Asymptotic Fermat for signatures \((p, p, 2)\) and \((p, p, 3)\) over totally real fields

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Abstract

Let \(K\) be a totally real number field and consider a Fermat-type equation \(Aa^p + Bb^q = Cc^r\) over \(K\). We call the triple of exponents \((p, q, r)\) the signature of the equation. We prove various results concerning the solutions to the Fermat equation with signature \((p, p, 2)\) and \((p, p, 3)\) using a method involving modularity, level lowering and image of inertia comparison. These generalize and extend the recent work of Işik, Kara and Ozman. For example, consider \(K\) a totally real field of degree \(n\) with \(2 \nmid h_K\) and 2 inert. Moreover, suppose there is a prime \(q \geq 5\) which totally ramifies in \(K\) and satisfies \(\gcd(n, q - 1) = 1\), then we know that the equation \(a^p + b^q = c^r\) has no primitive, non-trivial solutions \((a, b, c) \in \mathcal{O}_K^3\) with \(2 | b\) for \(p\) sufficiently large.

1 Introduction

1.1 Historical background

The study of Diophantine equations is of great interest in Mathematics and goes back to antiquity. The most famous example of a Diophantine equation appears in Fermat’s Last Theorem. This is the statement, asserted by Fermat in 1637 without proof, that the Diophantine equation \(a^n + b^n = c^n\) has no solutions in whole numbers when \(n\) is at least 3, other than the trivial solutions which arise when \(abc = 0\). Andrew Wiles famously proved the Fermat’s Last Theorem in 1995 in his paper "Modular elliptic curves and Fermat’s Last Theorem" [34]. The proof is by contradiction employing techniques from algebraic geometry and number theory to prove a special case of the modularity theorem for elliptic curves, which together with Ribet’s level lowering theorem gives the long-waited result. Since then, number theorists extensively studied Diophantine equations using Wiles’ modularity approach. Siksek gives a comprehensive survey about this method over the field of rationals in [25].

Even before Wiles announced his proof, various generalizations of Fermat’s Last Theorem had already been considered, which are of the shape

\[Aa^p + Bb^q = Cc^r\]  \hspace{1cm} (1)

for fixed integers \(A, B\) and \(C\). We call \((p, q, r)\) the signature of the equation \((1)\). A primitive solution \((a, b, c)\) is a solution where \(a, b\) and \(c\) are pairwise coprime and a non-trivial solution \((a, b, c)\) is a solution where \(abc \neq 0\).

In [16], Işik, Kara and Ozman list all known cases where equation \((1)\) has been solved over the rational integers in two tables (p.4). Table 1 contains...
all unconditional results for infinitely many primes. In Table 2, they give all conditional results. We highlight here one relevant family of solutions, namely \((n,n,k)\) where \(k \in \{2,3\}\). Darmon and Merel \cite{6} and Poonen \cite{19} proved the following theorem:

**Theorem 1** (Darmon and Merel).  
(i) The equation \(a^n + b^n = c^2\) has no non-trivial primitive integer solutions for \(n \geq 4\).

(ii) The equation \(a^n + b^n = c^3\) has no non-trivial primitive integer solutions for \(n \geq 3\).

Note that the above equations, typically have infinitely many non-primitive solutions. For example, if \(n\) is odd, and \(a\) and \(b\) are any two integers with \(a^n + b^n = c^2\), then \((ac)^n + (bc)^n = (c^{n+1})^2\) giving a rather uninteresting supply of solutions. Thus, we would only study the primitive solutions of the above equations.

A naive sketch of the proof of Theorem 1 is as follows. First note that it is enough to prove the assumption for \(n = p\) an odd prime. Suppose \(a, b, c \in \mathbb{Z}\) is a non-trivial, primitive solution to (i) or (ii). In each of the cases, we can associate a so-called Frey elliptic curve \(E_{a,b,c}/\mathbb{Q}\) and let \(\rho_{E,p}\) be its mod \(p\) Galois representation, where \(E = E_{a,b,c}\). Then \(\rho_{E,p}\) is irreducible by Mazur \cite{18} and modular by Wiles and Taylor \cite{34} and \cite{29}. Applying Ribet’s level lowering theorem \cite{21} one gets that that \(\rho_{E,p}\) arises from a weight 2 newform of level 32 for (i) and level 27 for (ii). These are closely related to the modular curves \(X_0(32)\) and \(X_0(27)\) which turn out to be elliptic curves with complex multiplication. Darmon and Merel prove in \cite{6}, by using the theory of complex multiplication that this implies \(j_E \in \mathbb{Z}[\frac{1}{p}]\) for \(p > 7\), which gives a contradiction. The cases when \(p \leq 7\) are treated in a more elementary way by Poonen \cite{19}.

Recently, important progress has been done towards generalisation of the modularity approach over larger number fields. In \cite{12} Freitas and Siksek proved the asymptotic Fermat’s Last Theorem (FLT) for certain totally real fields \(K\). That is, they showed that there is a constant \(B_K\) such that for any prime \(p > B_K\), the only solutions to the Fermat equation \(a^p + b^p + c^p = 0\) where \(a, b, c \in \mathcal{O}_K\) are the trivial ones i.e. the ones satisfying \(abc = 0\). Then, Deconinck \cite{7} extended the results of Freitas and Siksek \cite{12} to the generalized Fermat equation of the form \(Aa^p + Bb^p + Cc^p = 0\) where \(A, B, C\) are odd integers belonging to a totally real field. Later in \cite{24} Şengün and Siksek proved the asymptotic FLT for any number field \(K\) by assuming modularity. This result has been generalized by Kara and Ozman in \cite{15} to the case of the generalized Fermat equation. Also, recently in \cite{39} and \cite{41} Turcas studied Fermat equation over imaginary quadratic field \(\mathbb{Q}(\sqrt{-d})\) with class number one.

We now present a result by İşik, Kara and Ozman, proved in \cite{10} which serves as the starting point of this paper. It gives a computable criteria of testing if the asymptotic Fermat Last Theorem holds for certain type of solutions of the equations with signatures \((p,p,2)\). To state it, we need the following notation:

\[
S_K := \{ \mathfrak{p} : \mathfrak{p} \text{ is a prime of } K \text{ above } 2 \}, \\
T_K := \{ \mathfrak{p} \in S_K : f(\mathfrak{p}/2) = 1 \}, \\
W_K := \{(a,b,c) \in \mathcal{O}_K^3 : a^p + b^p = c^2 \text{ with } \mathfrak{p} \mid b \text{ for every } \mathfrak{p} \in T_K \};
\]

where \(f(\mathfrak{p}/2)\) denotes the residual degree of \(\mathfrak{p}\).
Theorem 2 (Işik, Kara and Ozman). Let $K$ be a totally real number field with narrow class number $h_K^+ = 1$. For each $a \in K(S_K, 2)$, let $L = K(\sqrt{a})$.

(A): Suppose that for every solution $(\lambda, \mu)$ to the $S_K$-unit equation
\[
\lambda + \mu = 1, \lambda, \mu \in \mathcal{O}_{S_K}^*
\]
there is some $\Psi \in T_K$ that satisfies $\max\{|v_\Psi(\lambda)|, |v_\Psi(\mu)|\} \leq 4v_\Psi(2)$.

(B): Suppose also that for each $L$, for every solution $(\lambda, \mu)$ of the $S_L$-unit equation $\lambda + \mu = 1, \lambda, \mu \in \mathcal{O}_{S_L}^*$, there is some $\Psi' \in T_L$ that satisfies $\max\{|v_{\Psi'}(\lambda)|, |v_{\Psi'}(\mu)|\} \leq 4v_{\Psi'}(2)$.

Then, there is a constant $B_K$ (depending only on $K$) such that for each $p > B_K$, the equation $a^p + b^p = c^2$ has no primitive, non-trivial solutions with $(a, b, c) \in W_K$ (i.e. the asymptotic Fermat holds for $W_K$).

1.2 Our results

We start by using the methods pioneered by Freitas and Siksek in [12] involving modularity, level lowering and image of inertia comparison to generalize Işik, Kara and Ozman’s Theorem 2. More precisely, we relax the assumption on the class group from $h_K^+ = 1$ to $Cl_{S_K}(K)[2] = \{1\}$. We use $Cl_S(K)$ to mean $Cl(K)/(\mathcal{P})|_{\mathcal{P} \in S}$ for $S$ a finite set of primes of $K$ and consequently, $Cl_S(K)[n]$ denotes its $n$-torsion points. Note that when all $\mathcal{P} \in S$ are principal, $Cl_S(K)$ is the usual $Cl(K)$, and hence we will drop the $S$ in the notation. Moreover, in this case, $Cl(K)[p] = \{1\}$ is equivalent to $p \mid h_K$, for $p$ prime.

Our main theorem regarding the Asymptotic Fermat Last Theorem for signature $(p, p, 2)$ reads as follows:

Theorem 3 (Main Theorem for $(p, p, 2)$). Let $K$ be a totally real number field with $Cl_{S_K}(K)[2] = \{1\}$ where $S_K = \{\mathcal{P} : \mathcal{P}$ is a prime of $K$ above 2). Suppose that there exists some distinguished prime $\mathcal{P} \in S_K$, such that every solution $(\alpha, \beta, \gamma) \in \mathcal{O}_{S_K} \times \mathcal{O}_{S_K}^* \times \mathcal{O}_{S_K}^*$ to the equation
\[
\alpha + \beta = \gamma^2
\]
that satisfies $|v_{\mathcal{P}}(\frac{\alpha}{\beta})| \leq 6v_{\mathcal{P}}(2)$. Then, there is a constant $B_K$ (depending only on $K$) such that for each rational prime $p > B_K$, the equation $a^p + b^p = c^2$ has no primitive, non-trivial solutions $(a, b, c) \in \mathcal{O}_K^3$ with $\mathcal{P}|b$.

Remark 4. By Theorem 41 the equation
\[
\alpha + \beta = \gamma^2, \quad (\alpha, \beta, \gamma) \in \mathcal{O}_{S_K}^* \times \mathcal{O}_{S_K}^* \times \mathcal{O}_{S_K}
\]
has finitely many solutions up to scaling by a square in $\mathcal{O}_{S_K}^*$, and these are effectively computable. Hence the criteria in Theorem 3 is testable in finite time.

Imposing local constraints, we get that for a totally real number field, in which 2 is inert, the following holds:

Theorem 5. Let $K$ be a totally real number field with $2 \mid h_K^+$ in which 2 is inert. Let $\mathcal{P}$ be the only prime above 2, and hence $S_K = \{\mathcal{P}\}$. Suppose that every solution $(\alpha, \gamma) \in \mathcal{O}_{S_K}^* \times \mathcal{O}_{S_K}$ with $v_{\mathcal{P}}(\alpha) \geq 0$ to the equation
\[
\alpha + 1 = \gamma^2
\]
(2)
satisfies $v_p(\alpha) \leq 6$. Then, there is a constant $B_K$ (depending only on $K$) such that for each rational prime $p > B_K$, the equation $a^p + b^p = c^2$ has no primitive, non-trivial solutions $(a,b,c) \in \mathcal{O}_K^3$ with $2|b$.

