Mary Somerville's Diophantine Equations

1 Introduction

Mary Somerville (born Mary Fairfax, 26 December 1780 - 29 November 1872) is celebrated as a Scottish polymath, whose knowledge spanned mathematics, astronomy, physics, molecular biology, geography and philosophy.

Somerville's mathematical translations and reflections on science made her "the queen of science"[1] or "the principal representative of science"[2] in the nineteenth century press. More recently, her portrait has featured on the current Scottish £10 note together with a quote from one of her most notable books "On the Connection of the Physical Sciences".

Mary Somerville's contributions to the scientific world are particularly remarkable in the context of a time and age when women were not receiving formal education in STEM subjects and were precluded from holding official positions in academia, thus their contributions to science were almost invisible. Nevertheless, Somerville is celebrated as one of the "most distinguished astronomers and philosophers" [3] of the nineteenth century alongside men contemporaries such as John Herschel, Alexander von Humboldt and David Brewster[4]. The later wrote that Mary Somerville was "certainly the most extraordinary woman in Europe - a mathematician of the very first rank with all the gentleness of a woman"[3]. This suggests that she was accepted in the prominent scientific societies not only for her intellectual abilities, but also for meeting the feminine expectations of good manners, being a wife and a mother.

Despite being actively discouraged in her intellectual endeavours by her family from a very young age, Somerville taught herself sufficient Latin to read the books in her home library in Burntisland, Scotland. During her early life, she sought every opportunity to get access to mathematical books and self-thought elementary algebra and geometry from Euclid's *Elements*

[1] The Morning Post, 29 November 1872

[2] Belfast Newsletter 1872

[3] Secord, 2004, Volume 1, II.29, [5]

[4] Stenhouse,2021, *Mary Somerville: Being and Becoming a Mathematician.*, Chapter 1, [7]

and *Algebra* by John Bonnycastle giving her the basis to understand astronomy and other sciences[5].
After a brief period of living in London during her first marriage (1804-1807), she returned to Scotland after her husband, Samuel Greig died. With the financial freedom of a widow, she continued her mathematical studies, learning about spherical trigonometry, conic sections and later reading Isaac Newton's *Principia* for the first time[6]. This marks the period when she started engaging with the scientific community. John Playfair, professor of natural philosophy at University of Edinburgh, encouraged her studies, and through him she began a correspondence with William Wallace (Playfair's former pupil and Professor of Mathematics at the Royal Military College), with whom she discussed mathematical problems[7].

This marked a turning point in Somerville's mathematical career. She started submitting solutions to mathematical puzzles proposed in *The New Series of the Mathematical Repository* under the pseudonym "A Lady"[8], becoming an active contributor of the mathematical community for the first time. She soon got recognition for her mathematical abilities, culminating with a silver medal in 1811, engraved with her real name, for the solution of a "Prize Question" featuring a Diophantine equation[9]. We will describe this problem along with Somerville's solution in Section 2 through the prism of modern algebraic geometry.

To be continued:
Second marriage + scientific community
Translation of "Mecanique Celeste", "On the Connection of the Physical Sciences" and meeting with Laplace.
Later years+other scientific writings+correspondence with Babbage
Old Age+Recollections

2 A Diophantine Equation

We shall now study the Prize Question which secured Mary Somerville a silver medal in 1811. It reads as follows:

XX. PRIZE QUESTION 310, by Mr. W. Wallace.[10]

[5] as described in her *Recollections*, [6]

[6] as described in her *Recollections*, [6]

[7] Stenhouse,2021, *Mary Somerville: Being and Becoming a Mathematician.*, Chapter 1, [7], O'Connor, Robertson *Mary Fairfax Greig Somerville*, [3]

[8] It was common practice to publish solutions under a pseudonym, but not submit them anonymously. Stenhouse,2021, *Mary Somerville: Being and Becoming a Mathematician.*, Chapter 1, [7]

[9] Prize Question 310, posed in Volume 3 of *The New Series of the Mathematical Repository.*

