Mary Somerville's Diophantine Equations

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1 Introduction

Figure 1: Photo credit: Somerville College, University of Oxford.

Mary Somerville (born Mary Fairfax, 26 December 1780 - 29 November 1872) is celebrated as a Scottish polymath, whose knowledge spanned mathematics, astronomy, physics, molecular biology, geography and philosophy.

Somerville's mathematical translations and reflections on science made her "the queen of science"[1] or "the principal representative of science"[2] in the nineteenth century press. More recently, her portrait has featured on the current Scottish £10 note together with a quote from one of her most notable books *On the Connection of the Physical Sciences.*

Mary Somerville's contributions to the scientific world are particularly remarkable in the context of a time and age when women were not receiving formal education in STEM subjects and were precluded from holding official positions in academia, thus their contributions to science were almost invisible. Nevertheless, Somerville is celebrated as one of the "most distinguished astronomers and philosophers" [3] of the nineteenth century alongside men contemporaries such as John Herschel, Alexander von Humboldt and David Brewster[4]. The later wrote that Mary Somerville was "certainly the most extraordinary woman

[1]The Morning Post, 29 November 1872.

[2]Belfast Newsletter 1872.

[3]Secord, 2004, Volume 1, II.29, [8].

[4]Stenhouse,2021, *Mary Somerville: Being and Becoming a Mathematician.*, Chapter 1, [10].

in Europe - a mathematician of the very first rank with all the gentleness of a woman"[5].

Despite being actively discouraged in her intellectual endeavours by her family from a very young age, Somerville taught herself sufficient Latin to read the books in her home library in Burntisland, Scotland. During her early life, she sought every opportunity to get access to mathematical books and self-thought elementary algebra and geometry from Euclid's *Elements* and *Algebra* by John Bonnycastle giving her the basis to understand astronomy and other sciences[5].
After a brief period of living in London during her first marriage (1804-1807), she returned to Scotland after her husband, Samuel Greig died. With the financial freedom of a widow, she continued her mathematical studies, learning about spherical trigonometry, conic sections and later reading Isaac Newton's *Principia* for the first time[6]. This marks the period when she started engaging with the scientific community. John Playfair, professor of natural philosophy at University of Edinburgh, encouraged her studies, and through him she began a correspondence with William Wallace (Playfair's former pupil and Professor of Mathematics at the Royal Military College), with whom she discussed mathematical problems[7].

This marked a turning point in Somerville's mathematical career. She started submitting solutions to mathematical puzzles proposed in *The New Series of the Mathematical Repository* under the pseudonym "A Lady"[8], becoming an active contributor of the mathematical community for the first time. She soon got recognition for her mathematical abilities, culminating with a silver medal in 1811, engraved with her real name, for the solution of a "Prize Question" featuring a Diophantine equation[9]. We will describe this problem along with Somerville's solution in Section 2 through the prism of modern algebraic geometry.

Despite never receiving a formal education, Somerville taught herself the Mathematics needed to answer questions from *The New Series of the Mathematical Repository* in various areas of mathematics such as Number Theory, Geometry and even Differential Geometry. The latter was viewed as a particularly difficult topic as the differential notation had not been adopted yet by the British mathematical community. Moreover, adopting continental knowledge of Mathematical Analysis was part of an act of reform in British Calculus at that time. Her correspondence with W. Wallace and

[5]as described in her *Recollections*, [9].

[6]as described in her *Recollections*, [9].

[7]Stenhouse,2021, *Mary Somerville: Being and Becoming a Mathematician.*, Chapter 1, [10], O'Connor, Robertson *Mary Fairfax Greig Somerville*, [5].

[8]It was common practice to publish solutions under a pseudonym, but not submit them anonymously. Stenhouse,2021, *Mary Somerville: Being and Becoming a Mathematician.*, Chapter 1, [10]

[9]Prize Question 310, posed in Volume 3 of *The New Series of the Mathematical Repository.*

Figure 2: The medal awarded to Mary Somerville for her solution of Prize Question. The medal is now held at Somerville College, Oxford and the inscription reads: Maria Greig, L.M.D; PALMAM QUI. MERUIT FERAT; T. Leybourn. L.M.D stands for Libens merito dedicavit, and the Latin loosely translates as 'deservedly won; let they who have earned the palm, bear it'. The palm signifies victory. Credits for figure and translation: Stenhouse, 2021, *Mary Somerville: Being and Becoming a Mathematician.*

J. Playfair [10] improved her knowledge in calculus of variation, trigonometry, integration, polar coordinates which enabled her to understand and translate Laplace's *Traité de mécanique céleste* on celestial mechanics. In 1826, Mary Somerville publishes her translation of this famous manuscript into a book entitled *Mechanism of the Heavens*, considered her most remarkable work as a mathematician.