More concretely, for quadratic totally real number fields $K$, Theorem 5 becomes:

**Theorem 6.** Let $d > 5$ be a rational prime satisfying $d \equiv 5 \pmod{8}$. Write $K = \mathbb{Q}(\sqrt{d})$. Then, there is a constant $B_K$ (depending only on $K$) such that for each rational prime $p > B_K$, the equation $a^p + b^p = c^2$ has no primitive, non-trivial solutions $(a,b,c) \in \mathcal{O}_K^3$ with $2|b$.

More generally, by employing additional local information, the following holds.

**Theorem 7.** Let $K$ be a totally real field of degree $n$, and let $q \geq 5$ be a rational prime.

(i) $2 \nmid h_K$,

(ii) $\gcd(n, q - 1) = 1$,

(iii) $2$ is inert in $K$,

(iv) $q$ totally ramifies in $K$.

Then, there is a constant $B_K$ (depending only on $K$) such that for each rational prime $p > B_K$, the equation $a^p + b^p = c^2$ has no primitive, non-trivial solutions $(a,b,c) \in \mathcal{O}_K^3$ with $2|b$.

**Remark 8.** A few examples of totally real fields $K$ satisfying the conditions above are the degree 3 extensions which are the splitting fields of:

$p_1(x) = x^3 - 51x - 85$, $p_2(x) = x^3 - x^2 - 40x + 13$, $p_3(x) = x^3 - x^2 - 38x - 75$ and $p_4(x) = x^3 - 17x - 17$.

We use the same methods to study the asymptotic behaviour of the analogue $(p,p,3)$ equation and we get the following:

**Theorem 9** (Main Theorem for $(p,p,3)$). Let $K$ be a totally real number field with $\mathcal{C}_L(K)[3] = \{1\}$ where $S_K := \mathfrak{P} : \mathfrak{P}$ is a prime of $K$ above 3. Suppose that there exists some distinguished prime $\mathfrak{P} \in S_K$ such that every solution $(\alpha, \beta, \gamma) \in \mathcal{O}_{S_K}^* \times \mathcal{O}_{S_K}^* \times \mathcal{O}_{S_K}$ to the $S_K$ equation

$$\alpha + \beta = \gamma^3$$

satisfies $|v_\mathfrak{P}(\frac{\alpha}{\beta})| \leq 3v_\mathfrak{P}(3)$. Then, there is a constant $B_K$ (depending only on $K$) such that for each rational prime $p > B_K$, the equation $a^p + b^p = c^3$ has no primitive, non-trivial solutions $(a,b,c) \in \mathcal{O}_K^3$ with $\mathfrak{P}|b$.

**Remark 10.** By Theorem 11 the equation

$$\alpha + \beta = \gamma^3, \quad (\alpha, \beta, \gamma) \in \mathcal{O}_{S_K}^* \times \mathcal{O}_{S_K}^* \times \mathcal{O}_{S_K}$$

has finitely many solutions up to scaling by a cube in $\mathcal{O}_{S_K}^*$, and these are effectively computable. Hence the criteria in Theorem 5 is testable in finite time.
Similarly to the $(p,p,2)$ case, the following hold when employing local information. We will consider various field extensions involving the primitive cube root of unity $\omega := \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right)$.

**Theorem 11.** Let $K$ be a totally real number field such that $3 \nmid h_K(\omega)$, $3 \nmid h_K$ and in which $3$ is inert. Let $\mathfrak{P}$ be the only prime above $3$, and hence $S_K = \{\mathfrak{P}\}$. Suppose that every solution $(\alpha, \gamma) \in \mathcal{O}_K^3 \times \mathcal{O}_K$ with $v_3(\alpha) \geq 0$ to the equation

$$\alpha + 1 = \gamma^3$$

satisfies $v_3(\alpha) \leq 3$. Then, there is a constant $B_K$ (depending only on $K$) such that for each rational prime $p > B_K$, the equation $a^p + b^p = c^3$ has no primitive, non-trivial solutions $(a, b, c) \in \mathcal{O}_K^3$ with $3|b$.

**Theorem 12.** Let $d$ a positive, square-free satisfying $d \equiv 2 \mod 3$. Write $K = \mathbb{Q}(\sqrt{d})$ and suppose $3 \nmid h_K(\omega)$, $3 \nmid h_K$. Then, there is a constant $B_K$ (depending only on $K$) such that for each rational prime $p > B_K$, the equation $a^p + b^p = c^3$ has no primitive, non-trivial solutions $(a, b, c) \in \mathcal{O}_K^3$ with $3|b$.

**Theorem 13.** Let $K$ be a totally real field of degree $n$, and let $q \geq 5$ be a rational prime. Suppose

(i) $3 \nmid h_K(\omega)$ and $3 \nmid h_K$,

(ii) $\gcd(n, q^2 - 1) = 1$,

(iii) $3$ is inert in $K$,

(iv) $q$ totally ramifies in $K$.

Then, there is a constant $B_K$ (depending only on $K$) such that for each rational prime $p > B_K$, the equation $a^p + b^p = c^3$ has no primitive, non-trivial solutions $(a, b, c) \in \mathcal{O}_K^3$ with $3|b$.

**Remark 14.** A few examples of totally real fields $K$ satisfying the conditions above are the degree $5$ extensions which are the splitting fields of: $p_1(x) = x^5 - 25x^3 - 10x^2 + 50x - 20$, $p_2(x) = x^5 - 30x^3 - 20x^2 + 160x + 128$, $p_3(x) = x^5 - 15x^3 - 10x^2 + 10x + 4$ and $p_4(x) = x^5 - 20x^3 - 15x^2 + 10x + 4$.

### 1.3 Notational conventions

We will follow the notational conventions in [12]. Throughout $p$ denotes a rational prime, and $K$ a totally real number field, with ring of integers $\mathcal{O}_K$. For a non-zero ideal $I$ of $\mathcal{O}_K$, we denote by $[I]$ the class of $I$ in the class group $\text{Cl}(K)$.

Let $G_K = \text{Gal}(\bar{K}/K)$. For an elliptic curve $E/K$, we write

$$\overline{p}_{E,p} : G_K \to \text{Aut}(E[p]) \simeq \text{GL}_2(\mathbb{F}_p)$$

for the representation of $G_K$ on the $p$-torsion of $E$. For a Hilbert eigenform $f$ over $K$, we let $\mathcal{Q}_f$ denote the field generated by its eigenvalues. In this situation $\mathfrak{p}$ will denote a prime of $\mathcal{Q}_f$ above $p$; of course if $\mathcal{Q}_f = \mathbb{Q}$ we write $p$ instead of $\mathfrak{p}$. All other primes we consider are primes of $K$. We reserve the symbol $\mathfrak{P}$ for primes belonging to $S$. An arbitrary prime of $K$ is denoted by $\mathfrak{q}$, and $G_\mathfrak{q}$ and $I_\mathfrak{q}$ are the decomposition and inertia subgroups of $G_K$ at $\mathfrak{q}$.

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2 Preliminaries

2.1 Elliptic Curves

We begin by collecting some useful results about elliptic curves, as they play a key role in the modular approach of solving Diophantine equations.

Lemma 15. Let $K$ be a field of char$(K) \neq 2,3$ and $E/K$ an elliptic curve. The following holds:

(i) If $E$ has a $K$-rational point of order 2, then $E$ has a model of the form

$$E : Y^2 = X^3 + aX^2 + bX. \quad (4)$$

Moreover, there is a bijection between

$$\{E/K \text{ with a } K\text{-torsion of order } 2 \text{ up to } \bar{K}\text{-isomorphism}\} \rightarrow \mathbb{P}^1(K) - \{4, \infty\}$$

via the map $E \rightarrow \lambda := \frac{a^2}{b}$.

(ii) If $E$ has a $K$-rational point of order 3, then $E$ has a model of the form

$$E : Y^2 + cXY + dY = X^3. \quad (5)$$

Moreover, there is a bijection between

$$\{E/K \text{ with a } K\text{-torsion of order } 3 \text{ up to } \bar{K}\text{-isomorphism}\} \rightarrow \mathbb{P}^1(K) - \{27, \infty\}$$

via the map $E \rightarrow \lambda := \frac{c^3}{d}$.

Proof. (i) The first part is a well-known result. For the second part, we are given an elliptic curve $E/K$ with a $K$-torsion point of order 2. After writing it as in (4), we make the assignment $E \mapsto \lambda := \frac{a^2}{b}$. As $\Delta_E = 2^4b^2(a^2 - 4b)$, non-singularity of $E$ gives $\lambda \in \mathbb{P}^1(K) - \{4, \infty\}$, which proves our map is well-defined. Moreover, any $\lambda \in \mathbb{P}^1(K) - \{4, \infty\}$ can be written as a ratio of the form $\frac{a^2}{b}$ with $b \neq 0$ and $a^2 \neq 4b$, and hence comes from an elliptic curve with a $K$-rational 2-torsion. Thus, our map is surjective. Injectivity follows from writing

$$j_E = 2^8 \frac{\lambda^3 (a^2 - 3b)}{b^2(a^2 - 4b)} = 2^8 \frac{(\lambda - 3)^3}{\lambda - 4}$$

and noting that $\lambda = \lambda'$ for given $E \rightarrow \lambda$, $E' \rightarrow \lambda'$ implies $j_E = j_{E'}$, which gives $E \simeq E'$.

(ii) If $E$ is in Weierstrass form we can translate the $K$-torsion point to $(0,0)$. This will give a model of the form

$$E : Y^2 + a_1XY + a_3Y = X^3 + a_2X^2 + a_4X$$

We now impose the condition that $(0,0)$ has order 3. First, we compute $-(0,0) = (0, -a_3)$ and note that we require $(0,0) \neq -(0,0) = (0, -a_3)$, so $a_3 \neq 0$. Now, by performing the change of variables

$$\begin{cases} Y \rightarrow (Y + \frac{a_4}{a_3}X) \\ X \rightarrow X \end{cases}$$

(6)
we get a model of the form

\[ E : Y^2 + cXY + dY = X^3 + eX^2 \] with \( d = a_3 \neq 0 \).

Finally, we make use of the order 3,

\[
\begin{align*}
(0,0) + (0,0) &= -(0,0) = (0, -d) \\
(0,0) + (0,0) &= (-e, -d)
\end{align*}
\] (7)

Hence, we need \( e = 0 \), and we get the desired form:

\[ E : Y^2 + cXY + dY = X^3. \]

For the second part, we are given an elliptic curve \( E/K \) with a \( K \)-torsion point of order 3. After writing it as in (5), we make the assignment \( \lambda : \mu := \frac{c^3 - 24d}{d(3c^3 - 27d)} \) with \( d = a_3 \neq 0 \) and \( c^3 \neq 27d \), and hence comes from an elliptic curve with a \( K \)-rational 3-torsion. Thus, our map is surjective.

Injectivity follows from writing

\[ j_E = 2^8 \frac{(\lambda - 3)^3}{\lambda - 4} \]

and noting that \( \lambda = \lambda' \) for given \( E \rightarrow \lambda \), \( E' \rightarrow \lambda' \) implies \( j_E = j_{E'} \), which gives \( E \simeq E' \).