[10] Volume 3 of *The New Series of the Mathematical Repository* [2]

Figure 1: The medal awarded to Mary Somerville for her solution of Prize Question. The medal is now held at Somerville College, Oxford and the inscription reads: Maria Greig, L.M.D; PALMAM QUI. MERUIT FERAT; T. Leybourn. L.M.D stands for Libens merito dedicavit, and the Latin loosely translates as 'deservedly won; let they who have earned the palm, bear it'. The palm signifies victory. Credits for figure and translation: Stenhouse, 2021, *Mary Somerville: Being and Becoming a Mathematician.*

Find such integer values of x, y, z as shall render the three expressions $x^2 + axy + y^2$, $x^2 + a'xz + z^2$, $y^2 + a''yz + z^2$ squares, a, a', a'' being given numbers.

Two solutions were published: the first one by "a Lady" (Mary Somerville) and the second one by "Mr. Lowry"(R. M. College).

2.1 Mary Somerville's solution

This section follows Mary Somerville's solution as published in *New Series of The Mathematical Repository: Volume 3* [2]. She starts her solution by assuming that x, y and z are parameterized by two new variables m, n in the following way:

$$\begin{cases} x = an^2 \pm 2mn, \\ y = m^2 - n^2, \\ z = a''n^2 + 2mn. \end{cases} \tag{1}$$

These make the first and third expression into a square as Somerville writes:

$$\begin{cases} x^2 + axy + y^2 = (m^2 \pm amn + n^2)^2, \\ y^2 + a''yz + z^2 = (m^2 + a''mn + n^2)^2. \end{cases} \tag{2}$$

Next, in order to make the last expression into a square, Somerville denotes by $r = an \pm 2m$ and by $s = a''n + 2m$ and by reparameterizing $r = p^2 - q^2$, $s = a'q^2 + 2pq$, she finally gets that

$$x^2 + a'xz + z^2 = n^2(p^2 + a'pq + q^2)^2.$$

The rest of her solution consisits in writing m, n in terms of the parameters p, q which after several simplifications leads to the following formulae:

1. if $x = an^2 + 2mn$

$$\begin{cases} m = a''p^2 - 2apq - (a'a + a'')q^2, \\ n = 2(a'+1)q^2 + 4pq - 2p^2. \end{cases} \tag{3}$$

2. if $x = an^2 - 2mn$

$$\begin{cases} m = -a''p^2 + 2apq - (a'a + a'')q^2, \\ n = 2(a'-1)q^2 + 4pq + 2p^2. \end{cases} \tag{4}$$

Somerville notices that only (4) gives non-trivial solutions when $a = a' = a''$. The article ends with three computed examples.

2.2 Note on Mr. Lowri's solution

At a first glance, Somerville's choices in the paramtrizations of x, y and z might seem surprising. We notice that the next published solution uses roughly the same parameterization up to a permutation of x, y, z. It turns out that Mr. Lowri's solution give us a deeper insight into Somerville's one, and perhaps into a common method of solving simultaneous quadratic equations of the nineteenth century in the British mathematical community. Mr. Lowri starts by dividing the first two expressions by x^2 and the third one by z^2. Hence, the Prize Question becomes equivalent to showing that the following are squares:

$$\begin{cases} 1 + a\frac{y}{x} + \frac{y^2}{x^2} \\ 1 + a'\frac{z}{x} + \frac{z^2}{x^2} \\ 1 + a''\frac{z}{y} + \frac{z^2}{y^2} \end{cases} \tag{5}$$

Next, Mr. Lowry assumes that $1 \mp \frac{vy}{ux}$ is a root for the first expression and $1 \mp \frac{mz}{nx}$ is a root for the second. Then he gets

$$\frac{y}{x} = \frac{u(au \pm 2v)}{v^2 - u^2}, \frac{z}{x} = \frac{n(a'n \pm 2m)}{m^2 - n^2}.$$

Therefore $\frac{z}{y} = \frac{n(a'n \pm 2m)(v^2-u^2)}{u(au \pm 2v)(m^2-n^2)}$. Lowri notices that this expression can be simplified if one assumes $v^2 - u^2 = m^2 - n^2$ and he writes down the most obvious way to do this: by letting $m = v$ and $n = u$. It turns out that this simplification gives exactly Somerville's first parameterization (1) up to a permutation of x, y, z. From here his method is very similar to Somerville's. This can also be seen by examining the common examples they both give: when $a = 1, a' = 3, a'' = 7$ they get the same result $x = 24, y = 11, z = 56$ and when $a = a' = a'' = -1$ they get the same result up to a sign $x = 832, y = 667, z = 520$.

