pi_1^*(z)) - c_1(\pi_2^*(z)) = x_1 + x_2$. This is probably the simplest example of a formal group law, and it is called additive formal group law.

If we take $E = K$, the spectrum representing the K-theory, we get
$m^*(z) = (z_1)(z_2)$ that is $m^*(\xi) = \xi_1 + \xi_2 - \xi_1 \xi_2$. This other formal group law is called the multiplicative formal group law.

4.3 THE LAZARD RING

Consider now, more abstractly, a formal group law over a ring R, that is a power series in two variables, with coefficients in R and with the proprieties 4.2, 4.3 and 4.4. A morphism of rings $\theta : R \to S$ carries a formal group law μ over R into a formal group law $\theta_* \mu$ over S simply by applying θ to the coefficients of the power series. In this section we discuss a universal commutative, unital ring L with a commutative formal group law μ^L with the property that for any other commutative, unital ring R with a commutative formal group law μ^R, we have a unique homomorphism $\theta : L \to R$ such that $\theta_* \mu^L = \mu^R$.

The ring L can be defined explicitly as a quotient of the free abelian group
$P = \mathbb{Z}[a_{11}, a_{12}, a_{21}, ..., a_{ij}, ...]$ for formal symbols a_{ij}. Set

$$\mu^L(x, y) = x + y + \sum_{i,j \geqslant 1} a_{ij} x^i y^j$$

and set

$$\mu^L(x, \mu^L(y, z)) - \mu^L(\mu^L(x, y), z) = \sum_{i,j,k} b_{ijk} x^i y^j z^k$$

so that b_{ijk} is a well-defined polynomial in the a_{ij}. Let $I \trianglelefteq P$ be the ideal generated by b_{ijk} and $a_{ij} - a_{ji}$. By construction, the ring $L = P/I$ with the formal group law $\mu^L \in L[[x, y]]$ has the desired property.

Therefore, giving a formal group law on a commutative unital ring R is the same as giving a homomorphism $L \to R$.

There is a very important theorem by Quillen regarding this universal ring L, namely, since the spectrum MU has a canonical orientation, the ring MU^* has a canonical formal group law in the way explained before. Therefore, we obtain a canonical map $\theta : L \to MU^*$

Theorem 5. θ *is an isomorphism.*

As a consequence, specifying a formal group law over R is the same as giving a homomorphism $MU^* \to R$.

But the coefficients ring of the spectrum MU is the complex cobordism ring Ω_U, therefore, if R is a $\mathbb{Q}$-algebra, then specifying a formal group law over R is the same as giving a U-genus $\phi : \Omega_U \otimes \mathbb{Q} = MU^* \otimes \mathbb{Q} \to R$.

5

MORPHISMS OF FGLs AND INVARIANT DIFFERENTIALS

In this chapter, we discuss some examples of formal group laws and exhibit some isomorphisms between them. The easiest one is probably the additive formal group law, defined by $F_a(x, y) = x + y$ that we already encountered at the end of section 4.2. Another fundamental formal group law is the multiplicative one defined by $F_m(x, y) = x + y - xy$ that we have seen arising from K-theory in section 4.2. Now we want to define morphisms between formal group laws and from our previous discussion about formal groups it is natural to require the commutativity of the diagrams

$$\begin{tikzcd}
\mathbb{G} \times_X \mathbb{G} \arrow[r, "m_{\mathbb{G}}"] \arrow[d, "\eta \times_X \eta"'] & \mathbb{G} \arrow[d, "\eta"] \\
\mathbb{H} \times_X \mathbb{H} \arrow[r, "m_{\mathbb{H}}"'] & \mathbb{H}
\end{tikzcd}$$

This can be translated in $\eta(F_{\mathbb{G}}(x, y)) = F_{\mathbb{H}}(\eta(x), \eta(y))$.
For example, consider the power series $f(x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} = \log(1 - x) \in \mathbb{Q}[[x]]$, then $f(F_m(x, y)) = \log((1 - x)(1 - y)) = f(x) + f(y) = F_a(f(x), f(y))$. Furthermore, $f(x)$ is a functional-invertible power series, in the sense that there exists $g \in \mathbb{Q}[[x]]$ such that $f \circ g = g \circ f = id$, thus, f is an isomorphism between the multiplicative and the additive formal group law (rationally). This is not a particular case, as the following result tells

Theorem 6. *Every commutative formal group law is rationally isomorphic to the additive one.*

Idea of the Proof. If R is a $\mathbb{Q}$-algebra, then to every formal group law $F \in R[[x, y]]$ we have an associated genus $\phi : MU_* \otimes \mathbb{Q} \to R$ by theorem 5, so we can form the corresponding power series $g'(x) \in R[[x]]$ by lemma 2 and integrate it to obtain the logarithm g of the genus. Then unrevealing all the one-to-one correspondences, one sees that $g(F(x, y)) = g(x) + g(y) = F_a(g(x), g(y))$. ■

In this situation, we will call g the logarithm of the formal group law F.
As we will see, this is one of the results that do not generalize to the equivariant case.

5.1 THE INVARIANT DIFFERENTIALS

Given a formal group $\mathbb{G} \in \hat{\mathcal{X}}_X$, there is a more intrinsic way to understand the isomorphism g of formal group laws from $F_{\mathbb{G}}$ to F_a in terms of the invariant differentials of $\mathbb{G}$.

We need some definitions, let $\mathbb{G}$ be an n-dimensional formal group over X and consider the kernel J of the multiplication map $\mathcal{O}_{\mathbb{G}}\hat{\otimes}_{\mathcal{O}_X}\mathcal{O}_{\mathbb{G}} \to \mathcal{O}_{\mathbb{G}}$. Call $\Omega^1_{\mathbb{G}/X} = J/J^2$, this is a module over $\mathcal{O}_{\mathbb{G}}\hat{\otimes}_{\mathcal{O}_X}\mathcal{O}_{\mathbb{G}}/J \cong \mathcal{O}_{\mathbb{G}}$. For $f \in \mathcal{O}_{\mathbb{G}}$, write $d(f) = f\hat{\otimes}1 - 1\hat{\otimes}f + J^2 \in \Omega^1_{\mathbb{G}/X}$, if $\mathbf{x} = (x_1, ..., x_n) : \mathbb{G} \to X \times \hat{\mathbb{A}}^n = X \times (\hat{\mathbb{A}}^1)^n$ is an isomorphism over X, then $\Omega^1_{\mathbb{G}/X}$ turns out to be a free module over $\mathcal{O}_{\mathbb{G}}$ generated by the elements $d(x_1), ..., d(x_n)$ (see [12] section 5.3).

We can pullback these differentials as in differential topology, that is, if we have a map from an n-dimensional formal group to a one-dimensional one over X, $f : \mathbb{H} \to \mathbb{G}$ and $\mathbf{y} = (y_1, ..., y_n) : \mathbb{H} \to X \times \hat{\mathbb{A}}^n$, $x : \mathbb{G} \to \mathbb{G} \times \hat{\mathbb{A}}^1$ are coordinates, then $x \circ f = \varphi(\mathbf{y}) \in \mathcal{O}_X[[y_1, ..., y_n]]$ for some power series φ and we define $f^*(r \cdot dx) = \sum_i r \circ f \frac{\partial \varphi(y)}{\partial y_i} \cdot dy_i$ for any $r \in \mathcal{O}_{\mathbb{G}}$.