**Lemma 16.** Let \( K \) be a number field and \( S \) a set of finite primes of \( K \). Then:

(i) If \( S \) contains the primes above 2 we get the following bijection

\[ \{ E/K \text{ with a } K\text{-torsion of order 2 with potentially good reduction outside } S \} \quad \longleftrightarrow \quad O_S^* \text{ via the map } E \rightarrow \mu := \lambda - 4 \in O_S^*, \text{ where } \lambda \text{ is as in Lemma 15 (i)}. \]

(ii) If \( S \) contains the primes above 3 we get the following bijection

\[ \{ E/K \text{ with a } K\text{-torsion of order 3 with potentially good reduction outside } S \} \quad \longleftrightarrow \quad O_S^* \text{ via the map } E \rightarrow \mu := \lambda - 27 \in O_S^*, \text{ where } \lambda \text{ is as in Lemma 15 (ii)}. \]

**Proof.** (i) Let \( E \) be an elliptic curve with a \( K \)-torsion point of order 2 with potentially good reduction outside \( S \). By Lemma 15 (i) \( E \) has a model

\[ E : Y^2 = X^3 + aX^2 + bX \]

with \( \lambda := \frac{a^2}{b} \) and \( \mu := \lambda - 4 = \frac{a^2 - 4b}{b} \). Thus

\[ j_E = 2^8 \frac{(\lambda - 3)^3}{\lambda - 4} = 2^8 \frac{(\mu + 1)^3}{\mu}. \] (8)

Good reduction outside \( S \) implies that \( v_q(j_E) \geq 0 \) for all \( q \notin S \), in other words \( j_{E'} \in O_S \). Consequently both \( \lambda \) and \( \mu \) satisfy monic equations with
coefficients in $\mathcal{O}_S$. Thus, we can conclude that $\lambda, \mu \in \mathcal{O}_S$. Moreover, by writing $j_E$ in terms of $\mu^{-1}$ and using the same reasoning, we deduce that also $\mu^{-1} \in \mathcal{O}_S$ and hence $\mu \in \mathcal{O}_S^*$ and so the assignment $E \mapsto \mu$ is well-defined.

Note that every $\mu \in \mathcal{O}_S^*$ can be written in the form $\mu = \frac{a^2}{b} - 4$ for some $a, b \in K$, thus coming from an elliptic curve with 2-torsion. Moreover, $\mu \in \mathcal{O}_S^*$ implies $j_E \in \mathcal{O}_S$, thus this represents a curve with potentially good reduction outside $S$, proving surjectivity.

Injectivity follows by noting that $\mu = \mu'$ implies $j_E = j'_E$ which gives $E \cong E'$. 

(ii) Let $E$ be an elliptic curve with a $K$-torsion point of order 3 with potentially good reduction outside $S$. By Lemma 15 (ii) $E$ has a model

$$E : Y^2 + cXY + dY = X^3$$

with $\lambda := \frac{c^3}{d}$ and $\mu = \lambda - 27 = \frac{c^3 - 27d}{d}$. Thus,

$$j_E = \frac{\lambda(\lambda - 24)^3}{\lambda - 27} = \frac{(\mu + 27)(\mu + 3)^3}{\mu}. \quad (9)$$

Same arguments as in the proof of (i) give $j_E, \lambda \in \mathcal{O}_S$ and $\mu \in \mathcal{O}_S^*$, giving $E \mapsto \mu$ is well-defined.

Surjectivity and injectivity follow exactly as in (i).

We say that a fractional ideal is an $S$-ideal if its decomposition into primes contains only primes in $S$.

**Lemma 17.** Let $K$ be a number field and $S$ a set of finite primes of $K$. Let $E/K$ be an elliptic curve with good reduction outside $S$.

(i) Suppose $S$ contains the primes above 2 and $E$ has a $K$-torsion point of order 2. Let $(\lambda, \mu) \in \mathcal{O}_S \times \mathcal{O}_S^*$ correspond to $E$ as in Lemma 16 (i) and therefore satisfy $\lambda - \mu = 4$. Then $(\lambda)\mathcal{O}_K = I^2J$ where $I, J$ are fractional ideals with $J$ being an $S$-ideal.

(ii) Suppose $S$ contains the primes above 3 and $E$ has a $K$-torsion point of order 3. Let $(\lambda, \mu) \in \mathcal{O}_S \times \mathcal{O}_S^*$ correspond to $E$ as in Lemma 16 (ii) and therefore satisfy $\lambda - \mu = 27$. Then $(\lambda)\mathcal{O}_K = I^3J$ where $I, J$ are fractional ideals with $J$ being an $S$-ideal.

**Proof.** (i) By Lemma 15 (i) $E$ has a model

$$E : Y^2 = X^3 + aX^2 + bX$$

with $\Delta_E = 2^4b^2(a^2 - 4b)$ and $c_4 = 2^4(a^2 - 3b)$. Good reduction outside $S$ implies that for a $q \notin S$ we have that $v_q(\Delta_{\text{min}}) = 0$ (where $\Delta_{\text{min}}$ is the minimal discriminant of $E$ viewed over the local field $K_q$). So, $q^{12k}\|\Delta_E$ and $q^{4k}|c_4$ for some integer $k$. This follows from standard results about
the minimal discriminant of an elliptic curve which can be found in [26, Ch. VII.1.] Therefore, \( q^{2k}|a \) and \( q^{4k}|b \). Hence,

\[
(a) \mathcal{O}_K = \prod_{q \notin S_K} q^{4k_q+2l_q} \prod_{\mathfrak{p} \in S_K} \mathfrak{p}^{a_{\mathfrak{p}}}, \quad (b) \mathcal{O}_K = \prod_{q \notin S_K} q^{4k_q} \prod_{\mathfrak{p} \in S_K} \mathfrak{p}^{b_{\mathfrak{p}}}. 
\]

Thus, as \( \lambda = \frac{a^2}{b} \), we get

\[
(\lambda) \mathcal{O}_K = I^2 J, \quad \text{where} \quad I := \prod_{q \notin S_K} q^{l_q}, \quad J := \prod_{\mathfrak{p} \in S_K} \mathfrak{p}^{2a_{\mathfrak{p}}-b_{\mathfrak{p}}}
\]

which makes \( J \) an \( S \)-ideal.

(ii) By Lemma 15 (ii) \( E \) has a model

\[
E : Y^2 + cXY + dY = X^3
\]

with \( \Delta_E = d^3(c^3 - 27d) \) and \( c_4 = c(c^3 - 24d) \). Good reduction outside \( S \) implies that for a \( q \notin S \) we have that \( v_q(\Delta_{\text{min}}) = 0 \) (where \( \Delta_{\text{min}} \) is the minimal discriminant of \( E \) viewed over the local field \( K_q \)). So, \( q^{12k}||\Delta_E \) and \( q^{4k}|c_4 \) for some integer \( k \). This follows from standard results about the minimal discriminant of an elliptic curve which can be found in [26, Ch. VII.1]. Therefore, \( q^{3k}|d \) and \( q^k||c \). Hence,

\[
(c) \mathcal{O}_K = \prod_{q \notin S} q^{k_q+l_q} \prod_{\mathfrak{p} \in S} \mathfrak{p}^{a_{\mathfrak{p}}}, \quad (d) \mathcal{O}_K = \prod_{q \notin S} q^{3k_q} \prod_{\mathfrak{p} \in S} \mathfrak{p}^{d_{\mathfrak{p}}}. 
\]

Thus, as \( \lambda = \frac{c^3}{d} \), we get

\[
(\lambda) \mathcal{O}_K = I^3 J, \quad \text{where} \quad I := \prod_{q \notin S} q^{l_q}, \quad J := \prod_{\mathfrak{p} \in S} \mathfrak{p}^{3c_{\mathfrak{p}}-d_{\mathfrak{p}}}
\]

which makes \( J \) an \( S \)-ideal.

\[\Box\]

### 2.2 Modularity Results

We now carefully formulate modularity in the context of a totally real field. Let us first recall that given \( K \) a totally real number field, \( G_K \) its absolute Galois group and \( E \) an elliptic curve over \( K \), we say that \( E \) is modular if there exists a Hilbert cuspidal eigenform \( f \) over \( K \) of parallel weight 2, with rational Hecke eigenvalues, such that the Hasse–Weil L-function of \( E \) is equal to the Hecke L-function of \( f \). A more conceptual way to phrase this is that there is an isomorphism of compatible systems of Galois representations

\[
\rho_{E,p} \cong \rho_{f,p}
\]

where the left-hand side is the Galois representation arising from the action of \( G_K \) on the \( p \)-adic Tate module \( T_p(E) \), while the right-hand side is the Galois representation associated to \( f \). A comprehensive definition of Hilbert modular forms and their associated representation can be found, for example in Wiles’ [33].

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We need the following theorem proved by Freitas, Hung and Siksek in [9]:

**Theorem 18.** Let $K$ be a totally real field. There are at most finitely many $\bar{K}$-isomorphism classes of non-modular elliptic curves $E$ over $K$. Moreover, if $K$ is real quadratic, then all elliptic curves over $K$ are modular.

Furthermore Derickx, Najman and Siksek have recently proved in [8]:

**Theorem 19.** Let $K$ be a totally real cubic number field and $E$ be an elliptic curve over $K$. Then $E$ is modular.

### 2.3 Irreductibility of mod $p$ representations of elliptic curves

We need the following theorem in the level lowering step of our proof. This was proved in [11, Theorem 2] and it is derived from the work of David and Momose who in turn built on Merel’s Uniform Boundedness Theorem.

**Theorem 20.** Let $K$ be a Galois totally real field. There is an effective constant $C_K$, depending only on $K$, such that the following holds. If $p > C_K$ is prime, and $E$ is an elliptic curve over $K$ which has multiplicative reduction at all $q | p$, then $\rho_{E,p}$ is irreducible.

**Remark 21.** The above theorem is also true for any totally real field by replacing $K$ by its Galois closure.

### 2.4 Level lowering

We present a level lowering result proved by Freitas and Siksek in [12] derived from the work of Fujira [13], Jarvis [14], and Rajaei [20]. Let $K$ be a totally real field and $E/K$ be an elliptic curve of conductor $N_E$. Let $p$ be a rational prime. Define the following quantities:

$$M_p = \prod_{q || N_E, \frac{v_q(\Delta_q)}{p} \mid q} q, \quad N_p = \frac{N_E}{M_p}$$

(10)

where $\Delta_q$ is the minimal discriminant of a local minimal model for $E$ at $q$. For a Hilbert eigenform $f$ over $K$, we write $Q_f$ for the field generated by its eigenvalues.

**Theorem 22.** With the notation above, suppose the following statements hold:

(i) $p \geq 5$, the ramification index $e(q/p) < p - 1$ for all $q | p$, and $Q(\zeta_p) \nsubseteq K$,

(ii) $E$ is modular,

(iii) $\bar{\rho}_{E,p}$ is irreducible,

(iv) $E$ is semistable at all $q | p$,

(v) $p | v_q(\Delta_q)$ for all $q | p$.