Later on, Mr. Lowri considers a generalisation of this method, where he assumes $v^2 - u^2 = m^2 - n^2$ by letting $v = \frac{r^2+s^2}{2rs}m + \frac{r^2-s^2}{2rs}n$ and $u = \frac{r^2-s^2}{2rs}m + \frac{r^2+s^2}{2rs}n$. Perhaps not surprising, this leads to a more complicated formula for the solutions. Moreover, Mr Lowri proposes an even more general solution which we will discuss in section 2.5.

He ends the discussion by giving solutions to the generalized problem that asks for simultaneous zeros of $x^2 + axy + by^2$, $x^2 + a'xz + b'z^2$, and $y^2 + a''yz + b''z^2$.

2.3 Prize Question Solution via Algebraic Geometry

We can rewrite this statement in the following way. Find such integer values of x, y, z such that:

$$\begin{cases} x^2 + axy + y^2 = w^2 \\ x^2 + a'xz + z^2 = w'^2 \\ y^2 + a''yz + z^2 = w''^2. \end{cases} \tag{6}$$

As the above equations are homogeneous, a natural way to view their common solutions is as a projective variety in $\mathbb{P}^5$. More concretely, consider the following homogeneous polynomials of degree 2:

$$\begin{cases} f_1(x, y, z, w, w', w'') = x^2 + axy + y^2 - w^2 \\ f_2(x, y, z, w, w', w'') = x^2 + a'xz + z^2 - w'^2 \\ f_3(x, y, z, w, w', w'') = y^2 + a''yz + z^2 - w''^2. \end{cases} \tag{7}$$

Define the projective variety:

$$S_{a,a',a''} := \{\mathbf{p} = [x : y : z : w : w' : w''] \in \mathbb{P}^5 : f_1(\mathbf{p}) = 0, f_2(\mathbf{p}) = 0, f_3(\mathbf{p}) = 0\}.$$

Consider the set of rational points $S_{a,a',a''}(\mathbb{Q}) = S_{a,a',a''} \cap \mathbb{P}^5(\mathbb{Q})$. For the rest of this section we would like to describe the geometry of $S_{a,a',a''}(\mathbb{Q})$.

Proposition 2.1 (Dimension)**.** *The projective variety $S_{a,a',a''}$ has dimension 2.*

Proof. The dimension of $S_{a,a',a''}$ can be computed as the degree of the Hilbert polynomial of the corresponding homogeneous ideal $I = I(S_{a,a',a''}) := \langle f_1, f_2, f_3 \rangle$. The Hilbert polynomial can be computed to be $4x^2 + 2$, hence the dimension is 2. □

Proposition 2.2 (Irreducibility)**.** *The number of irreducible components depend on the choices of (a, a', a'') in the following way:*

1. *If $a \neq \pm 2$, $a' \neq \pm 2$ and $a'' \neq \pm 2$, then $S_{a,a',a''}$ is irreducible.*

2. *If exactly one of a, a' or a'' is ± 2, then $S_{a,a',a''}$ has 2 irreducible components, that intersect in a one dimensional variety.*

3. *If exactly two of a, a' or a'' are ± 2, then $S_{a,a',a''}$ has 4 irreducible components, regardless of what combination of signs we choose.*

4. *If all of a, a' and a'' are ± 2, then $S_{a,a',a''}$ has 8 irreducible components, regardless of what combination of signs we choose.*

Proof. We first note that $S = V(f_1, f_2, f_3) = V(f_1) \cap V(f_2) \cap V(f_3)$. We will prove the following.