We say that $\omega \in \Omega^1_{\mathbb{G}/X}$ is an invariant differential if $m^*\omega = \pi_1^*(\omega)+\pi_2^*(\omega) \in \Omega^1_{\mathbb{G}\times_X\mathbb{G}}$, where $m : \mathbb{G} \times_X \mathbb{G} \to \mathbb{G}$ is the multiplication and π_i are the projections. This condition can be checked with the corresponding power series, namely, $\omega(x)dx \in \Omega^1_{\mathbb{G}/X} = \mathcal{O}_{\mathbb{G}}\cdot dx \cong \mathcal{O}_X[[x]]\cdot dx$ is an invariant differential if and only if $\omega(F(x,y))\frac{\partial F(x,y)}{\partial x}dx + \omega(F(x,y))\frac{\partial F(x,y)}{\partial y}dy = \omega(x)dx + \omega(y)dy$. Since we are dealing with commutative formal group laws, this is equivalent to asking that $\omega(F(x,y))\frac{\partial F(x,y)}{\partial x} = \omega(x)$. We are now ready to prove the following result

Theorem 7. *Given a one-dimensional formal group* $\mathbb{G}$ *over a scheme* X*, if* $F_{\mathbb{G}}(x,y) \in \mathcal{O}_X[[x,y]]$ *is its formal group law and* $g(x) \in \mathbb{Q} \otimes \mathcal{O}_X[[x]]$ *is the logarithm of* F*, then* $g'(x)dx$ *is the invariant differential.*

Remark 2. *We use the expression the invariant differential because the set of the invariant differentials is a free module of rank* 1 *over* $\mathcal{O}_X$ *(see [12], proposition 7.2) so, for example, if the ground ring is a field then an invariant differential is unique up to scalar multiplication.*

Proof. Since $g(F(x,y)) = g(x) + g(y)$, by differentiating with respect to x we find that $g'(F(x,y))\frac{\partial F(x,y)}{\partial x} = g'(x)$ that is exactly the condition to check. ■

We observe that since $1 - \exp(x)$ is the functional inverse of $\log(1 - x)$, then every commutative formal group law F with logarithm g is isomorphic to the multiplicative formal group law through $h = (1 - \exp(x)) \circ g$.

Theorem 8. *In the hypothesis of the previous theorem, if* $h = (1 - \exp(x)) \circ g$ *then* $\frac{h'(x)}{1-h(x)}dx$ *is the invariant differential.*

Proof. Since $\log(1 - h(x)) = g(x)$, $\frac{h'(x)}{1-h(x)}dx = -g'(x)dx$. ■

6

THE EQUIVARIANT CASE

In this chapter we consider A-equivariant formal group laws where A is an abelian compact Lie group. We will indicate with $A^* = \hom(A, S^1)$ the dual of A.
We introduce the concept of a complete A-universe $\mathcal{U}$ as an infinite-dimensional complex vector space that contains a countable infinite amount of copies of each irreducible A-representation. we define a complete flag for $\mathcal{U}$ as a sequence of vector spaces $V^0 \subset V^1 \subset V^2 \subset \ldots$ such that $\dim_{\mathbb{C}}(V^i) = i$ and $\bigcup_{i \geqslant 0} V^i = \mathcal{U}$.
Inspired by our previous discussion about the ring of functions of a formal group over a scheme X, we define an A-equivariant formal group law in terms of the ring of functions of a corresponding A-equivariant formal group.

Definition 4. *An A-equivariant formal group law over a commutative ring k is*

- *A complete topological Hopf k-algebra R with*
- *A homomorphism $\theta : R \to k^{A^*}$ of topological Hopf k-algebras so that the topology on R is defined by the finite intersections of kernels of its components*
- *A regular element $y(\varepsilon) \in R$ that generates the kernel of the ϵth component of θ, that is*

$$0 \longrightarrow R \xrightarrow{y(\varepsilon)} R \xrightarrow{\theta_\varepsilon} k \longrightarrow 0$$

  is a short exact sequence.

We are giving to k^{A^*} the natural structure of Hopf k-algebra where the coproduct is induced by the group multiplication and the counit by the inclusion of the identity of the group. In particular, from the diagram

$$\begin{tikzcd}
R \arrow[r, "\theta"] \arrow[d, "\text{counit}"'] & k^{A^*} \arrow[d, "\pi_\epsilon"] \\
k \arrow[r, "\mathrm{id}_k"'] & k
\end{tikzcd}$$

we see that θ_ϵ must be the counit of the Hopf algebra R. Therefore, we may think of θ_ϵ as the map corresponding to the unit of the respective formal group $X \to \mathbb{G}$ in $\widehat{\mathcal{X}}_X$.
The augmentation map θ can be viewed as the effect on the ring of functions of the group action, indeed the following diagram in the category $\widehat{\mathcal{X}}_X$

$$\begin{tikzcd}
\mathbb{G} & \mathbb{G} \times_X \mathbb{G} \arrow[l, "m"'] & X \times_X \mathbb{G} \arrow[l, "\zeta_\alpha \times_X \mathbb{G}"'] \arrow[r, phantom, "\cong"] & \mathbb{G}
\end{tikzcd}$$

at ring of functions level, becomes

$$\begin{tikzcd} R \arrow[r, "\Delta"] & R\hat{\otimes}_k R \arrow[r, "\theta_{\alpha^{-1}}\hat{\otimes}R"] & k\hat{\otimes}_k R \arrow[r, phantom, "\cong"] & R \end{tikzcd}$$

Thus, we may define the action of the representation $\alpha \in A^*$ on an element $r \in R$ by $l_\alpha(r) = (\theta(\alpha^{-1})\hat{\otimes} id)\Delta(r)$. We define the elements $y(\alpha) = l_\alpha(y(\epsilon))$ so we have that $y(\alpha)$ is regular and generates the kernel of θ_α.

Note that, with these definitions, the topology of R is defined by all the finite products ideals $\left(\prod_i y(\alpha_i)\right)$.

We now define new elements that have no analogous in the non-equivariant case, these are the Euler classes, defined by $e(\alpha) = \theta(\epsilon)(y(\alpha))$ and, in general, if $V = \bigoplus_i \alpha_i$, $e(V) = \prod_i e(\alpha_i)$.

By the unital axiom of the formal group $\mathbb{G}$ we have the commutativity of the diagram

$$\begin{tikzcd} X \arrow[r, "\zeta_\epsilon"] \arrow[rrrrr, bend right, "\zeta_\alpha"'] & \mathbb{G} \arrow[r, phantom, "\cong"] & \mathbb{G} \times_X X \arrow[r, "\mathbb{G}\times_X\zeta_\alpha"] & \mathbb{G} \times_X \mathbb{G} \arrow[r, "m"] & \mathbb{G} \end{tikzcd}$$

From which we expect the formula $\theta_\epsilon \circ l_\alpha = \theta_{\alpha^{-1}}$ to hold. Indeed, using the axioms of equivariant formal group law, we have $\theta_\epsilon \circ l_\alpha = \theta_\epsilon \circ (\theta_{\alpha^{-1}}\hat{\otimes} id) \circ \Delta = (\theta_{\alpha^{-1}}\hat{\otimes}\theta_\epsilon) \circ \Delta = \theta_{\alpha^{-1}}$. As a corollary, we have that $e(\alpha) = \theta(\alpha^{-1})(y(\epsilon))$.

The definition of equivariant formal group law we have given is a natural generalization of the non-equivariant definition, however, we also give an equivalent one that is more explicit and is usually more convenient to perform calculations, as we will see.

Given a ring k and a complete flag $F = (0 = V^0 \subset V^1 \subset V^2 \subset \ldots)$ for $\mathcal{U}$, we write $k\{\{F\}\} = k\{\{1, y(V^1), y(V^2), \ldots\}\}$ for the inverse limit of the free k-modules with basis $1, y(V^1), y(V^2), \ldots, y(V^s)$. Call α_i the quotient V^i/V^{i-1}.