Then, there is a Hilbert eigenform $f$ of parallel weight 2 that is new at level $N_p$ and some prime $\varpi$ of $Q_f$ such that $\varpi | p$ and $\bar{\rho}_{E,p} \sim \bar{\rho}_{f,\varpi}$.

**Proof.** A proof is given in [12, p. 8]. \qed
2.5 Eichler-Shimura

For totally real fields, modularity reads as follows.

**Conjecture 23** (Eichler-Shimura). Let $K$ be a totally real field. Let $\mathfrak{f}$ be a Hilbert newform of level $\mathcal{N}$ and parallel weight 2, and rational eigenvalues. Then there is an elliptic curve $E_{\mathfrak{f}}/K$ with conductor $\mathcal{N}$ having the same $L$-function as $\mathfrak{f}$.

Freitas and Siksek obtain the following theorem [12] from works of Blasius [3], Darmon [5] and Zhang [35].

**Theorem 24.** Let $E$ be an elliptic curve over a totally real field $K$, and $p$ be an odd prime. Suppose that $\rho_{E,p}$ is irreducible, and $\rho_{E,p} \sim \rho_{f,\chi}$ for some Hilbert newform $f$ over $K$ of level $\mathcal{N}$ and parallel weight 2 which satisfies $Q_f = Q$. Let $q \nmid p$ be a prime ideal of $\mathcal{O}_K$ such that:

(i) $E$ has potentially multiplicative reduction at $q$,

(ii) $p \nmid \# \rho_{E,p}(I_q)$,

(iii) $p \nmid (\text{Norm}_{K/Q}(q) + 1)$.

Then there is an elliptic curve $E_{\mathfrak{f}}/K$ of conductor $\mathcal{N}$ with the same $L$-function as $\mathfrak{f}$.

3 Signature $(p, p, 2)$

Let $K$ be a totally real field. Recall the set $S_K = \{ \mathfrak{p} : \mathfrak{p} \text{ is a prime of } K \text{ above } 2 \}$. Throughout this section we denote by $(a, b, c) \in \mathcal{O}_K^3$ a non-trivial, primitive solution of $a^p + b^p = c^2$.

3.1 Frey Curve

For $(a, b, c) \in \mathcal{O}_K^3$ as described above we associate the following Frey elliptic curve defined over $K$:

$$E : Y^2 = X^3 + 4cX^2 + 4a^pX. \quad (11)$$

We compute the arithmetic invariants:

$$\Delta_E = 2^{12}(a^2b)^p, c_4 = 2^6(4b^p + a^p) \text{ and } j_E = 2^6 \frac{(4b^p + a^p)^3}{(a^2b)^p}.$$ 

**Lemma 25.** Let $(a, b, c)$ be the non-trivial, primitive solution to the equation $a^p + b^p = c^2$. Let $E$ be the associated Frey curve (11) with conductor $\mathcal{N}_E$. Then, for all primes $q \notin S_K$, the model $E$ is minimal, semistable and satisfies $p | v_q(\Delta_E)$. Moreover

$$\mathcal{N}_E = \prod_{\mathfrak{p} \in S_K} \mathfrak{p}^{r_{\mathfrak{p}}} \prod_{q | ab, q \notin S_K} q, \quad \mathcal{N}_p = \prod_{\mathfrak{p} \in S_K} \mathfrak{p}^{r_{\mathfrak{p}}'} \quad (12)$$

where $0 \leq r_{\mathfrak{p}}' \leq r_{\mathfrak{p}} \leq 2 + 6v_{\mathfrak{p}}(2)$. 

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Proof. Let \( q \) be an odd prime of \( K \). The invariants of the model \( E \) are \( \Delta_E = 2^{12}(a^2b)^p \) and \( c_4 = 2^9(4b^p + a^p) \). Suppose that \( q \) divides \( \Delta_E \), so \( q | ab \). Since \( a \) and \( b \) are relatively prime, \( q \) divides exactly one of \( a \) and \( b \). Therefore, \( q \) does not divide \( c_4 \). In particular, the model is minimal at \( q \) and has multiplicative reduction. Hence \( p | v_q(\Delta_E) = v_q(\Delta_q) \). On the other hand \( \mathfrak{p} \in S_K \) is an even prime, so we have \( r_{\mathfrak{p}} \equiv v_{\mathfrak{p}}(N_E) \leq 2 + 6v_{\mathfrak{p}}(2) \) by \([27]\) Theorem IV.10.4]. The definition of \( N_E \) gives the desired form in \([12]\). Then, use \((10)\) to get \( N_{\mathfrak{p}} \) and observe that \( r_{\mathfrak{p}}' = r_{\mathfrak{p}} \) unless \( E \) has multiplicative reduction at \( \mathfrak{p} \) and \( p | v_{\mathfrak{p}}(\Delta_{\mathfrak{p}}) \) in which case \( r_{\mathfrak{p}} = 1 \) and \( r_{\mathfrak{p}}' = 0 \).

Lemma 26. Let \( K \) be a totally real field. There is some constant \( A_K \) depending only on \( K \), such that for any non-trivial, primitive solution \( (a, b, c) \) of \( a^p + b^p = c^2 \) and \( p > A_K \), the Frey curve given by \((11)\) is modular.

Proof. By Theorem \([18]\) there are at most finitely many possible \( K \)-isomorphism classes of elliptic curves over \( E \) which are not modular. Let \( j_1, j_2, \ldots, j_n \in K \) be the \( j \)-invariants of these classes. Define \( \lambda := b^p/a^p \). The \( j \)-invariant of \( E \) is

\[
j(\lambda) = 2^6(4\lambda + 1)^3\lambda^{-1}.
\]

We can assume \( \lambda \notin \{0, \pm 1\} \) as these \( \lambda \) lead to \( j(\lambda) \in \mathbb{Q} \) and we know that all rational elliptic curves are modular. Each equation \( j(\lambda) = j_i \) has at most three solutions \( \lambda \in K \). Thus there are values \( \lambda_1, \ldots, \lambda_m \in K \) (where \( m \leq 3n \)) such that \( \lambda \neq \lambda_k \) for all \( k \), then the elliptic curve \( E \) with \( j \)-invariant \( j(\lambda) \) is modular.

If \( \lambda = \lambda_k \) then \( (b/a)^p = \lambda_k \), but the polynomial \( x^p + \lambda_k \) has a root in \( K \) if and only if \( \lambda_k \in (K^*)^p \) because \( K \) is totally real and \( \lambda_k \notin \{0, \pm 1\} \). Hence we get a lower bound on \( p \) for each \( k \), and by taking the maximum of these bounds we get \( A_K \).

Remark 27. The constant \( A_K \) is ineffective as the finiteness of Theorem \([18]\) relies on Falting’s Theorem (which is ineffective). See \([9]\) for more details. Note that if \( K \) is quadratic or cubic we get \( A_K = 0 \) (by the last part of Theorem \([18]\) and Theorem \([19]\)).

3.2 Images of Inertia

We gather information about the images of inertia \( \overline{\rho}_{E,p}(I_q) \). This is a crucial step in applying Corollary \([24]\) and for controlling the behaviour at the primes in \( S_K \) of the newform obtained by level lowering.

Lemma 28. Let \( E \) be an elliptic curve over \( K \) with \( j \)-invariant \( j_E \). Let \( p \geq 5 \) and let \( q \nmid p \) be a prime of \( K \). Then \( p | \#\overline{\rho}_{E,p}(I_q) \) if and only if \( E \) has potentially multiplicative reduction at \( q \) (i.e. \( v_q(j_E) < 0 \)) and \( p \nmid v_q(j_E) \).

Proof. See \([12]\) Lemma 3.4].

Lemma 29. Let \( q \nmid 2 \) and let \( (a, b, c) \) be a non-trivial, primitive solution to the equation \( a^p + b^p = c^2 \) with the prime exponent \( p \geq 5 \), such that \( q \nmid p \). Let \( E \) be the Frey curve in \((11)\). Then \( p \nmid \#\overline{\rho}_{E,p}(I_q) \).
Proof. Using Lemma 28, it is enough to show that at all \( q \mid 2 \) and \( q \nmid p \) either \( v_q(j_E) \geq 0 \) or \( p|v_q(j_E) \). If \( q \mid N \), then \( E \) has good reduction at \( q \), so \( v_q(j_E) \geq 0 \). If \( q \not| N \) then \( q \mid \Delta_E \). Thus \( q \) divides exactly one of \( a \) and \( b \). This implies that \( q \mid c_4 \), i.e. \( v_q(c_4) = 0 \). Thus, \( v_q(j_E) = -v_q(a^2b) \), i.e. \( p|v_q(j_E) \). \( \square \)

Lemma 30. Let \( \mathfrak{P} \in S_K \) and \( (a,b,c) \) a non-trivial, primitive solution to \( ap^b + bp^c = c^2 \) with \( \mathfrak{P}|b \) and prime exponent \( p > 6v_\mathfrak{P}(2) \). Let \( E \) be the Frey curve in \( \mathbb{Q} \) with \( j \)-invariant \( j_E \). Then \( E \) has potentially multiplicative reduction at \( \mathfrak{P} \) and \( p|\#\mathfrak{P}_{E,p}(I_{\mathfrak{P}}) \).

Proof. Assume that \( \mathfrak{P} \in S_K \) with \( v_\mathfrak{P}(b) = k \). Then \( v_\mathfrak{P}(j_E) = 6v_\mathfrak{P}(2) - pk \). Since \( p > 6v_\mathfrak{P}(2) \), it follows that \( v_\mathfrak{P}(j_E) < 0 \) and clearly \( p \nmid v_\mathfrak{P}(j_E) \). This implies that \( E \) has potentially multiplicative reduction at \( \mathfrak{P} \) and by Lemma 28 we get \( p|\#\mathfrak{P}_{E,p}(I_{\mathfrak{P}}) \). \( \square \)

3.3 Level Lowering and Eichler Shimura

This is a key result in the proof of Theorem 3, for which we have prepared the ingredients in the previous sections. We will follow the corresponding proofs in [12] and [16].

Theorem 31. Let \( K \) be a totally real number field and assume it has a distinguished prime \( \mathfrak{P} \in S_K \). Then there is a constant \( B_K \) depending only on \( K \) such that the following hold. Suppose \( (a,b,c) \in \mathcal{O}_K^3 \) is a non-trivial, primitive solution to \( ap^b + bp^c = c^2 \) with prime exponent \( p > B_K \) such that \( \mathfrak{P}|b \). Write \( E \) for the Frey curve in \( \mathbb{Q} \). Then, there is an elliptic curve \( E' \) over \( K \) such that:

(i) the elliptic curve \( E' \) has good reduction outside \( S_K \);

(ii) \( \mathfrak{P}_{E,p} \sim \mathfrak{P}_{E',p} \);

(iii) \( E' \) has a \( K \)-rational point of order 2;

(iv) \( E' \) has multiplicative reduction at \( \mathfrak{P} \) \( (v_\mathfrak{P}(j_E) < 0 \) where \( j_E \) is the \( j \)-invariant of \( E' \)).