Claim 1. The projective variety $V(f_1)$ is

1. either reducible, with exactly two irreducible components when $a = \pm 2$;

2. or irreducible when $a \neq \pm 2$.

By symmetry, the claim is true for $V(f_2)$ and $V(f_3)$. Then, the proposition follows by counting the number of irreducible components in the intersection $V(f_1, f_2, f_3) = V(f_1) \cap V(f_2) \cap V(f_3)$.

Proof of Claim 1.1. If $a = \pm 2$, then

$$f_1 = x^2 \pm 2xy + y^2 - w^2 = (x \pm y - w)(x \pm y + w)$$

hence giving $V(f_1) = V(x \pm y - w) \cup V(x \pm y + w)$. Note that the two components are irreducible as they have degree 1.
If $a \neq \pm 2$ we consider f_1 as a polynomial in w by fixing the other variables. This factorizes if and only if $x^2 + axy + y^2$ is a square. Consider now the degree two the polynomial

$$p(x, y) = x^2 + axy + y^2.$$

By fixing y, we get that $x^2+axy+y^2$ is a square if and only if its discriminant $\Delta_p = a^2y^2 - 4y^2$ is a square. This is equivalent to $a^2 - 4 = t^2$, for some integer t. This is equivalent to solving $(a - t)(a + t) = 4$ in the ring of integers. By considering the parities of the factorisation of 4 we get that the only possibilities are

$$\begin{cases} a - t = \pm 2 \\ a + t = \pm 2 \end{cases}$$

These give $a \pm 2$, a contradiction. Hence f_1 does not factorize if $a \neq \pm 2$.

□

Proposition 2.3 (Singularities)**.** *For any integer triple (a, a', a''), $S_{a,a',a''}$ is singular and the singularities depend on the choices of (a, a', a'') in the following way:*

1. *If $a \neq \pm 2$, $a' \neq \pm 2$ and $a'' \neq \pm 2$, then the singularities of $S_{a,a',a''}$ consist in 12 isolated singularities given by:*

$$\begin{aligned}&[-1:0:0:-1:1:0],[-1:0:0:1:1:0],[0:-1:0:-1:0:1],\\&[0:-1:0:1:0:1],[0:0:-1:0:-1:1],[0:0:-1:0:1:1],\\&[0:0:1:0:-1:1],[0:0:1:0:1:1],[0:1:0:-1:0:1],\\&[0:1:0:1:0:1],[1:0:0:-1:1:0],[1:0:0:1:1:0].\end{aligned} \tag{8}$$

2. *If $a = \pm 2$, $a' \neq \pm 2$ and $a'' \neq \pm 2$ with $a' \neq \mp a''$ (or any other permutation of (a, a', a'') which respects these conditions), then the singularities of $S_{a,a',a''}$ consist in:*

   (a) *points on an 1-dimensional variety (corresponding to the intersection of the 2 irreducible components);*

   (b) *8 isolated points given by:*

$$\begin{aligned}&[-1:0:0:-1:1:0],[-1:0:0:1:1:0],[0:-1:0:-1:0:1],\\&[0:-1:0:1:0:1],[0:1:0:-1:0:1],\\&[0:1:0:1:0:1],[1:0:0:-1:1:0],[1:0:0:1:1:0].\end{aligned} \tag{9}$$

3. *If $a = \pm 2$, $a' \neq \pm 2$ and $a'' \neq \pm 2$ with $a' = \mp a''$ (or any other permutation of (a, a', a'') which respects these conditions), then the singularities of $S_{a,a',a''}$ consist in:*

   (a) *points on two distinct 1-dimensional varieties (corresponding to the intersection of the 2 irreducible components) which intersect in two points if $(a')^2 - 4$ is a square or are disjoint otherwise;*

   (b) *the 8 isolated points described in (9).*

4. *If $a = \pm 2$, $a' = \pm 2$ and $a'' \neq \pm 2$ with (or any other permutation of (a, a', a'') which respects these conditions, regardless of the signs we choose), then the singularities of $S_{a,a',a''}$ consist in:*