Definition 5. *An* (A, F)*-equivariant formal group law over a commutative ring* k *relative to a complete flag* F *is the topological* k*-module* $k\{\{F\}\}$ *with a continuous product, a continuous coproduct and a continuous action of* A^* *satisfying the following conditions.*

- *The product is commutative, associative, and unital.*
- *The action is through ring homomorphisms, associative and unital.*
- *The coproduct is through ring homomorphisms, equivariant (that is* $\Delta \circ l_{\alpha\beta} = (l_\alpha \hat{\otimes} l_\beta) \circ \Delta$*), commutative, associative and unital.*
- $y(\alpha_{j+1})y(V^j) = y(V^{j+1})$.
- *For each* i *the ideal* $(y(V^i))$ *has topological basis* $y(V^i), y(V^{i+1}), y(V^{i+2}), \ldots$.

The continuity conditions can be made slightly more explicit by specifying the product, the action, and the coproduct with respect to a topological basis, indeed

$$y(V^i)y(V^j) = \sum_{s\geqslant 0} b_s^{i,j} y(V^s) \qquad l_\alpha y(V^i) = \sum_{s\geqslant 0} d(\alpha)_s^i y(V^s) \qquad \Delta y(V^i) = \sum_{s,t\geqslant 0} f_{s,t}^i y(V^s)\hat{\otimes} y(V^t)$$

for suitable structure constants $b_s^{i,j}$, $d(\alpha)_s^i$ and $f_{s,t}^i$ in k. In these terms, the continuity of the product is equivalent to asking that for fixed i, s the elements $b_s^{i,j}$ vanish for j sufficiently large, and the same must be true with i and j exchanged. The continuity of the action is equivalent to $d(\alpha)_s^i$ vanishing for fixed α and s and sufficiently large i and finally the continuity of the coproduct is just the vanishing of $f_{s,t}^i$ for fixed s, t and sufficiently large i. Now we give an idea about the equivalence of the above definitions of A-equivariant formal group law, more details can be found in [4].

Idea of the Proof. To pass from an (A, F) formal group law $k\{\{F\}\}$ to an A-equivariant formal group law in the sense of the first definition, we take $R = k\{\{F\}\}$ and define the elements $y(\alpha) = l_\alpha y(\epsilon)$. We already have the comultiplication and, assuming that F starts with ϵ, we define $\theta : k\{\{F\}\} \to k^{A^*}$ by taking $\theta(r)(\alpha)$ to be the constant coefficient in $l_{\alpha^{-1}}(r)$. On the other hand we can pass from an A-equivariant formal group law R to an (A, F)-equivariant one by defining $y(V) = y(\alpha_1)y(\alpha_2)\cdots y(\alpha_n)$ where $V = \alpha_1 \oplus \alpha_2 \oplus \ldots \oplus \alpha_n$. ■

From this fact, we deduce that an (A, F)-equivariant formal group law is essentially independent of the complete flag F.
Using the definition relative to a flag it is possible to prove the existence of a representing ring also for equivariant formal group laws, the key is the following lemma that makes uniform the continuity conditions of the product, the action, and the coproduct.

Lemma 3. *For any* (A, F)*-equivariant formal group law over* k *we have the following explicit vanishing conditions:*

- $b_s^{i,j} = 0$ *for* $s < i$ *or* $s < j$
- $d(\alpha)_s^i = 0$ *for* $V^i \otimes \alpha \geqslant V^{s+1}$
- $f_{s,t}^i = 0$ *for* $V^i \geqslant \alpha_1^{-1} V^{s+1} \otimes V^{t+1}$.

see [4] proposition 14.1 for a proof.

Corollary 1. *There is a representing ring* $L_A(F)$ *for* (A, F)*-equivariant formal group laws, it is constructed as the* $\mathbb{Z}$*-algebra with generators* $b_s^{i,j}$, $d(\alpha)_s^i$ *and* $f_{s,t}^i$ *subject to the relations specified by the first four axioms of definition 5 and the three uniform continuity conditions.*

We can now formalize the independence of the flag by saying that if F and F' are two complete flags for the complete A-universe $\mathcal{U}$, then there is a canonical isomorphism $L_A(F) \cong L_A(F')$. In view of this isomorphism, we will assume from now that our complete flag F starts with the trivial representation ϵ if not differently specified. We can use the universal ring to perform some 'universal' calculations and obtain relations that hold in every equivariant formal group law, as an example we prove the following lemma.

Lemma 4.

$$d(\beta)^i_k = \sum_j f^i_{j,k} e(\beta \otimes V^j).$$

Proof. By definition, we have

$$l_\beta = (\theta_{\beta^{-1}} \hat{\otimes} \mathrm{id}) \circ \Delta$$

and applying it to $y(V^i)$ we obtain

$$\sum_s d(\beta)^i_s y(V^s) = (\theta_{\beta^{-1}} \hat{\otimes} \mathrm{id})(\sum_{j,k} f^i_{j,k} y(V^j) \hat{\otimes} y(V^k))$$

$$\sum_s d(\beta)^i_s y(V^s) = \sum_{j,k} f^i_{j,k} e(\beta V^j) y(V^k).$$

■

From the proof, we see that the formula holds for any complete flag, however, when the complete flag begins with $V^1 = \epsilon$ we observe the following simplifications.

Since $e(\epsilon) = \theta_\epsilon(y(\epsilon))$, if V is any A-representation that contains a trivial direct summand, then $e(V) = 0$. By the counitality of the Hopf algebra, $(\mathrm{id}_R \hat{\otimes} \theta_\epsilon) \circ \Delta = \mathrm{id}_R$, evaluating this equality at $y(V^i)$ gives $y(V^i) = (\mathrm{id}_R \hat{\otimes} \theta_\epsilon)\left(\sum_{j,k} f^i_{j,k} y(V^j) \hat{\otimes} y(V^k)\right) = \sum_{j,k} f^i_{j,k} e(V^k) y(V^j) = \sum_j f^i_{j,0} y(V^j)$ so, when our flag begins with $V^1 = \epsilon$, we must have $f^i_{j,0} = \delta^i_j$ and similarly $f^i_{0,j} = \delta^i_j$.

6.1 Equivariant formal group laws from G-spectra

We have seen in chapter 4.2 that formal group laws naturally arise in algebraic topology from oriented cohomology theory, in this chapter we show that the same thing happens in the equivariant case.

For each complex A-representation V we form the A-space $\mathbb{C}P(V)$ of complex lines in V. If α is one-dimensional, we have a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C}P(V) \arrow[r] & \mathbb{C}P(V \oplus \alpha) \arrow[r] & S^{V \otimes \alpha^{-1}} \arrow[r] & 0 \end{tikzcd}$$

where the first map is the inclusion $[v_1 : \ldots : v_n] \mapsto [v_1 : \ldots : v_n : 0]$. Its cokernel is topologically a $\dim_{\mathbb{C}}(V)$-dimensional sphere where every point, except the basepoint, can be written as $[v_1 : \ldots : v_n : a] \sim [v_1/a : \ldots : v_n/a : 1]$ so we see that the group action on the sphere is the one given by $V \otimes \alpha^{-1}$.

The A-invariant complex lines are exactly the subrepresentations of V, so we have

$$\mathbb{C}P(V)^A = \amalg_\alpha \mathbb{C}P(V_\alpha)$$

where $V_\alpha = \mathrm{Hom}_A(\alpha, V)$ is the α-isotropical part of V. We fix the complete A-universe $\mathcal{U} = \bigoplus_{k \geq 0} \bigoplus_{\alpha \in A^*} \alpha$ and consider $\mathbb{C}P(\mathcal{U})$ with its topology as a colimit of its subspaces $\mathbb{C}P(V)$ with V finite-dimensional.