Proof. We first observe that by Lemma 30 that \( E \) has multiplicative reduction outside \( S_K \). By taking \( B_K \) sufficiently large, we see from Lemma 20 that \( E \) is modular and by Theorem 21 that \( \mathfrak{P}_{E,p} \) is irreducible. Applying Theorem 22 and Lemma 28 we see that \( \mathfrak{P}_{E,p} \sim \mathfrak{P}_{f,\mathfrak{P}} \) for a Hilbert newform \( f \) of level \( N_p \) and some prime \( \mathfrak{P} \) of \( \mathbb{Q}_f \). Here \( \mathbb{Q}_f \) denotes the field generated by the Hecke eigenvalues \( f \). Next we reduce to the case when \( \mathbb{Q}_f = \mathbb{Q} \), after possibly enlarging \( B_K \). This step uses standard ideas originally due to Mazur that can be found in [2] Section 4], [4] Proposition 15.4.2], and so we omit the details.

Next we want to show that there is some elliptic curve \( E'/K \) of conductor \( N_p \) having the same \( L \)-function as \( f \). We apply Lemma 20 with \( \mathfrak{P} = \mathfrak{P} \) and get that \( E \) has potentially multiplicative reduction at \( \mathfrak{P} \) and \( p|\#\mathfrak{P}_{E,p}(I_{\mathfrak{P}}) \). The existence of \( E' \) follows from Theorem 24 after possibly enlarging \( B_K \) to ensure that \( p \nmid (\text{Norm}_{K/Q}(\mathfrak{P}) \pm 1) \). By putting all the pieces together we can conclude that there is an elliptic curve \( E'/K \) of conductor \( N_p \) satisfying \( \mathfrak{P}_{E,p} \sim \mathfrak{P}_{E',p} \). This proves (i) and (ii).

To prove (iii) we use that \( \mathfrak{P}_{E,p} \sim \mathfrak{P}_{E',p} \) for some \( E'/K \) with conductor \( N_p \). After enlarging \( B_K \) by an effective amount, and possibly replacing \( E' \) by an
Finally, implies that \( I \) by dividing \( I \). Thus \( \lambda \) is an isogenous curve, we may assume that \( E' \) has full 2-torsion. This uses standard ideas which can be found, for example, in \cite{IV} Section IV-6.

Now let \( j_{E'} \) be the \( j \)-invariant of \( E' \). As we have already seen, Lemma \ref{lem:Eisogenous} implies \( p | \# \mathcal{P}_{E,p}(I_{P}) \), hence \( p | \# \mathcal{P}_{E',p}(I_{P}) \), thus by Lemma \ref{lem:Divisibility} we get that \( E' \) has multiplicative reduction at \( \mathfrak{p} \) and so \( v_{\mathfrak{p}}(j_{E'}) < 0 \).

\[ \square \]

### 3.4 Proof of Theorem \ref{thm:Main}

**Proof.** So far, we have shown that for a primitive, non-trivial solution \((a, b, c)\) such that \( \mathfrak{p} | b \) with a prime exponent \( p \) we associate the Frey elliptic curve in \((\ref{eq:curve})\). By Theorem \ref{thm:primitive} for there exists \( B_K \) such that for all \( p > B_K \) we can find an elliptic curve \( E' \) which is related to \( E \) by \( \mathcal{P}_{E,p} \sim \mathcal{P}_{E',p} \) and has a \( K \)-rational point of order 2. Hence by Theorem \ref{thm:valuation} \( i \) we get a model

\[ E': Y^2 = X^3 + a'X^2 + b'X \]

with arithmetic invariants \( \Delta_{E'} = 2^4b'^2(a'^2 - 4b'), \ j_{E'} = 2^8 \left( \frac{a'^2 - 4b'}{6} \right)^2 \). Moreover, by Theorem \ref{thm:valuation} \( i \), we know that \( E' \) has good reduction outside \( S_K \) which implies that \( v_{\mathfrak{p}}(j_{E'}) \geq 0 \) for \( \mathfrak{p} \notin S_K \). Therefore, \( j_{E'} \in \mathcal{O}_{S_K} \). Consider \( \lambda := \frac{a'^2}{4} \) and \( \mu := 4 - \frac{a'^2 - 4b'}{2} \). Next, we need to show that \( \lambda \) can be written as \( \lambda = u\gamma^2 \), where \( u \) is an \( S_K \)-unit. By Lemma \ref{lem:valuation} \( i \) applied to \( E' \) we get that

\[ (\lambda)\mathcal{O}_K = \lambda^2 J \text{ where } J \text{ is an } S\text{-ideal.} \]

Thus \( [I] = [J] \) as elements of the class group \( \text{Cl}(K) \) and \( [J] \in \langle \mathfrak{p} \rangle \mathcal{P}_{E,p} \). This implies that \( [I] \in \text{Cl}_{S_K}(K)[9] \) and by our assumption on \( K \) that \( \text{Cl}_{S_K}(K)[9] \) is trivial, we get that \( [I] \in \langle \mathfrak{p} \rangle \mathcal{P}_{E,p} \), i.e. \( I = \gamma I \), where \( I \) is an \( S \)-ideal and \( \gamma \in \mathcal{O}_K \). Consequently,

\[ (\lambda)\mathcal{O}_K = (\gamma)^2 \hat{J} \text{ where both } \hat{I} \text{ and } J \text{ are } S\text{-ideals.} \]

Finally, \( (\frac{1}{\mathfrak{p}})\mathcal{O}_K \) is an \( S \)-ideal, which implies that \( u := \frac{A}{u} \) is an \( S \)-unit. Now, by dividing \( \mu + 4 = \lambda \) by \( u \), we get

\[ \alpha + \beta = \gamma^2, \quad \alpha := \frac{\mu}{u} \in \mathcal{O}_{S_K}^*, \quad \beta := \frac{4}{u} \in \mathcal{O}_{S_K}^*. \quad (\ref{eq:alpha_beta}) \]

Now, suppose that there is some \( \mathfrak{q} \in S_K \) that satisfies \( |v_{\mathfrak{p}}(\frac{u}{\mathfrak{q}})| \leq 6v_{\mathfrak{p}}(2) \). We will show that \( v_{\mathfrak{p}}(j_{E'}) \geq 0 \), contradicting Theorem \ref{thm:valuation} \( iv \) and hence we can conclude the proof. By using \((\ref{eq:alpha_beta})\) we can rewrite the assumption \( |v_{\mathfrak{p}}(\frac{u}{\mathfrak{q}})| \leq 6v_{\mathfrak{p}}(2) \) in terms of the valuation of \( \mu \), using that \( \frac{\mu}{2} = \frac{\beta}{2} \):

\[ -4v_{\mathfrak{p}}(2) \leq v_{\mathfrak{p}}(\mu) \leq 8v_{\mathfrak{p}}(2). \]

Note that \( j_{E'} = 2^8(\mu + 1)^4\mu^{-1} \), hence

\[ v_{\mathfrak{p}}(j_{E'}) = 8v_{\mathfrak{p}}(2) + 3v_{\mathfrak{p}}(\mu + 1) - v_{\mathfrak{p}}(\mu). \]

There are three cases according to the valuation of \( \mathfrak{p} \) at \( \mu \):

- **Case (1):** Suppose \( v_{\mathfrak{p}}(\mu) = 0 \). This implies that \( v_{\mathfrak{p}}(\mu + 1) \geq 0 \), thus \( v_{\mathfrak{p}}(j_{E'}) \geq 0 \), a contradiction.
- **Case (2):** Suppose \( v_{\mathfrak{p}}(\mu) > 0 \). This implies \( v_{\mathfrak{p}}(\mu + 1) = 0 \), thus, by using \( v_{\mathfrak{p}}(\mu) \leq 8v_{\mathfrak{p}}(2) \) we get again \( v_{\mathfrak{p}}(j_{E'}) \geq 0 \).
- **Case (3):** Finally, suppose \( v_{\mathfrak{p}}(\mu) < 0 \). This implies \( v_{\mathfrak{p}}(\mu + 1) = v_{\mathfrak{p}}(\mu) \), thus, by using \( -4v_{\mathfrak{p}}(2) \leq v_{\mathfrak{p}}(\mu) \), we get one last time \( v_{\mathfrak{p}}(j_{E'}) \geq 0 \).

All three cases lead to contradictions and hence we conclude the proof. \( \square \)
3.5 Proof of Theorem 5

Proof. We want to apply Theorem 3 with \( \overline{\mathfrak{O}} = \mathfrak{O} \) and \( S_K = \{ \mathfrak{P} \} \). Note that \( 2 \nmid h_K^2 \) implies that \( Cl_{S_K}(K)[2] \) is trivial. As \( 2 \) is inert, we get \( v_\mathfrak{P}(2) = 1 \).

Now, let us consider the equation \( \alpha + \beta = \gamma^2 \), with \( \alpha, \beta \in \mathcal{O}_{S_K}^* \). By scaling the equation by even powers of \( 2 \) and swapping \( \alpha \) and \( \beta \) if necessary, we may assume \( 0 \leq v_\mathfrak{P}(\beta) \leq v_\mathfrak{P}(\alpha) \) with \( v_\mathfrak{P}(\beta) \in \{0,1\} \).

**Case (1):** Suppose \( v_\mathfrak{P}(\beta) = 1 \). If \( v_\mathfrak{P}(\alpha) \geq 2 \), then \( v_\mathfrak{P}(\gamma^2) = v_\mathfrak{P}(\alpha + \beta) = 1 \), which leads to a contradiction as \( v_\mathfrak{P}(\gamma^2) \) must be even. Thus, \( v_\mathfrak{P}(\alpha) = v_\mathfrak{P}(\beta) = 1 \) and \( v_\mathfrak{P}(\frac{\gamma}{2}) = 0 \leq 6 \).

**Case (2):** Suppose \( v_\mathfrak{P}(\beta) = 0 \) with \( \beta \) not a square. If \( v_\mathfrak{P}(\alpha) > 6 \), then \( v_\mathfrak{P}(\gamma^2) = v_\mathfrak{P}(\alpha + \beta) = 0 \) and \( \beta \equiv \gamma^2 \mod 2^6 \). Consider the field extension \( L = K(\sqrt{\beta}) \). We will show that \( L \) is unramified at \( 2 \), hence contradicting \( 2 \nmid h_K^2 \).

Consider the element \( \delta := \frac{\gamma + \sqrt{\beta}}{2} \). Its minimal polynomial is

\[
m_\mathfrak{P}(X) = X^2 - \gamma X + \frac{\gamma^2 - \beta}{4}.
\]

This belongs to \( \mathcal{O}_K[X] \) and has odd discriminant \( \Delta = \beta \), proving that \( L \) is unramified at \( 2 \). Thus, we must have \( v_\mathfrak{P}(\alpha) \leq 6 \), giving \( v_\mathfrak{P}(\frac{\gamma}{2}) = v_\mathfrak{P}(\alpha) \leq 6 \).

**Case (3):** Suppose \( \beta \) is a square. By dividing everything through \( \beta \), we may assume \( \beta = 1 \). Then, by the hypothesis of the theorem we get \( v_\mathfrak{P}(\frac{\gamma}{2}) = v_\mathfrak{P}(\alpha) \leq 6 \).