   (a) *points on four distinct 1-dimensional varieties V_1, V_2, V_3, V_4 (corresponding to the intersection of the 4 irreducible components) which intersect as follows. For every $i \neq j$, $V_i \cap V_j = \bigcap_i V_i$ which intersect in two points if $(a' + a'')$ is a square or are dijoint otherwise.*

   (b) *4 isolated points given by:*

$$\begin{aligned}&[-1:0:0:-1:1:0],[-1:0:0:1:1:0],\\&[1:0:0:-1:1:0],[1:0:0:1:1:0].\end{aligned} \tag{10}$$

5. *If $a = \pm 2$, $a' = \pm 2$ and $a'' = \pm 2$, then the singularities of $S_{a,a',a''}$ consist in 12 lines $L_1, \cdots L_{12}$ (corresponding to the intersection of the irreducible components) which intersect as follows.*

   (a) *If we have $a = 2$, $a' = 2$ and $a'' = 2$ or two out of a, a', a'' is -2 and the other one is 2, they intersect as in the graph below:*
   To be drawn!

   (b) *In any other case, for $i \neq j$, $L_i \cap L_j = \bigcap_i L_i$ which intersect in the single point $[1 : -1 : -2 : 0 : 0 : 0]$ (or 3-cycle permutations of this point).*

Proof. We are going to use the Jacobian condition for smoothness to compute the singularities of the projective variety $S_{a,a',a''}$. In our case, this reduces to finding the points $\mathbf{p} \in S_{a,a',a''}$, such that the rank of the Jacobian matrix evaluated at $\mathbf{p}$ is less than 3, in particular all 3×3 minors of $J(\mathbf{p})$ vanish, where $J(\mathbf{p})$ is the Jacobian matrix evaluated at $\mathbf{p} = [x : y : z : w : w' : w'']$.

$$J(\mathbf{p}) = \begin{pmatrix} 2x + ay & ax + 2y & 0 & -2w & 0 & 0 \\ 2x + a'z & 0 & a'x + 2z & 0 & -2w' & 0 \\ 0 & 2y + a''z & a''y + 2z & 0 & 0 & -2w'' \end{pmatrix}$$

By computing the 3×3 minors of $J(\mathbf{p})$ and imposing that they are 0, we get different solutions according to whether a, a', a'' are ± 2, as described in the statement of the proposition. □

The next proposition tells us that the singularities described above are the "nicest" possible ones, when we restrict to irreducible components. We checked that this is indeed the case using MAGMA [1].

Proposition 2.4. *Each of the singularities described in Proposition 2.3 is at most a double rational point inside its irreducible component.*

Theorem 2.5. *Let k be a field of characteristic 0. Assume that X is a surface over k of one of the following three types:*

1. *a quartic surface in $\mathbb{P}^3_k$,*
2. *an intersection of a cubic and a quadric hypersurface in $\mathbb{P}^4_k$,*
3. *an intersection of three quadrics in $\mathbb{P}^5_k$.*

Furthermore, assume that all singularities of X are rational double points. Then the minimal regular model of X is a K3-surface.

Proof. This is a well-known result. A proof can be found in the Appendix of [4]. □

Corollary 2.6. *The minimal regular model of $S_{a,a',a''}$ in the most general case (i.e. when $a \neq \pm 2$, $a' \neq \pm 2$ and $a'' \neq \pm 2$) is a K3-surface.*

Proof. By Proposition 2.2 we get that $S_{a,a',a''}$ is irreducible, and then by Proposition 2.4 it has only double rational points. Thus by Theorem 2.5 applied to an intersection of three quadrics in $\mathbb{P}^5$ we get the desired result. □

In the cases where $S_{a,a',a''}$ is reducible, we get that each irreducible component is a rational surface by MAGMA [1].