Now since $\mathbb{C}P(\mathcal{U})$ classifies line bundles, in the sense that every A-line bundle is a pullback of the tautological A-line bundle z over $\mathbb{C}P(\mathcal{U})$, we have that $\mathbb{C}P(\mathcal{U})$ is an abelian group object up to homotopy, and the inclusion of fixed points is a group homomorphism. Indeed, we can use the equivariant version of the multiplication map m defined in chapter 4.2, this defines a commutative (associative) product because the tensor product of line bundles is commutative (associative) up to homotopy. We can take $\mathbb{C}P(\epsilon)$ as the unit, as we can deduce from the diagram

$$\begin{tikzcd}
z \otimes \mathbb{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \pi_1^*(z) \otimes \pi_2^*(z) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & z \arrow[d] \\
\mathbb{C}P(\mathcal{U}) \times \{*\} \arrow[r, "\mathrm{Id} \times e"'] \arrow[rr, bend right, "\mathrm{Id}"'] & \mathbb{C}P(\mathcal{U}) \times \mathbb{C}P(\mathcal{U}) \arrow[r, "m"'] & \mathbb{C}P(\mathcal{U})
\end{tikzcd}$$

Note that we have an equivariant isomorphism

$$A^* \times \mathbb{C}P(\mathcal{U}_\epsilon) \cong \amalg_\alpha \mathbb{C}P(\mathcal{U}_\alpha)$$

given by $(\alpha, W) \mapsto \alpha \otimes W$, thus, since $\mathbb{C}P(\mathcal{U}_\alpha)$ is connected, there is a unique homotopy class of maps $A^* \to (\mathbb{C}P(\mathcal{U}))^A$ splitting the natural augmentation $(\mathbb{C}P(\mathcal{U}))^A \to A^*$ and it is a group homomorphism up to homotopy. In particular A^* acts on $\mathbb{C}P(\mathcal{U})$ through A-maps by $\alpha \cdot L = \alpha \otimes L$ and there is the group homomorphism $\mathbb{C}P(\mathcal{U}_\epsilon) \hookrightarrow \mathbb{C}P(\mathcal{U})$.

Now we introduce our cohomology theories. First, a genuine equivariant cohomology theory $E_A^*(-)$ is an exact contravariant functor on A-spaces, which admits a $RO(G)$-graded extension so that we have coherent suspension isomorphisms

$$\tilde{E}_A^{V+n}(S^V \wedge X) \cong \tilde{E}_A^n(X)$$

for all real representations V. We are interested in the following subfamily of them

Definition 6. *A genuine equivariant cohomology theory* $E_A^*(-)$ *is said to be complex stable if the stronger stability property*

$$\tilde{E}_A^{|V|+n}(S^V \wedge X) \cong \tilde{E}_A^n(X)$$

holds.

For any complex representation V, complex stability provides an element $\lambda(V) \in \tilde{E}_A^{|V|}(S^V)$ corresponding to the unit $1 \in \tilde{E}_A^0$ and $\tilde{E}_A^*(S^V)$ is a free E_A^*-module on this generator. Since complex stability isomorphisms are give by multiplication by $\lambda(V)$, we have $\lambda(V \oplus W) = \lambda(V)\lambda(W)$.

We define the Euler classes as $\chi(V) = e_V^*(\lambda(V)) \in E_A^{|V|}$, where $e_V : S^0 \to S^V$ is the inclusion, as a consequence $\chi(V \oplus W) = \chi(V)\chi(W)$.

Now we need a concept analogous to the orientation that we have used in the non-equivariant case in chapter 4.2.

Definition 7. $x(\epsilon) \in E^*_A(\mathbb{C}P(\mathcal{U}), \mathbb{C}P(\epsilon))$ *is an orientation if for all one-dimensional representations* $\alpha \in A^*$,

$$\mathit{res}^{\mathcal{U}}_{\epsilon\oplus\alpha} x(\epsilon) \in E^*_A(\mathbb{C}P(\epsilon \oplus \alpha), \mathbb{C}P(\epsilon)) \cong \tilde{E}^*_A(S^{\alpha^{-1}})$$

is a generator.

From this definition, we see that if $x(\epsilon)$ is an orientation then for all irreducible representations α, we have $\mathrm{res}^{\mathcal{U}}_{\epsilon\oplus\alpha} x(\epsilon) = u_{\alpha^{-1}}\lambda(\alpha^{-1})$ for some unit $u_{\alpha^{-1}}$. We use these units to define elements $e(\alpha^{-1}) = u_{\alpha^{-1}}\chi(\alpha^{-1}) \in E^*_A$.

We call $x(\alpha) \in E^*_A(\mathbb{C}P(\mathcal{U}), \mathbb{C}P(\alpha))$ the pullback of $x(\epsilon)$ through the map $l_\alpha := (\otimes\alpha^{-1})^* : (\mathbb{C}P(\mathcal{U}), \mathbb{C}P(\alpha)) \to (\mathbb{C}P(\mathcal{U}), \mathbb{C}P(\epsilon))$. Taking external products, we can define also $x(V) \in E^*_A(\mathbb{C}P(\mathcal{U}), \mathbb{C}P(V))$ and forgetting the subspace we obtain just $y(V) \in E^*_A(\mathbb{C}P(\mathcal{U}))$. By the short exact sequence

$$0 \longleftarrow E^*_A(\mathbb{C}P(W)) \longleftarrow E^*_A(\mathbb{C}P(V \oplus W)) \longleftarrow E^*_A(\mathbb{C}P(V \oplus W), \mathbb{C}P(V)) \longleftarrow 0$$

we see that x determines y and vice versa and from the properties of x we deduce that $y(0) = 1, y(V \oplus W) = y(V)y(W)$ and $(\alpha^{-1})^* y(V) = y(V \otimes \alpha)$.

By the commutativity of the diagram

$$\begin{tikzcd}
(\mathbb{C}P(\alpha \oplus \epsilon), \mathbb{C}P(\epsilon)) \arrow[r] & (\mathbb{C}P(\mathcal{U}), \mathbb{C}P(\epsilon)) \\
\mathbb{C}P(\alpha \oplus \epsilon) \arrow[u] & \mathbb{C}P(\mathcal{U}) \arrow[u] \\
\mathbb{C}P(\alpha) \arrow[u] \arrow[r, equal] & \mathbb{C}P(\alpha) \arrow[u]
\end{tikzcd}$$

we see that

$$\mathrm{res}^{\mathcal{U}}_{\alpha} y(\epsilon) = e(\alpha^{-1}) \in E^*_A(\mathbb{C}P(\alpha)) = E^*_A.$$

Theorem 9. *A complete flag* $F = (V^0 \subset V^1 \subset V^2 \subset \ldots)$ *for* $\mathcal{U}$ *specifies a basis of* $E^*_A(\mathbb{C}P(\mathcal{U}))$ *as follows*

$$E^*_A(\mathbb{C}P(\mathcal{U})) = E^*_A\{\{y(V^0) = 1, y(V^1), y(V^2), \ldots\}\}.$$

Proof. The cofiber sequence

$$(S^{\alpha^{-1}_{n+1}}) \quad = \quad (\mathbb{C}P(V^{n+1}), \mathbb{C}P(V^n)) \longrightarrow (\mathbb{C}P(\mathcal{U}), \mathbb{C}P(V^n)) \longrightarrow (\mathbb{C}P(\mathcal{U}), \mathbb{C}P(V^{n+1}))$$

splits in cohomology by $x(V^n)$. ■

Finally, we can state our result as a corollary.

Corollary 2. *A complex oriented cohomology theory* $E^*_A(-)$ *gives rise to an* A*-equivariant formal group law with*

- $k = E^*_A$

- $R = E^*_A(\mathbb{C}P(\mathcal{U}))$
- *the coproduct* $\Delta : R \longrightarrow R\hat{\otimes}R$ *induced by* $\otimes : \mathbb{C}P(\mathcal{U}) \times \mathbb{C}P(\mathcal{U}) \longrightarrow \mathbb{C}P(\mathcal{U})$
- *the map* $\theta : R \longrightarrow k^{A^*}$ *induced by* $A^* \longrightarrow \mathbb{C}P(\mathcal{U})$
- *the coordinate* $y(\epsilon)$ *obtained from the orientation* $x(\epsilon)$.