All of the possible three cases lead to \( v_\mathfrak{P}(\frac{\gamma}{2}) = 6 = 6v_\mathfrak{P}(2) \), so we can conclude the proof by Theorem 3.

\[\square\]

3.6 Proof of Theorem 6

Proof. Note that the assumption \( d \equiv 5 \mod 8 \) gives that \( 2 \) is inert in the quadratic field \( K = \mathbb{Q}(\sqrt{d}) \), take \( \mathfrak{P} \) to be the unique prime above \( 2 \) and denote \( S_K = \{ \mathfrak{P} \} \). Moreover, \( d \) prime is equivalent to \( 2 \nmid h_K^2 \) by [17, Section 1.3.1]. By Theorem 3, it is enough to check that every solution \( (\alpha, \gamma) \in \mathcal{O}_{S_K} \times \mathcal{O}_{S_K} \) with \( v_\mathfrak{P}(\alpha) \geq 0 \) to the equation \( \alpha + 1 = \gamma^2 \) satisfies \( v_\mathfrak{P}(\alpha) \leq 6 \). Rearranging the above we get that \( (\gamma + 1)(\gamma - 1) = \alpha \). Denote \( x = \frac{(\gamma + 1)}{2} \) and \( y = \frac{(1-\gamma)}{2} \). Note that since \( (\gamma + 1), (\gamma - 1) \in \mathcal{O}_{S_K} \) and they are factors of the \( S_K \)-unit \( \alpha \), they must be \( S_K \)-units, consequently \( x, y \in \mathcal{O}_{S_K}^* \).

In [12, p. 15], it is proved that the only solutions of \( S_K \)-unit equation \( x + y = 1 \), where \( K = \mathbb{Q}(\sqrt{d}) \) with \( d \equiv 5 \mod 8 \), \( d > 5 \) and \( S_K = \{ \mathfrak{P} \} \) are the so-called irrelevant solutions \((-1,2), (1/2,1/2), (2,-1)) \). This leads to \( \alpha \in \{ -1, 8 \} \), and hence \( v_\mathfrak{P}(\alpha) \in \{ 0, 3 \} \), proving \( v_\mathfrak{P}(\alpha) \leq 6 \). Hence we can conclude the proof by Theorem 3.

\[\square\]

3.7 Proof of Theorem 7

Proof. We will take \( \mathfrak{P} \) to be the unique prime above \( 2 \) and denote \( S_K = \{ \mathfrak{P} \} \).

By Theorem 3, it is enough to check that every solution \( (\alpha, \gamma) \in \mathcal{O}_{S_K} \times \mathcal{O}_{S_K} \) with \( v_\mathfrak{P}(\alpha) \geq 0 \) to the equation \( \alpha + 1 = \gamma^2 \) satisfies \( v_\mathfrak{P}(\alpha) \leq 6 \). Rearranging as in (3.6) we get an \( S_K \)-unit equation \( x + y = 1 \) such that \( \alpha = -4xy \).

We will now use a result proved in [10, p.5] saying that if \( K \) satisfies the hypothesis of Theorem 7 it follows that every solution \( (x, y) \) of the \( S_K \)-unit
equation satisfies \( \max\{v_p(x), v_p(y)\} < 2v_p(2) = 2 \). Thus,
\[
v_p(\alpha) = 2v_p(2) + v_p(x) + v_p(\beta) < 2 + 2 + 2 = 6.
\]
Hence we can conclude the proof by Theorem \[5\] \qed

4 Signature \((p, p, 3)\)

Let \(K\) be a totally real field. Recall the set \(S_K = \{\mathfrak{p} : \mathfrak{p} \) is a prime of \( K \) above \( 3\}\). Throughout this section we denote by \((a, b, c) \in \mathcal{O}_K^3\) a non-trivial, primitive solution of \(a^p + b^p = c^3\).

4.1 Frey Curve

For \((a, b, c) \in \mathcal{O}_K^3\) as described above we associate the following Frey elliptic curve defined over \(K\):
\[
E : Y^2 + 3cXY + a^pY = X^3.
\]
(14)

We compute the arithmetic invariants:
\[
\Delta_E = 3^3(a^3b)^p, \quad c_4 = 3^2c(9b^p + a^p) \quad \text{and} \quad j_E = 3^3c^3(9b^p + a^p)^3/(a^3b)^p.
\]

Lemma 32. Let \((a, b, c)\) be the non-trivial, primitive solution to the equation \(a^p + b^p = c^3\). Let \(E\) be the associated Frey curve \((14)\) with conductor \(N_E\). Then, for all primes \(q \notin S_K\), the model \(E\) is minimal, semistable and satisfies \(p | v_q(\Delta_E)\). Moreover
\[
N_E = \prod_{\mathfrak{p} \in S_K} \mathfrak{p}^{r_{\mathfrak{p}}} \prod_{\mathfrak{q} \mid ab} \mathfrak{q}, \quad N_p = \prod_{\mathfrak{p} \in S_K} \mathfrak{p}^{r_{\mathfrak{p}}} (15)
\]
where \(0 \leq r_{\mathfrak{p}} \leq r_{\mathfrak{q}} \leq 2 + 3v_p(3)\).

Proof. The proof follows exactly like the proof of Lemma \[25\] \qed

Lemma 33. Let \(K\) be a totally real field. There is some constant \(A_K\) depending only on \(K\), such that for any non-trivial, primitive solution \((a, b, c)\) of \(a^p + b^p = c^3\) the Frey curve given by \((14)\) is modular.

Proof. The proof follows exactly like the proof of Lemma \[26\] \qed

4.2 Images of Inertia

Lemma 34. Let \(q \nmid 3\) and let \((a, b, c)\) be a non-trivial, primitive solution to the equation \(a^p + b^p = c^3\) with the prime exponent \(p \geq 5\), such that \(q \nmid p\). Let \(E\) be the Frey curve in \((14)\). Then \(p \nmid \#\mathcal{E}_p(I_q)\).

Proof. The proof follows exactly like the proof of Lemma \[29\] \qed

Lemma 35. Let \(\mathfrak{p} \in S_K\) and \((a, b, c)\) with \(\mathfrak{p} | b\) and prime exponent \(p > 3v_p(3)\). Let \(E\) be the Frey curve in \((14)\) with \(j\)-invariant \(j_E\). Then \(E\) has potentially multiplicative reduction at \(\mathfrak{p}\) and \(p | \#\mathcal{E}_p(I_{\mathfrak{p}}\).

Proof. The proof follows exactly like the proof of Lemma \[30\] \qed
4.3 Level Lowering and Eichler Shimura

As in the previous section, the crucial level lowering theorem reads as follows:

**Theorem 36.** Let $K$ be a totally real number field and assume it has a distinguished prime $\mathfrak{P} \in S_K$. Then there is a constant $B_K$ depending only on $K$ such that the following hold. Suppose $(a,b,c) \in \mathcal{O}_K^3$ is a non-trivial, primitive solution to $a^p + b^p = c^p$ with prime exponent $p > B_K$ such that $\mathfrak{P} | b$. Write $E$ for the Frey curve (14). Then, there is an elliptic curve $E'$ over $K$ such that:

1. the elliptic curve $E'$ has good reduction outside $S_K$,
2. $\mathfrak{P}_{E,p} \sim \mathfrak{P}_{E',p}$,
3. $E'$ has a $K$-rational point of order 3,
4. $E'$ has multiplicative reduction at $\mathfrak{P}$ ($v_{\mathfrak{P}}(j_{E'}) < 0$ where $j_{E'}$ is the $j$-invariant of $E'$).

**Proof.** The proof follows exactly like the proof of Theorem \[31\] \[\square\]

4.4 Proof of Theorem \[9\]

**Proof.** So far, we have shown that for a primitive, non-trivial solution $(a,b,c)$ such that $\mathfrak{P} | b$ with a prime exponent $p$ we associate the Frey elliptic curve in (14). By Theorem \[36\] for $p > B_K$ we can find an elliptic curve $E'$ which is related to $E$ by $\mathfrak{P}_{E,p} \sim \mathfrak{P}_{E',p}$ and has a $K$-rational point of order 3. Hence by Theorem \[15\] (ii) we get a model

$$E': Y^2 + c'XY + d'Y = X^3$$

with arithmetic invariants $\Delta_{E'} = d'^3(c'^3 - 27d'^3)$ and $j_{E'} = \frac{c'^3(c'^3 - 24d'^3)^3}{d'^6(c'^3 - 27d'^3)}$.

Moreover, by Theorem \[31\] (i), we know that $E'$ has good reduction outside $S_K$ which implies that $v_q(j_{E'}) \geq 0$ for $q \notin S_K$. Therefore, $j_{E'} \in \mathcal{O}_{S_K}$. Consider $\lambda := \frac{c^3}{d'}$ and $\mu := \lambda - 27 = \frac{c^3 - 27d}{d'}$. Next, we need to show that $\lambda$ can be written as $\lambda = u\gamma^3$, where $u$ is an $S_K$-unit. By Lemma \[17\] (ii) applied to $E'$ we get that

$$L)\mathcal{O}_K = I^3J$$

where $J$ is an $S$-ideal.

Thus $[I]^3 = [J]$ as elements of the class group $\text{Cl}(K)$ and $[J] \in \langle [\mathfrak{P}] \rangle_{\mathfrak{P} \in S_K}$. This implies that $[I] \in \text{Cl}_{S_K}(K)[3]$ and by our assumption on $K$ that $\text{Cl}_{S_K}(K)[3]$ is trivial, we get that $[I] \in \langle [\mathfrak{P}] \rangle_{\mathfrak{P} \in S_K}$, i.e. $I := \gamma \tilde{I}$, where $\tilde{I}$ is an $S$-ideal and $\gamma \in \mathcal{O}_K$. Consequently,

$$L)\mathcal{O}_K = (\gamma)^3\tilde{I}^3J$$

where both $\tilde{I}$ and $J$ are $S$-ideals.

Finally, $(\frac{1}{\gamma})\mathcal{O}_K$ is an $S$-ideal, which implies that $u := \frac{1}{\gamma}$ is an $S$-unit. Now, by dividing $\mu + 27 = \lambda$ by $u$, we get

$$\alpha + \beta = \gamma^3 \quad \alpha := \frac{\mu}{u} \in \mathcal{O}_{S_K}^* \quad \beta := \frac{27}{u} \in \mathcal{O}_{S_K}^*$$

(16)

Now, suppose that there is some $\mathfrak{P} \in S_K$ that satisfies $|v_\mathfrak{P}(\frac{u}{\gamma})| \leq 3v_\mathfrak{P}(3)$. We will show that $v_\mathfrak{P}(j_{E'}) \geq 0$, contradicting Theorem \[36\] (iv) and hence we can
conclude the proof. By using \((16)\) we can rewrite the assumption \(|v_{\overline{P}}(\frac{3}{\mu})| \leq 3v_{\overline{P}}(3)\) in terms of the valuation of \(\mu\), using that \(\frac{3}{\mu} = \frac{3}{v_{\overline{P}}(\mu)}\):

\[
0 \leq v_{\overline{P}}(\mu) \leq 6v_{\overline{P}}(3).
\]

Note that \(j_{E'} = (\mu + 27)(\mu + 3)^3\mu^{-1}\), hence

\[
v_{\overline{P}}(j_{E'}) = v_{\overline{P}}(\mu + 27) + 3v_{\overline{P}}(\mu + 3) - v_{\overline{P}}(\mu).
\]

There are three cases according to the valuation of \(\overline{P}\) at \(\mu\):

**Case (1):** Suppose \(0 \leq v_{\overline{P}}(\mu) \leq v_{\overline{P}}(3)\). This implies that \(v_{\overline{P}}(\mu + 27) = v_{\overline{P}}(\mu)\) and \(v_{\overline{P}}(\mu + 3) \geq v_{\overline{P}}(\mu)\), thus \(v_{\overline{P}}(j_{E'}) \geq 0\).