2.4 Back to Somerville's solution

In the language of Algebraic Geometry, Somerville's solution proposes a rational map

$$F \colon \mathbb{P}^1 \dashrightarrow S_{a,a',a''}$$

defined over $\mathbb{Q}$, given as a composition of a couple explicit rational maps. More precisely, $F = f \circ g$, where f is given by (1) and g is given by (3) or (4) depending on the choice of a, a', a''.

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r, dashed, "g"] \arrow[rr, dashed, bend left, "F"] & \mathbb{P}^1 \arrow[r, dashed, "f"] & S_{a,a',a''} \end{tikzcd}$$

$$[p,q] \xrightarrow{\ g\ } [m,n] \xrightarrow{\ f\ } [x,y,x,w,w',w'']$$

This turns out to be a birational map onto its image, as both of f and g have rational inverses.

We check that in the general case, Somerville's solution avoids the singularities of $S_{a,a',a''}$ described in Proposition 2.3. This implies that Somerville's curve is birationally equivalent to a rational curve on the minimal resolution of $S_{a,a',a''}$. In conclusion, for the general case, a modern algebraic geometer would say that Somerville constructs a rational curve on the K3-surface $S_{a,a',a''}$.

2.5 Alternative Rational Curve

Inspired by Mr. Lowri's comments, we might want to generalize Somerville's solution in (1) in the following way:

$$\begin{cases} x = (au^2 \pm 2uv)(m^2 - n^2), \\ y = (m^2 - n^2)(v^2 - u^2), \\ z = (a''n^2 + 2mn)(v^2 - u^2). \end{cases} \tag{11}$$

As in Somerville's solution, these choices make sure that the first and third expressions are squares, giving:

$$\begin{cases} x^2 + axy + y^2 = (v^2 \pm auv + u^2)^2(m^2 - n^2)^2, \\ y^2 + a''yz + z^2 = (m^2 + a''mn + n^2)^2(v^2 - u^2)^2. \end{cases} \tag{12}$$

Now, we have to impose that also $x^2 + a'xz + z^2$ is also a square. After writing this condition in terms of m, n, u, v and simplifying calculations we note that the following choice for u and v gives $x^2 + a'xz + z^2$ as a square:

$$\begin{cases} u = 4n(a''n + 2m)(an(a''n + 2m) - a'(m^2 - n^2)), \\ v = (4 - a'^2)(m^2 - n^2)^2 + 2aa'n(a''n + 2m)(m^2 - n^2) - (4 + a^2)n^2(a''n + 2m)^2. \end{cases} \tag{13}$$

We make the quick note that Somerville's solution takes $u = n, v = m$ and imposes constrains on m, n as discussed in Section 2.1. In the language of algebraic geometry this gives another rational map

$$G\colon \mathbb{P}^1 \dashrightarrow S_{a,a',a''}$$

defined over $\mathbb{Q}$ which is birational onto its image by the same reasons as before.

3 Another Diophantine equation

We move our attention to a second Diophantine problem solved by Somerville in Volume 3 of *The New Series of the Mathematical Repository* [2]. It reads as follows:

I. QUESTION 311, by Mr. John Hynes, Dublin.
To divide a given square number n^2, into two such parts that the sum of their squares and the sum of their cubes may both be rational squares.

Three solutions were published: the first one by "a Lady" (Mary Somerville), the second one by by Mr. Cunliffe, (R. M. College) and the third one by "Mr. Lowry" (R. M. College).

3.1 Somerville's Solution

Somerville begins her solution by denoting the first part by x. Hence, in her notation, the problem asks to find such x, n that make $x^2 + (n - x)^2$ and $x^3 + (n - x)^3$ into squares. Expanding she gets that $x^2 + (n - x)^2 = n^4 - 2n^2x + 2x^2$ and $x^3 + (n - x)^3 = n^2(n^4 - 3n^2x + 3x^2)$ must be squares and notices that the later reduces to $n^4 - 3n^2x + 3x^2$ being a square.