Idea of the Proof. The first two conditions of definition 4 descend from the Künneth theorem, together with some properties of $\mathbb{C}P(\mathcal{U})$ and the last from the particular basis. ■

6.2 EQUIVARIANT K-THEORY

As a concrete example, we see more in detail the equivariant formal group law arising from equivariant K-theory. This cohomology theory is complex stable in view of Bott periodicity, and we can work entirely in degree 0. We can view the projective space $\mathbb{C}P(V)$ as a quotient of the sphere $S(V \otimes z)$ and write

$$K_{A\times\mathbb{T}}(S(V \otimes z)) = K_A(\mathbb{C}P(V))$$

where z is the natural representation of the circle group $\mathbb{T}$. Now consider the based cofiber sequence $S(V)_+ \longrightarrow D(V)_+ \longrightarrow S^V$, the associate long exact sequence in cohomology has the form

$$\cdots \longleftarrow K_{A\times\mathbb{T}}(s(V \otimes z)) \longleftarrow R(A)[z, z^{-1}] \xleftarrow{K_{A\times\mathbb{T}}(j)} K_{A\times\mathbb{T}}(S^{V\otimes z}) \longleftarrow \cdots.$$

But since $D(V)$ is contractible, $K_{A\times\mathbb{T}}(j)$ is based homotopic to $\left(e^{V\otimes z}\right)^*$, therefore we have the following isomorphism of long exact sequences

$$\begin{tikzcd}
\cdots & K_{A\times\mathbb{T}}(s(V \otimes z)) \arrow[l] \arrow[d, equal] & R(A)[z, z^{-1}] \arrow[l] \arrow[d, equal] & K_{A\times\mathbb{T}}(S^{V\otimes z}) \arrow[l, "\left(e^{V\otimes z}\right)^*"'] & \cdots \arrow[l] \\
\cdots & K_{A\times\mathbb{T}}(s(V \otimes z)) \arrow[l] & R(A)[z, z^{-1}] \arrow[l] & R(A)[z, z^{-1}] \arrow[l, "\chi(V\otimes z)"] \arrow[u, "\cdot\lambda(V\otimes z)"'] & \cdots \arrow[l]
\end{tikzcd}$$

To understand who is $\chi(V \otimes z)$ we have to unravel the stability isomorphism.
By **[segal1966equivariant]** proposition 3.1 we are allowed to identify complex A-vector bundles over a locally compact A-space X and complexes of vector bundles over X that are non-exact only on a compact subset of X. Consider the trivial A-vector bundle $p : \alpha \longrightarrow \{*\}$, by **[segal1966equivariant]** proposition 3.2, the Thom isomorphism $K^*_A(\{*\}) \longrightarrow K^*_A(\alpha)$ maps an A-vector bundle over the point (that is just an A-representation) $F^\bullet$ to the tensor product $\Lambda^\bullet_\alpha \otimes p^*F^\bullet$ where $\Lambda^\bullet_\alpha$ is the Koszul complex on the A-space α that over the point $x \in \alpha$ is given by

$$\cdots \longrightarrow 0 \longrightarrow \epsilon \xrightarrow{\cdot x} \alpha \longrightarrow 0 \longrightarrow \cdots. * +$$

So we have that the unit of K^*_A is sent to $p^*(\epsilon^\bullet) \otimes \Lambda^\bullet_\alpha = \Lambda^\bullet_\alpha$ as a complex, that corresponds to $1 - \alpha$ as a vector bundle over S^α. In this way we see that $\lambda(\alpha) = 1 - \alpha$ and pulling back with respect to the inclusion $e^{V\otimes z}$ we obtain the bundle $1-\alpha$ over the point, so, finally, we have that $\chi(\alpha)=1-\alpha$ and $\chi(V\otimes z)=\prod_i(1-\alpha_i z)$ where $V=\bigoplus_i \alpha_i$. In particular, since the multiplication by $\chi(V\otimes z)$ is injective, we can turn the previous long exact sequence in a short exact sequence and deduce that

$$K_A(\mathbb{C}P(V)) = K_{A\times\mathbb{T}}(S(V\otimes z)) = R(A)[z,z^{-1}]/\chi(V\otimes z) = R(A)[z]/\chi(V\otimes z)$$

where the last equality follows from the fact that z is already invertible in $R(A)[z]/\chi(V\otimes z)$, indeed

$$1-\chi(V\otimes z) = z\cdot(V+\text{higher terms})$$

in particular, we see that the groups $K_A(\mathbb{C}P(V\oplus V\oplus\ldots\oplus V))$ satisfy the Mittag-Leffler condition, thus we can pass to the inverse limit and obtain that

$$K_A(\mathbb{C}P(\infty V)) = R(A)[z]^{\wedge}_{(\chi(V\otimes z))}.$$

We may take as orientation $y=1-z$, indeed this expression makes sense as an element of $K_A(\mathbb{C}P(V))$ for any V, and $1-z$ visibly generates the kernel of

$$K_A(\mathbb{C}P(\epsilon\oplus\alpha)) = R(A)[z]/(1-z)(1-\alpha z)\longrightarrow R(A)[z]/(1-z) = K_A(\mathbb{C}P(\epsilon)).$$

The element z, regarded as an element of $K_A(\mathbb{C}P(V))$, is the tautological line bundle over $\mathbb{C}P(V)$. From the diagram

$$\begin{tikzcd}
z\otimes\alpha \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & z \arrow[d] \\
\mathbb{C}P(\mathcal{U}) \arrow[r, "\otimes\alpha^{-1}"'] & \mathbb{C}P(\mathcal{U})
\end{tikzcd}$$

we can identify the A^* action on $K_A(\mathbb{C}P(V))$ as $l_\alpha z=\alpha z$, so $y(\alpha)=l_\alpha(1-z)=1-\alpha z$. Now assume that A is finite and let V be the regular representation. Define $\Pi=\chi(V\otimes\alpha)$, then the inclusion $j: A^*\times\mathbb{C}P(\mathcal{U}_\epsilon)\to\mathbb{C}P(\mathcal{U})$ induces a map

$$j^*: R(A)[z]^{\wedge}_{\Pi}\longrightarrow\prod_\alpha R(A)[z]^{\wedge}_{(1-\alpha z)}$$

whose αth component is induced by completing the identity map of $R(A)[z]$ with respect to Π in the domain and $(1-\alpha z)$ in the codomain, as is legitimate since $(1-\alpha z)$ divides Π. Note that if we invert the Euler classes $e(\alpha)$ then we have that $y(\alpha)-\alpha\beta^{-1}y(\beta)$ becomes invertible, so all the ideals $(1-\alpha z)$ are coprime and we can apply the Chinese Remainder Theorem to conclude that j^* is an isomorphism. On the other hand, working modulo the Euler classes, we have $1-\alpha z=1-z$ so that j^* is just the diagonal inclusion.

6.3 EXAMPLES AND COMPUTATIONS

In this section we see some examples of equivariant formal group laws, first, we prove the following lemma.

Lemma 5. $e(\alpha\beta)=e(\alpha)+e(\beta)+e(\alpha)e(\beta)\left(\sum_{i,j\geqslant 1} f^1_{i,j}e(\alpha V^i/\alpha)e(\beta V^j/\beta)\right)$.