**Case (2):** Suppose \(v_{\overline{P}}(3) < v_{\overline{P}}(\mu) \leq 3v_{\overline{P}}(3)\). This implies that \(v_{\overline{P}}(\mu + 27) \geq v_{\overline{P}}(3)\) and \(v_{\overline{P}}(\mu + 3) = v_{\overline{P}}(3)\), thus we get again \(v_{\overline{P}}(j_{E'}) \geq 0\).

**Case (3):** Suppose \(3v_{\overline{P}}(3) < v_{\overline{P}}(\mu) \leq 6v_{\overline{P}}(3)\). This implies that \(v_{\overline{P}}(\mu + 27) = 3v_{\overline{P}}(3)\) and \(v_{\overline{P}}(\mu + 3) = v_{\overline{P}}(3)\), thus we get one last time \(v_{\overline{P}}(j_{E'}) \geq 0\). All three cases lead to contradictions and hence we conclude the proof.

4.5 Proof of Theorem 11

**Proof.** We want to apply Theorem 9 with \(\overline{P} = P\) and \(S_K = \{\overline{P}\}\). As 3 is inert, we get \(v_{\overline{P}}(3) = 1\).

Now, let us consider the equation \(\alpha + \beta = \gamma^3\), with \(\alpha, \beta \in \mathcal{O}_{S_K}\). By scaling the equation by triple powers of 3 and swapping \(\alpha\) and \(\beta\) if necessary, we may assume \(0 \leq v_P(\beta) \leq v_P(\alpha)\) with \(v_P(\beta) \in \{0, 1, 2\}\). Also, we can assume that \(\beta\) is positive, otherwise we multiply everything by \(-1\).

**Case (1):** Suppose \(v_P(\beta) = 2\). If \(v_P(\alpha) \geq 3\), then \(v_P(\gamma^3) = v_P(\alpha + \beta) = 2\), which leads to a contradiction as \(v_P(\gamma^3)\) must be a multiple of 3. Thus, \(v_P(\alpha) = v_P(\beta) = 2\) and \(v_P(\frac{3}{\beta}) = 0 < 3\).

**Case (2):** Suppose \(v_P(\beta) = 1\). If \(v_P(\alpha) \geq 3\), then \(v_P(\gamma^3) = v_P(\alpha + \beta) = 1\), which leads to a contradiction as \(v_P(\gamma^3)\) must be a multiple of 3. Thus, \(v_P(\alpha) = v_P(\beta) = 1\) and \(v_P(\frac{3}{\beta}) = 0 < 3\).

**Case (3):** Suppose \(v_P(\beta) = 0\) with \(\beta\) not a cube. If \(v_P(\alpha) > 3\), then \(v_P(\gamma^3) = 0\) and \(\beta \equiv \gamma^3 \pmod{3^4}\). Consider the field extension \(L = K(\sqrt[3]{\beta}, \omega)\) of \(K(\omega)\). We will show that \(L\) is unramified at 3, hence contradicting \(3 \nmid h_{K(\omega)}\).

Consider the element \(\delta := \frac{\gamma^3 - \gamma^3}{3} + \frac{\gamma^3 - \beta^3}{3} X^2 - \frac{\gamma^3 - \beta^3}{3^2} X + \frac{\gamma^3 - \beta^3}{27}\). Its minimal polynomial is

\[
m_\delta(X) = X^3 + \frac{\gamma^3 - \beta^3}{3} X^2 - \frac{\gamma^3 - \beta^3}{3^2} X + \frac{\gamma^3 - \beta^3}{27}.
\]

This belongs to \(\mathcal{O}_K[X]\) and has discriminant

\[
\Delta = -2\gamma^3(\gamma^3 - \beta^3)^3 - 4\gamma^3(\gamma^3 - \beta^3)^5 + 4\gamma^6(\gamma^3 - \beta^3)^2 - 4\gamma^6(\gamma^3 - \beta^3)^3.
\]

We can deduce that \(\Delta \equiv -4\gamma^6 \pmod{3}\), proving that \(L\) is unramified at 3. Thus, we must have \(v_P(\alpha) \leq 3\), giving \(v_P(\frac{3}{\beta}) = v_P(\alpha) \leq 3\).

**Case (4):** Suppose \(\beta\) is a cube. By dividing everything through \(\beta\), we can assume that \(\beta = 1\). Then by the hypothesis of the theorem, we get \(v_P(\frac{3}{\beta}) = v_P(\alpha) \leq 3\).

All of the possible four cases lead to \(v_P(\frac{3}{\beta}) \leq 3 = 3v_P(3)\), so we can conclude the proof by Theorem 9.
4.6 Proof of Theorem 12

Proof. Note that \( d \equiv 2 \mod 3 \) gives that 3 is inert in the quadratic field \( K = \mathbb{Q}(\sqrt{d}) \), take \( \mathfrak{P} \) to be the unique prime above 3 and denote \( S_K = \{ \mathfrak{P} \} \). By Theorem 11 it is enough to check that every solution \((\alpha, \gamma) \in \mathcal{O}_K^* \times \mathcal{O}_{S_K} \) with \( v_{\mathfrak{P}}(\alpha) \geq 0 \) to the equation \( \alpha + 1 = \gamma^3 \) satisfies \( v_{\mathfrak{P}}(\alpha) \leq 3 \).

Assume by a contradiction that we have a solution \( \alpha \) to the above equation such that \( v_{\mathfrak{P}}(\alpha) > 3 \). This implies that \( v_{\mathfrak{P}}(\gamma) = 0 \), giving \( \gamma \in \mathcal{O}_K \).

Rearranging we get that \((\gamma - 1)(\gamma - \omega)(\gamma - \omega^2) = \alpha \) when viewed over \( L := K(\omega) \).

In the new field extension \( L \) we have that \((3)\mathcal{O}_L = (\omega - 1)^2\mathcal{O}_L \). We take \( p = (\omega - 1)\mathcal{O}_L \) and \( S_L = \{ p \} \). Denote \( x = \gamma - 1 \) and \( y = \gamma - \omega, z = \gamma - \omega^2 \) and observe that

\[
\begin{cases}
x - y = (\omega - 1) \\ y - z = \omega(\omega - 1)
\end{cases}
\]  

(17)

Note that \( x, y, z \in \mathcal{O}_{S_L} \) and they are factors of the \( S_K \)-unit \( \alpha \), hence they must be \( S_L \)-units.

Consider \( \tau \in \text{Gal}(L/K) \) such that \( \tau(\omega) = \omega^2 \). It is easy to see that

\[ \tau(x) = x, \quad \tau(y) = z \quad \text{and} \quad \tau(p) = p. \]

This implies that \( v_p(y) = v_p(z) =: r \). We will show that \( r = 1 \). First note that by (17) we get that \( 1 = v_p(\omega(\omega - 1)) = v_p(y - z) \geq r \). Suppose \( r \leq 0 \). Then \( v_p(x) \geq v_p(xyz) = v_p(\alpha) \geq 8 \) since \( 3^4 | \alpha \). Then, by using (17) again, we will get \( 1 = v_p(\omega - 1) = v_p(x - y) = r \leq 0 \), a contradiction. So, \( r \) must be exactly 1. As \( v_p(\alpha) = v_p(xyz) = 8 \), we must have \( v_p(x) = 6 \). Consider now

\[ u := \frac{x}{\omega - 1} \quad \text{and} \quad v = \frac{-y}{\omega - 1}. \]

By the above discussion, we will get that \( p^8 | u \) and \( v \in \mathcal{O}_L^* \). Denote \( F := \mathbb{Q}(\omega) \). As \( v \) is a unit, we must have

\[ \text{Norm}_{L/F}(v) \in \mathcal{O}_F^* = \langle \omega + 1 \rangle \]  

(18)

As \( u + v = 1 \), we get that \( v \equiv 1 \mod 3 \). Let \( \sigma \) be the generator of \( \text{Gal}(L/F) \). By noting that \( 3|\sigma(u) \), we get that \( \sigma(v) \equiv 1 \mod 3 \) and consequently \( \text{Norm}_{L/F}(v) = v\sigma(v) \equiv 1 \mod 3 \). This and (18) give \( \text{Norm}_{L/F}(v) = 1 \). Suppose that \( v \in \mathcal{O}_L^* \setminus \mathcal{O}_K^* = \omega\mathcal{O}_K^* \), then \( \omega|\text{Norm}_{L/F}(v) \) contradicting \( \text{Norm}_{L/F}(v) = 1 \). Thus \( v \in \mathcal{O}_K^* \), giving \( u = 1 - v \in \mathcal{O}_K \) which is a contradiction as \( u \) is a ratio of a \( K \)-integer and \( \omega - 1 \notin K \).

\[ \square \]

4.7 Proof of Theorem 13

We first need to prove some preliminary lemmas. Throughout this section, \( K \) denotes a totally real field of degree \( n \), \( L := K(\omega) \) and \( F := \mathbb{Q}(\omega) \). Moreover, \( K \) satisfies the conditions (i), (ii), (iii) and (iv) in the statement of Theorem 13. More precisely let \( q \) be the prime which totally ramifies in \( K \). Note that \( q \geq 5 \) so it is inert in \( F \). Denote \( \mathfrak{q} := (q)\mathcal{O}_F \) and take \( \mathfrak{q} \) to be the unique prime above \( q \) in \( L \), so that \( (q)\mathcal{O}_L = \mathfrak{q}\mathcal{O}_L \). Take \( \mathfrak{P} \) to be the unique prime above 3 in \( K \) and denote \( S_K = \{ \mathfrak{P} \} \). In \( L \) we have that \((3)\mathcal{O}_L = (\omega - 1)^2\mathcal{O}_L \). We take \( p = (\omega - 1)\mathcal{O}_L \) and \( S_L = \{ p \} \).
Lemma 37. Let $\lambda \in \mathcal{O}_L$, then there exists $\beta \in \mathbb{Z}[\omega]$ such that $\lambda \equiv b \mod q$ and
\[
\text{Norm}_{L/F}(\lambda) \equiv b^n \mod \tilde{q}.
\] (19)

Proof. Note that $\mathcal{O}_L/q\mathcal{O}_L \cong F_q(\omega) \cong \mathbb{Z}[\omega]/q\mathbb{Z}[\omega]$. Thus, there exists $b \in \mathbb{Z}[\omega]$ such that $\lambda \equiv b \mod q$. Let $\tilde{L}$ be the normal closure of $L$. Take $\sigma \in \text{Gal}(\tilde{L}/F)$. Note that
\[
(\sigma(q\mathcal{O}_L))^n = \sigma(q\mathcal{O}_L) = q\mathcal{O}_L = (q\mathcal{O}_L)^n.
\]
Thus, by the unique factorisation of ideals we get $\sigma(q\mathcal{O}_L) = q\mathcal{O}_L$. Moreover, by applying $\sigma$ to $\lambda \equiv b \mod q$ we get that $\sigma(\lambda) \equiv b \mod q\mathcal{O}_L$. Finally multiplying everything together we get
\[
\text{Norm}_{L/F}(\lambda) = \prod_{\sigma} \sigma(\lambda) \equiv b^n \mod q\mathcal{O}_L.
\]

As $\lambda \in \mathcal{O}_L$, it follows that $\text{Norm}_{L/F}(\lambda) \in \mathcal{O}_F$. Also $b^n \in \mathcal{O}_F$. Thus, $\text{Norm}_{L/F}(\lambda) - b^n \in \mathcal{O}_F \cap q\mathcal{O}_L = q\mathcal{O}_F$. Hence (19) holds.