Consequently, she assumes that

$$\begin{cases} -3n^2 + 3x = -2n^2px + p^2x^2 \\ -2n^2 + 2x = -2n^2q + q^2x \end{cases} \tag{14}$$

which will give $n^4 - 2n^2x + 2x^2 = (n^2 - qx)^2$ and $n^4 - 3n^2x + 3x^2 = (n^2 - px)^2$. Then she solves both equations in (14) for x and equalizes them, and hence getting

$$\frac{3n^2 - 2n^2p}{3 - p^2} = \frac{2n^2 - 2n^2q}{2 - q^2}.$$

Here, she makes the strict assumptions that $3 - 2p = 2 - 2q$ and $3 - p^2 = 2 - q^2$, which gives her $q = \frac{3}{4}$ and $p = \frac{5}{4}$. Finally, these values give the desired values $x = \frac{8n^2}{23}$ and $n^2 - x = \frac{15n^2}{23}$. The last part of her solution consists in a few examples for different n.

3.2 Somerville's solution - the modern perspective

We will first make the substitution $y = n^2$. We can rewrite the problem as finding the common zeros of the following two equations:

$$\begin{cases} y^2 - 2yx + 2x^2 = w^2 \\ y^2 - 3yx + 3x^2 = w'^2 \end{cases} \tag{15}$$

Let's denote by C the vanishing set of these two equations in $\mathbb{P}^3$. It can be shown (for example in MAGMA [1]) that this is a non-singular curve of genus 1 which has an obvious rational point $[0 : 1 : 1 : 1]$. Thus C is an elliptic curve.

Somerville's solution computes a rational point on this elliptic curve, namely $S := [8 : 23 : 17 : 13]$. Note that we don't have to worry about $y = n^2$ as we can just assume $y = n = 1$.

A modern algebraic geometer would perhaps want to investigate how other points on this curve look like and maybe find a bigger set of solutions to the equation. We computed with MAGMA [1] an elliptic curve E in Weierstras form which is birationally equivalent to C and has $[0 : 1 : 1 : 1]$ as the point at infinity:

$$E : y^2 = x^3 + 8x^2 + 12x.$$

Somerville's point $S = [8 : 23 : 17 : 13]$ corresponds to the affine point $S_E = (48, 360)$ on E. We can now make use of the group law on E. We compute using MAGMA that the order of S infinite. Thus, it is a source of infinite points on our curve, namely $[k]S_E$ for k a natural number. Consequently, it gives rise to infinitely many solutions to the equation.

For example, $[2]S_E = (36481/3600, 9620479/216000)$ which corresponds to the point $[10130640 : 18240049 : 12976609 : 9286489]$ on the initial curve C. Now, since we assumed that $n = y = 1$, we read this projective point as

$$[\frac{10130640}{18240049}, 1, \frac{12976609}{18240049}, \frac{9286489}{18240049}].$$

A simple check shows that $n = 1$ and $x = \frac{10130640}{18240049}$ are solutions to the initial problem. Moreover, $n = k^2$ and $x = \frac{10130640}{18240049}k^2$ will also work as solutions.

References

[1] Bosma W., Cannon J., Playoust C. *The Magma algebra system. I.The user language. J. Symbolic Comput.*

[2] Leybourn, Thomas. *New Series of The Mathematical Repository: Volume 3.* Digitalized by Google here: `https://play.google.com/books/reader?id=Ja42AAAAMAAJ&pg=GBS.RA1-PA150&hl=en`

[3] O'Connor, John J.; Robertson, Edmund F. *Mary Fairfax Greig Somerville.* MacTutor History of Mathematics archive, University of St Andrews.

[4] Pannekoek, René. *Diophantus Revisited: On Rational Surfaces and K3 Surfaces in the "Arithmetica"* .

[5] Secord, James A. *Collected Works of Mary Somerville.* Thoemmes Continuum, 2004.

[6] Somerville, Martha (Mary's daughter). *Personal Recollections, from Early Life to Old Age, of Mary Somerville: With Selections from Her Correspondence.*

[7] Stenhouse, Brigitte. *Mary Somerville: Being and Becoming a Mathematician.* PhD thesis The Open University, 2021.