Proof.

$$\begin{aligned}
e(\alpha\beta) &= \theta_{\alpha^{-1}\beta^{-1}}(y(\epsilon)) \\
&= \Delta \circ \theta_{\alpha^{-1}\beta^{-1}}(y(\epsilon)) \\
&= (\theta_{\alpha^{-1}}\hat{\otimes}\theta_{\beta^{-1}}) \circ \Delta(y(\epsilon)) \\
&= \sum_{i,j} f^1_{j,i}\theta_{\alpha^{-1}}(y(V^j))\theta_{\beta^{-1}}(y(V^i)) \\
&= \sum_{i,j} f^1_{j,i}e(\alpha V^j))e(\beta V^i) \\
&= e(\alpha) + e(\beta) + e(\alpha)e(\beta)\left(\sum_{i,j\geqslant 1} f^1_{i,j}e(\alpha V^i/\alpha)e(\beta V^j/\beta)\right).
\end{aligned}$$

■

6.3.1 THE ADDITIVE EQUIVARIANT FGL

In the equivariant setting, the additive formal group law is one for which

$$\Delta_a(y(\epsilon)) = y(\epsilon)\hat{\otimes}1 + 1\hat{\otimes}y(\epsilon).$$

The difference from the non-equivariant case is that this form of the coproduct now has more implications, for examples by the lemma we have just proven, we must have $e(\alpha\beta) = e(\alpha) + e(\beta)$. This is an important observation because it means that it is improbable for an equivariant formal group law to be isomorphic to the additive one. Indeed, if A is finite, say of order n, then in the additive equivariant formal group law we have

$$0 = e(\epsilon) = e(\alpha^n) = ne(\alpha).$$

Thus, an additive equivariant formal group law over the rationals has zero Euler classes, which means, by lemma 4, that $l_\alpha = \mathrm{id}_R$ for all α, hence the A^*-action is trivial. Indeed, since we have $f^1_{1,0} = f^1_{0,1} = 1$ and all the others $f^1_{j,k} = 0$, we obtain $d(\beta)^1_0 = e(\beta)$, $d(\beta)^1_1 = 1$ and $d(\beta)^1_k = 0$ for $k \geqslant 2$. In other words $y(\beta) = e(\beta) + y(\epsilon)$.

6.3.2 THE MULTIPLICATIVE EQUIVARIANT FGL

As in the non-equivariant case, we define a multiplicative equivariant formal group law as one for which

$$\Delta_m(y(\epsilon)) = y(\epsilon)\hat{\otimes}1 + 1\hat{\otimes}y(\epsilon) - y(\epsilon)\hat{\otimes}y(\epsilon)$$

or equivalently, $f^1_{1,0} = f^1_{0,1} = 1$, $f^1_{1,1} = -1$ and all the others $f^1_{j,k} = 0$. We have seen a concrete example of such an equivariant formal group law in section 6.2. In this case, by lemma 4 we obtain the relations

$$y(\beta) = e(\beta) + (1 - e(\beta))y(\epsilon)$$

and by applying $\theta_\epsilon \circ l_\alpha$

$$1 - e(\alpha\beta) = (1 - e(\alpha))(1 - e(\beta)).$$

6.3.3 The group of order 2

If we consider the example in which A has order 2 we can perform some more explicit calculations to understand exactly who is R.
First we fix the complete flag $(0 \subset \epsilon \subset \epsilon \oplus \alpha \subset \epsilon \oplus \alpha \oplus \epsilon \subset \epsilon \oplus \alpha \oplus \epsilon \oplus \alpha \subset ...)$ and we call $y = y(\epsilon)$ and $x = y(\epsilon)y(\alpha)$. We know that a topological basis of R is given by $1, y, x, yx, x^2, yx^2, x^3, ...$
If we apply l_α to the equation

$$y(\alpha) = d(\alpha)^1_0 + d(\alpha)^1_1 y(\epsilon) + d(\alpha)^1_2 y(\epsilon)y(\alpha) + ...$$

we obtain

$$y(\epsilon) = d(\alpha)^1_0 + d(\alpha)^1_1 y(\alpha) + d(\alpha)^1_2 y(\alpha)y(\epsilon) + ...$$

and multiplying by $y(\epsilon)$

$$y(\epsilon)^2 = d(\alpha)^1_0 y(\epsilon) + d(\alpha)^1_1 y(\epsilon)y(\alpha) + d(\alpha)^1_2 y(\epsilon)y(\alpha)y(\epsilon) + ...$$

that is

$$\begin{aligned} y^2 &= d(\alpha)^1_0 y + d(\alpha)^1_1 x + d(\alpha)^1_2 yx + d(\alpha)^1_3 x^2 + ... \\ &= y(d(\alpha)^1_0 + d(\alpha)^1_2 x + d(\alpha)^1_4 x^2 + ...) + x(d(\alpha)^1_1 + d(\alpha)^1_3 x + d(\alpha)^1_5 x^2 + ...) \\ &=: yp(x) + xq(x). \end{aligned}$$

So in this case, we have

$$R = k[[x]][y]/(y^2 = yp(x) + xq(x)).$$

In particular, it is not just a power series ring, this is another difference from the non-equivariant case.

6.3.4 The Euler-invertible case

We now discuss a situation in which we can describe our equivariant formal group law in terms of a non-equivariant one very easily.
When the Euler classes are invertible, the ideals of R generated by $y(\alpha)$ for $\alpha \in A^*$ are all coprime, simply because, by lemma 4, we have $y(\alpha) = e(\alpha\beta^{-1}) + ry(\beta)$ for some $r \in R$. By the Chinese Reminder Theorem, we deduce that the natural map

$$R \longrightarrow \prod_{\alpha \in A^*} R^{\wedge}_{(y(\alpha))}$$

is an isomorphism, furthermore, by completing the map $l_{\alpha^{-1}\beta} : R \to R$ with respect to $(y(\alpha))$ in the domain and $(y(\beta))$ in the codomain, we obtain an isomorphism $\hat{l}_{\alpha^{-1}\beta} : R^{\wedge}_{(y(\alpha))} \to R^{\wedge}_{(y(\beta))}$ so that R splits as a product of power series rings and A acts simply by permuting them isomorphically. Each of these power series rings is a non-equivariant formal group law, indeed on the ϵth component we can use the image of $y(\epsilon)$ as a coordinate and the split surjection $R \to R^{\wedge}_{(y(\epsilon))}$ to define the coproduct and the counit. They have the desired proprieties in view of the commutative diagram

$$\begin{tikzcd}
R \arrow[r, "\Delta"] \arrow[rr, bend left, "\mathrm{Id}_R"] \arrow[d, two heads] & R\hat{\otimes}R \arrow[r, "R\hat{\otimes}\theta_\epsilon"] \arrow[d, two heads] & R \arrow[d, two heads] \\
R^{\wedge}_{(y(\epsilon))} \arrow[r, "\tilde{\Delta}"'] \arrow[rr, bend right, "\mathrm{Id}_{R^{\wedge}_{(y(\epsilon))}}"'] \arrow[u, bend left] & R^{\wedge}_{(y(\epsilon))}\hat{\otimes}R^{\wedge}_{(y(\epsilon))} \arrow[r, "R^{\wedge}_{(y(\epsilon))}\hat{\otimes}\tilde{\theta}"'] \arrow[u, bend left] & R^{\wedge}_{(y(\epsilon))} \arrow[u, bend left]
\end{tikzcd}$$

The other components are isomorphic to the ϵth one so they have the formal group law structure induced by the isomorphism.

Notice that the isomorphism $\hat{\iota}_{\alpha^{-1}\beta} : R^{\wedge}_{(y(\alpha))} \to R^{\wedge}_{(y(\beta))}$ maps the coordinate $y(\alpha)$ of $R^{\wedge}_{(y(\alpha))}$ to the coordinate $y(\beta)$ of $R^{\wedge}_{(y(\beta))}$, as a consequence, the coproduct in all these non-equivariant formal group laws have the same formal expression.