Lemma 38. Suppose $\lambda \in \mathcal{O}_L^*$ and (ii) holds, i.e. $\gcd(n,q^2-1) = 1$. Then $(\lambda \mod q) \in \langle \omega + 1 \rangle = \{\pm 1, \pm(\omega + 1), \pm \omega\}$.

Proof. Let $b \in \mathbb{Z}[\omega]$ with $\lambda \equiv b \mod q$ as in Lemma 37. This gives us $\text{Norm}_{L/F}(\lambda) \equiv b^n \mod \tilde{q}$. However, as $\lambda$ is a unit, we must have
\[
\text{Norm}_{L/F}(\lambda) \in \mathcal{O}_F^* = \langle \omega + 1 \rangle.
\]

Putting these together we get that $b^n \equiv (\omega + 1)^i \mod \tilde{q}$. On the other hand, $b \in \mathcal{O}_F$ and maps to a non-zero element of $\mathcal{O}_F/\tilde{q}\mathcal{O}_F \cong \mathbb{F}_q^2$ thus $b^{q^2-1} \equiv 1 \mod \tilde{q}$. The assumption $\gcd(n,q^2-1) = 1$ is equivalent to the existence of integers $u,v$ so that $un + v(q^2-1) = 1$. It follows that
\[
b = (b^n)^u(b^{q^2-1})^v \equiv (\omega + 1)^iu \mod \tilde{q}.
\]

Thus, $(\lambda \mod q) \in \langle \omega + 1 \rangle = \{\pm 1, \pm(\omega + 1), \pm \omega\}$.

Proof of Theorem 13. We will reduce the problem to a simpler one as described in Section 14. More precisely, by using Theorem 11 and then rewriting the equation into an $S_K$-unit equation, we get that it is enough to show that there are no solutions to
\[
u + v = 1
\]
with $(u,v) \in \mathcal{O}_S^* \times \mathcal{O}_L^*$ such that $p^5|u$. We will show the slightly stronger statement that there are no solutions to (20) such that $9|u$.

Note that by (20) it follows that $v \equiv 1 \mod 9$. Thus $\sigma(v) \equiv 1 \mod 9$ for all conjugates $\sigma(v)$ of $v$ in $\text{Gal}(L/F)$, where $L$ is the normal closure of $L$. Hence, $\text{Norm}_{L/F}(v) \equiv 1 \mod 9$. As $v$ is a unit, we get $\text{Norm}_{L/F}(v) \in \mathcal{O}_F^* = \langle \omega + 1 \rangle$. Thus, the only possibility is
\[
\text{Norm}_{L/F}(v) = 1.
\] (21)

By Lemma 38 applied with $\lambda = v$ we get that
\[
(v \mod q) \in \langle \omega + 1 \rangle = \{\pm 1, \pm(\omega + 1), \pm \omega\}.
\] (22)
If \( v \equiv 1 \mod q \), then \( u = 1 - v \equiv 0 \mod q \), so \( q \mid u \), but this is false as \( u \) is an \( S_L \)-unit and \( p \) and \( q \) are different primes.

Thus \( (v \mod q) \in \{-1, \pm(\omega + 1), \pm\omega\} \). Then

\[
(\text{Norm}_{L/F}(v) \mod q) \in \{(-1)^n, (\pm(\omega + 1))^n, (\pm\omega)^n\}.
\]

(23)

Since \( \gcd(n, q^2 - 1) = 1 \) and \( q \geq 5 \) is a prime, it follows in particular that \( 2 \nmid n \) and \( 3 \nmid n \). This observation along with (23) proves that \( \text{Norm}_{L/F}(v) \mod q \not\equiv 1 \), contradicting (21).

\[\Box\]

5 S-unit equations and computability

Finally, we will describe how to algorithmically check the hypotheses in our two main Theorems 3 and 9 by studying how to compute solutions of certain (linear) \( S \)-unit equations over the number field \( K \), i.e. equations of the form

\[ ax + by = 1 \text{ where } a, b \in K^* \text{ with solutions } x, y \in O_S^*. \]

Throughout this section \( S \) denotes a finite set of prime ideals of \( K \).

**Theorem 39** (Siegel). Let \( K \) be a number field and \( S \subset O_K \) a finite set of prime ideals, and let \( a, b \in K^* \). Then, the equation

\[ ax + by = 1 \]

has only finitely many solutions in \( O_S^* \).

**Remark 40.** Methods of effectively computing solutions to \( S \)-unit were pioneered by De Weger’s famous thesis [32] for \( K = \mathbb{Q} \). His method of lattice approximation reduction algorithms was later generalized for all number fields by others, see for example Smart’s [28]. Moreover, an \( S \)-unit solver for \( a = b = 1 \) has been implemented in the free open-source mathematics software, Sage by A. Alvarado, A. Koutsianas, B. Malmeskog, C. Rasmussen, D. Roe, C. Vincent, M. West in [1].

We will now study two non-linear equations involving \( S \)-units which are going to play a crucial role in checking the hypothesis of our Theorems 3 and 9. Let \( K \) be a number field and \( S \) a finite set of prime ideals. Consider the equation

\[ \alpha + \beta = \gamma^i, \alpha, \beta \in O_S^*, \gamma \in O_S. \]

There is a natural scaling action of \( O_S^* \) on the solutions. We regard two solutions \( (\alpha_1, \beta_1, \gamma_1) \sim_1 (\alpha_2, \beta_2, \gamma_2) \) as equivalent if there is some \( \epsilon \in O_S^* \) such that \( \alpha_2 = \epsilon^i \alpha_1, \beta_2 = \epsilon^i \beta_1 \) and \( \gamma_2 = \epsilon \gamma_1 \).

**Theorem 41.** Let \( K \) be a number field and \( S \) a finite set of prime ideals. Consider the equation

\[ \alpha + \beta = \gamma^i, \alpha, \beta \in O_S^*, \gamma \in O_S. \]

For \( i = 2, 3 \), the equation has a finite number of solutions up to the equivalence relation \( \sim_1 \). Moreover, these are effectively computable.
Proof. Let \( i = 2 \) and \( (\alpha, \beta, \gamma) \in \mathcal{O}_S^* \times \mathcal{O}_S^* \times \mathcal{O}_S^* \) a solution to \( \alpha + \beta = \gamma^2 \). By Dirichlet Unit Theorem \( \mathcal{O}_S^* \) is finitely generated, and hence \( \mathcal{O}_S^*/(\mathcal{O}_S^*)^2 \) is finite. Fix a set of representatives \( \beta_1, \beta_2, ..., \beta_l \). We may scale our solution so that \( \beta \in \{\beta_1, \beta_2, ..., \beta_l\} \). Thus, there are finitely many choices of \( \beta \) (up to \( \sim_2 \) equivalence) and we fix one of them. We next show that for each such choice of \( \beta \), there is a finite number of distinct \( \alpha \), and thus, a finite number of triples \( (\alpha, \beta, \gamma) \) up to \( \sim_2 \) equivalence.

We rewrite the equation as
\[
(\gamma + \sqrt{\beta})(\gamma - \sqrt{\beta}) = \alpha \text{ over } L. 
\]

where \( L := K(\sqrt{\beta}) \). Denote by \( x := \gamma + \sqrt{\beta}, y := \gamma - \sqrt{\beta} \) and consider \( S' := \{\mathfrak{P}_L \text{ prime of } L : \mathfrak{P}_L|\mathfrak{P}_K, \text{ for some } \mathfrak{P}_K \in S\} \). We claim that
\[
\frac{1}{2\sqrt{\beta}} x - \frac{1}{2\sqrt{\beta}} y = 1
\]
and \( x, y \) are both \( S' \)-units in \( L \). By Theorem 39, we get finitely many \( x, y \), and thus finitely many possibilities for \( \alpha = xy \). Moreover, these are computable by Remark 40. The claim that \( x, y \) are \( S' \)-units follows by considering the valuation of the product in (24) at the primes of \( L \) outside the set \( S' \). Then, we use the definition of \( S' \) and the fact that \( \alpha \) is an \( S \)-unit in \( K \).

For \( i = 3 \), the argument works in a similar manner. Fixing a representative \( \beta \) of the finite quotient \( \mathcal{O}_S^*/(\mathcal{O}_S^*)^3 \), we rewrite the equation as
\[
(\gamma - \sqrt[3]{\beta})(\gamma - \omega \sqrt[3]{\beta})(\gamma - \omega^2 \sqrt[3]{\beta}) = \alpha \text{ over } L 
\]

where \( L = K(\omega, \sqrt[3]{\beta}) \) and \( \beta \neq -1 \). Denote by \( x := \gamma - \sqrt[3]{\beta}, y := \gamma - \omega \sqrt[3]{\beta}, S' := \{\mathfrak{P}_L \text{ prime of } L : \mathfrak{P}_L|\mathfrak{P}_K, \text{ for some } \mathfrak{P}_K \in S\} \).

We make the quick note that for \( \beta = -1 \) we take \( x := \gamma + 1, y = \gamma + \omega, L := K(\omega) \) and the rest of the argument follows the same, so it is omitted.

As in the previous case, by examining the product in (25) we get that \( x, y \) are both \( S' \)-units in \( L \) and
\[
\frac{1}{(\omega - 1) \sqrt[3]{\beta}} x - \frac{1}{(\omega - 1) \sqrt[3]{\beta}} y = 1.
\]

Thus, by Theorem 39, Remark 40 and the observation that \( \alpha = xy(y - \omega(\omega - 1) \sqrt[3]{\beta}) \), we can conclude the proof.

\[\square\]

**Remark 42.** In the hypotheses of Theorems 3 and 9 one needs to examine the local behaviour of \( \frac{x}{2} \) which, by the above theorem, can only take a finite, computable number of values.

**References**