Summarizing, we have proved the following localization theorem

Theorem 10. *If* R *is an* A*-equivariant formal group law with all the Euler classes (except* $e(\epsilon)$*) invertible, then* R *splits as a product of isomorphic non-equivariant formal group laws, one for each element of* A^**, and* A^* *acts by permuting these copies.*

7

Building blocks & isomorphisms of equivariant FGLs

In this chapter, we will prove a weaker version of theorem 6 for the A-equivariant case when A is the group with two elements. Indeed, by section 6.3.1 we know that, over the rationals, an A-equivariant formal group law with non-trivial A^* action cannot be isomorphic to the additive A-equivariant formal group law. In this section, A will always be cyclic of order a prime p, and all the formal group laws are rationalized (every integer is invertible in their ground rings).

Lemma 6. *Let A be a cyclic group of order a prime p and let R be an A-equivariant formal group law. If one Euler class $e(\alpha)$ of R is invertible, then all the Euler classes of R, except $e(\epsilon)$, are invertible.*

Proof. Let $\beta \neq \epsilon$, since both α and β are generators of A we can write $\alpha = \beta^i$ and $\beta = \alpha^j$ for some i and j. Now, by applying lemma 5, we find that $e(\alpha) = r \cdot e(\beta)$ and $e(\beta) = s \cdot e(\alpha)$ for some $r, s \in k$. In particular $e(\alpha) = r \cdot s \cdot e(\alpha)$ but since $e(\alpha)$ is invertible, we see that r and s are units of the ground ring, therefore $e(\beta)$ is the product of two invertible. ■

Lemma 7. *If A is finite of prime order p, then, over the rationals, every equivariant formal group law decomposes as a direct product of one with invertible Euler classes and one with zero Euler classes.*

Proof. Let α be a generator of A^*, by lemma 5, $0 = e(\alpha^p) = pe(\alpha) + e(\alpha)^2 r$ for some $r \in k$. Thus, if we define $f = -e(\alpha)r/p$ we have $f^2 = f$ and $(1-f)^2 = 1-f$. Then, we can use these idempotents to obtain the decomposition

$$\begin{aligned} R &\cong f \cdot R \times (1-f) \cdot R \\ &\cong (f \cdot k)\{\{y(V^0), y(V^1), ...\}\} \times ((1-f) \cdot k)\{\{y(V^0), y(V^1), ...\}\} \\ &\cong (k/(1-f))\{\{y(V^0), y(V^1), ...\}\} \times (k/(f))\{\{y(V^0), y(V^1), ...\}\}. \end{aligned}$$

All the operations of equivariant formal group law factor through this decomposition, giving both components the structure of equivariant formal group law.
Because $f \cdot e(\alpha) = e(\alpha)$, the image of $e(\alpha)$ in the first component is just $e(\alpha)$ that, in the first component, is invertible since f is. On the other hand, $(1-f) \cdot e(\alpha) = 0$, therefore, its image in the second component vanishes. Now by lemma 7 we can conclude that all the Euler classes, except $e(\epsilon)$, are invertible in the first component. In the second component, by lemma 4, we have that l_α acts trivially, as a consequence also l_{α^i} acts trivially, in particular $0 = d(\alpha^i)_0^1 = e(\alpha^i)$. ■

When A is cyclic of order p we will refer to the components of this decomposition of an A-equivariant formal group law R by R_1 and R_2 respectively, and we will indicate with f_R the idempotent described in the proof that allows this decomposition. Note that R_1 and R_2 may be zero.

Remark 3. *By putting together lemma 7 and the subsection 6.3.4, we see that, for any* A-*equivariant formal group law* R *with* A *of prime order, the component* R_2 *is isomorphic to* $|A|$ *copies of a non-equivariant formal group law* $(R_2)^{\wedge}_{(y(\epsilon))}$ *and* A^* *acts through isomorphisms simply by permuting the coordinates of these copies. These non-equivariant formal group laws are all isomorphic to the additive one by theorem 6. However, since the different copies are isomorphic through the isomorphisms* $\hat{l}_{\alpha^{-1}\beta} : R^{\wedge}_{(y(\alpha))} \to R^{\wedge}_{(y(\beta))}$ *that just map the coordinate of one copy to the coordinate of another copy, we can choose the same logarithm for all the copies. As a consequence,* R_2 *is isomorphic to* $|A|$ *copies of the additive non-equivariant formal group law, and* A^* *still acts by permuting the coordinates of these additive formal group laws.*

Corollary 3. *Let* A *be a cyclic group of order* p *and let* R *and* S *be two* A-*equivariant formal group laws over the same ground ring. Then, as a consequence of the previous remark, if* $f_R = f_S$ *then* R *and* S *are isomorphic as equivariant formal group laws.*

We now prove the following lemma, which, at this point, is the last missing piece. The proof involves a lot of straightforward but long and tedious verifications. At each step, we mention what we have to check and why, but we omit some of the non-exciting calculations.

Lemma 8. *Let* A *be the group with two elements. Given any ground ring* k *and any* $e, v \in k$ *such that* $2e + ve^2 = 0$, *there is an* A-*equivariant* v-*multiplicative formal group law, that is one with coproduct* $\Delta(y(\epsilon)) = y(\epsilon)\hat{\otimes}1 + 1\hat{\otimes}y(\epsilon) + vy(\epsilon)\hat{\otimes}y(\epsilon)$, *over that ground ring with Euler class* $e(\alpha) = e$.

Proof. First fix the complete flag $F = (0 \subset \epsilon \subset \epsilon \oplus \alpha \subset \epsilon \oplus \alpha \oplus \epsilon \subset \ldots)$ and define R as the k-module $k\{\{F\}\}$. If we want an A-equivariant formal group law structure on R, then, by [4] appendix C, we must have that the structure constants relative to the product are

$$b^{i,j}_k = \delta^{i+j}_k + e(\alpha^{ij})f^1_{1,k-i-j+1} = \delta^{i+j}_k(ve(\alpha^{ij}) + 1) + \delta^{i+j}_{k+1}e(\alpha^{ij}).$$

By using them to define our product, we automatically obtain a commutative unital and associative product on R. Again by [4] we must have the following structure constants relative to the A-action

$$d(\alpha)^n_k = \delta^n_k + e(\alpha^n)f^1_{1,k-n+1} = \delta^n_k(1 + ve(\alpha^n)) + \delta^n_{k+1}e(\alpha^n).$$

In particular, $d(\alpha)^1_0 = e(\alpha)$, $d(\alpha)^1_1 = 1 + ve(\alpha)$ and $d(\alpha)^1_i = 0$ for $i \geqslant 2$.
By the discussion in section 2, we can say that $R = k[[x]][y]/(y^2 = (1 + ve)x + ey)$ where we have defined $y = y(\epsilon)$, $x = y(\epsilon)y(\alpha)$ and $e = e(\alpha)$. With this description, we see that the coefficients $d(\alpha)^i_j$ we have defined, are specifying the map $l_\alpha : R \to R$ given by $y \mapsto e + (1 + ve)y$, $x \mapsto x$ and extended in order to respect the product: $x^n \mapsto x^n$ and

$x^n y \mapsto x^n(e + (1 + \nu e)y)$. This map extends to a ring homomorphism $R \longrightarrow R$, indeed, the equation $2e + \nu e^2 = 0$ ensures that $l_\alpha(y)^2 = l_\alpha((1 + \nu e)x + ey)$ so that the map respects the relation of R. From the explicit definition, it is easy to verify that $l_\alpha \circ l_\alpha = \mathrm{id}$ so that $l : A^* \longrightarrow \mathrm{End}(R)$ defines an action of A^* on R trough ring homomorphisms. This action is clearly associative and unital.

We define the coproduct on the generators of R by

$$\Delta(y) = y\hat{\otimes}1 + 1\hat{\otimes}y + \nu y\hat{\otimes}y$$

$$\Delta(x) = (y\hat{\otimes}y)(2 + e\nu) + 1\hat{\otimes}x + x\hat{\otimes}1 + (x\hat{\otimes}x + x\hat{\otimes}y)(2\nu + e\nu^2) + (x\hat{\otimes}x)(\nu^2 + \nu^3 e).$$

We can check that $(\Delta(y))^2 = e(y\hat{\otimes}1 + 1\hat{\otimes}y) + (2 + 2\nu e)(y\hat{\otimes}y) + (1 + \nu e)(x\hat{\otimes}1 + 1\hat{\otimes}x) + \nu^2(x\hat{\otimes}x) + (2\nu + e\nu^2)(x\hat{\otimes}y + y\hat{\otimes}x) = \Delta((1 + \nu e)x + ey)$, so that our Δ defined on the generators can be extended to a ring homomorphism $\Delta : R \longrightarrow R\hat{\otimes}R$. This coproduct is clearly commutative and counital, with counit $\theta_\epsilon : R \longrightarrow k$ defined by $\theta_\epsilon(x) = \theta_\epsilon(y) = 0$, indeed from the definitions it is apparent that the equalities $(\theta_\epsilon\hat{\otimes}\mathrm{id}) \circ \Delta(x) = x$, $(\theta_\epsilon\hat{\otimes}\mathrm{id}) \circ \Delta(y) = y$ and the symmetrical ones hold.

For the associativity, it follows from the definitions that $(\Delta\hat{\otimes}\mathrm{id}) \circ \Delta(x) = (\mathrm{id}\hat{\otimes}\Delta) \circ \Delta(x)$ and that $(\Delta\hat{\otimes}\mathrm{id}) \circ \Delta(y) = (\mathrm{id}\hat{\otimes}\Delta) \circ \Delta(y)$ and for the equivariance, again with the definition it is easy to check that the operators $(l_\alpha\hat{\otimes}\mathrm{id}) \circ \Delta$, $(\mathrm{id}\hat{\otimes}l_\alpha) \circ \Delta$ and $\Delta \circ l_a$ agree on the generators. The last two axioms of definition 5 are clear by construction, therefore we have verified all the five axioms of definition 5.

Finally, we have to make sure that our operators are continuous, so we can use the continuity conditions. Using the explicit description of the structure constants $d(\alpha)^i_j$ and $b^{i,j}_k$ that we have that the continuity conditions for the action and the product are obvious. Understanding the symbols $f^i_{i,j}$ is a bit more complicated, but having already verified that the five axioms of definition 5 hold, we can apply the proof of proposition 14.1 of [4] that gives $f^i_{s,t} = 0$ if $V^i \geqslant V^{s+1} \otimes V^{t+1}$. Hence, we also have the continuity of the coproduct and the proof is complete.

■

Putting all the pieces together, we can now prove the final theorem.

Theorem 11. *Let* A *be the group with two elements, then any* A*-equivariant formal group law is rationally isomorphic to the* ν*-multiplicative one for some* ν *in its ground ring.*

Proof. By lemma 5, we know that in any A-equivariant formal group law R, the relation $2e(\alpha) + e(\alpha)^2 f^1_{1,1} = 0$ holds, therefore if we put $\nu = f^1_{1,1}$, by lemma 8, we can find an A-equivariant ν-multiplicative formal group law M with the same ground ring as R and with the same Euler class $e_M = e(\alpha)_R$. Furthermore, the idempotents f_R and f_M of the two must agree, because $f_R = -e(\alpha)_R f^1_{1,1}/2 = -e_M\nu/2 = f_M$, thus, we can apply corollary 3. ■

A

A DIFFERENT PROOF OF A WEAKER VERSION OF THEOREM 11

Lemma 8 is essential for our purpose, however its proof is not that satisfying. There are particular cases of the lemma in which we do not need to construct a multiplicative formal group law by hand. For example, if we start with an A-equivariant formal group law R for which the structure constant $f^1_{1,1}$ is -1, then we can use the A-equivariant multiplicative formal group law M arising from equivariant K-theory discussed in section 6.2.

In this appendix, we briefly sketch how the calculation goes in this particular case.

Once again, let A be the group of order 2 and fix the complete flag $F = (0 \subset \epsilon \subset \epsilon \oplus \alpha \subset \epsilon \oplus \alpha \oplus \epsilon \subset \ldots)$.

Consider an equivariant formal group law R with $\Delta_R(y(\epsilon)_R) = \sum_{j,k}(f^1_{j,k})_R y(V^j)_R \hat{\otimes} y(V^k)_R$ for which $(f^1_{1,1})_R = -1$.

By lemma 5, we have

$$2e(\alpha)_R = e(\alpha)_R^2 \Rightarrow 1 = (1 - e(\alpha)_R)^2. \tag{A.1}$$

Now there are two different cases, namely

1. $1 - e(\alpha)_R \in \mathbb{Q} \subseteq k$
2. $1 - e(\alpha)_R \in k - \mathbb{Q}$.

(We are assuming $\mathbb{Q} \subseteq k$ because we are working rationally). Consider the multiplicative A-equivariant formal group law M arising from A-equivariant K-theory. Then we have the decomposition of lemma 7 that in this case gives

$$\mathbb{Q} \otimes R(A)\{\{F\}\} \cong (\mathbb{Q} \otimes R(A)/(\alpha = 1))\{\{F\}\} \times (\mathbb{Q} \otimes R(A)/(\alpha = -1))\{\{F\}\}.$$

Indeed, in equivariant K-theory, $e(\alpha)_M = 1 - \alpha$, therefore, the idempotent element f_M of lemma 7 is $(1 - \alpha)/2$, and $1 - f_M$ is $(1 + \alpha)/2$.

In the case 2, we have an isomorphism of ground rings

$$\begin{aligned} k \to {} & (k \otimes \mathbb{Q} \otimes R(A)) / ((1 - e(\alpha)_R) \otimes 1 \otimes 1 = 1 \otimes 1 \otimes \alpha) \\ = {} & \big(k \otimes (\mathbb{Q}[\alpha]/(\alpha^2 = 1))\big) / ((1 - e(\alpha)_R) \otimes 1 = 1 \otimes \alpha). \end{aligned}$$

Basically we are tensoring k and $\mathbb{Q} \otimes R(A)$ over the two isomorphic copies of the $\mathbb{Q}$-linear subspace $\mathrm{span}_{\mathbb{Q}}(1, 1 - e(\alpha)_R)$ in k and $\mathrm{span}_{\mathbb{Q}}(1, \alpha)$ in $\mathbb{Q} \otimes R(A)$. These subspaces are isomorphic also as subrings since both $1 - e(\alpha)_R$ and α square to 1, hence the operation is legitimate.

Note that this isomorphism of ground rings maps $1 - e(\alpha)_R$ to α and thus, also the idempotent $f_R = -(f^1_{1,1})_R e(\alpha)_R/2 = e(\alpha)_R/2$ is mapped to $e(\alpha)_M/2 = -(f^1_{1,1})_M e(\alpha)_M/2 = f_M$. From here we can conclude by applying lemma 3 as before.

In the case 1, that is if $e(\alpha)_R \in \mathbb{Q}$, the only possibilities to satisfy A.1 are that $e(\alpha)_R = 0$ or $e(\alpha)_R = 2$. If $e(\alpha)_R = 0$ then the action of A^* is trivial thus, by theorem 6, R is isomorphic to the non-equivariant multiplicative formal group law (which is a particular case of the equivariant multiplicative formal group law).

If $e(\alpha)_R = 2$, then we can quotient the equivariant multiplicative formal group law M arising from equivariant K-theory by adding the relation $\alpha = -1$ in the ground ring and proceed as before. Just notice that this quotient is still an equivariant multiplicative formal group law, indeed it is just the second block M_2 of the decomposition of M with respect to lemma 7.

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