Her second marriage was very different to the first. Whereas she felt isolated and not encouraged in doing science by her first husband, her second one, William Somerville (which she married in 1812) was not only supporting her, but also acted as her representative. He was allowed to have memberships to scientific societies while for a woman this was impossible, hence he was providing her with books and made sure her writings were well-received by these communities. He opened the door for her to meet famous personalities from the scientific circles such as William Herschel (astronomer), Leonard Horner (geologist), Alexander and Jane Marcet (physician and chemist). She travelled a lot during her second marriage. She participated in lectures at prestigious universities and became acquainted with many famous academics such as Claude Louis Mathieu, Simeon-Denis Poisson and most notably Pierre-Simon Laplace, who hosted the Somervilles in 1817 at his estate in Arcueil [11]. Here, Somerville talked to Laplace about his scientific works, including potential improvements in analytical methods regarding the convergence of series[12]. Later, Laplace wrote in his letter to

[10] Stenhouse,2021, *Mary Somerville: Being and Becoming a Mathematician.*, Chapter 2, [10]

[11] Stenhouse,2021, *Mary Somerville: Being and Becoming a Mathematician.*, Chapter 3, [10].

[12] Patterson, 1985, p. 360 [7].

Somerville that “the interest which you deign to take in my work flatters me all the more as there are few other readers and judges so enlightened” [13].

After publishing *Mechanism of the Heavens*, she wrote *Connection of the Physical Sciences* (1834, 2 vol.) which explores astronomy, physics, geography, and meteorology. She updated it nine times during her lifetime. The comments she made about a hypothetical planet perturbing the computations of Uranus' exact location allegedly inspired British astronomer John Couch Adams to discover Neptune. In 1848 she wrote *Physical Geography* (2 vol., underwent six editions) which was the first book in English on the subject and was very well-received. During her life, Mary Somerville had never stopped contributing to science, publishing her last book *On Molecular and Microscopic Science* at 89 years old. She wrote an autobiography on her life as a polymath while navigating gender barriers entitled *Personal Recollections, from Early Life to Old Age* which was edited and published by her daughter, Martha after her death.

Despite Somerville's impressive contribution to many areas of academia, this article focuses on her mathematical achievements. More precisely, we will focus on two Diophantine equations that Mary Somerville solved in Volume 3 of *The New Series of the Mathematical Repository* [4], firstly on the Prize Question 310 (section 2), for which she got the above-mentioned silver medal and secondly, Question 311 (section 3). In both cases, we first begin by studying Mary's solution and then put the equations into a modern framework. Moreover, we will try to generalize her solutions in the style of algebraic geometry.

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2 Prize Question: a K3-surface

We shall now study the Prize Question which secured Mary Somerville a silver medal in 1811. It reads as follows:

XX. PRIZE QUESTION 310, by Mr. W. Wallace.[14]
Find such integer values of x, y, z as shall render the three expressions $x^2 + axy + y^2$, $x^2 + a'xz + z^2$, $y^2 + a''yz + z^2$ squares, a, a', a'' being given numbers.

Two solutions were published: the first one by "a Lady" (Mary Somerville) and the second one by "Mr. Lowry"(R. M. College).

[13] (Hahn, 2013, pp. 1250–1, Stenhouse,2021, *Mary Somerville: Being and Becoming a Mathematician.*, Chapter 3, [10].

[14] Volume 3 of *The New Series of the Mathematical Repository* [4].

2.1 Mary Somerville's solution

This section follows Mary Somerville's solution as published in *New Series of The Mathematical Repository: Volume 3* [4]. She starts her solution by assuming that x, y and z are parameterized by two new variables m, n in the following way:

$$\begin{cases} x = an^2 \pm 2mn, \\ y = m^2 - n^2, \\ z = a''n^2 + 2mn. \end{cases} \tag{1}$$

These make the first and third expression into a square as Somerville writes:

$$\begin{cases} x^2 + axy + y^2 = (m^2 \pm amn + n^2)^2, \\ y^2 + a''yz + z^2 = (m^2 + a''mn + n^2)^2. \end{cases} \tag{2}$$

Next, in order to make the last expression into a square, Somerville denotes by $r = an \pm 2m$ and by $s = a''n + 2m$ and by reparameterizing $r = p^2 - q^2$, $s = a'q^2 + 2pq$, she finally gets that

$$x^2 + a'xz + z^2 = n^2(p^2 + a'pq + q^2)^2.$$

The rest of her solution consisits in writing m, n in terms of the parameters p, q which after several simplifications leads to the following formulae:

1. if $x = an^2 + 2mn$

$$\begin{cases} m = a''p^2 - 2apq - (a'a + a'')q^2, \\ n = 2(a' + 1)q^2 + 4pq - 2p^2. \end{cases} \tag{3}$$

2. if $x = an^2 - 2mn$

$$\begin{cases} m = -a''p^2 + 2apq - (a'a + a'')q^2, \\ n = 2(a' - 1)q^2 + 4pq + 2p^2. \end{cases} \tag{4}$$

Somerville notices that only (4) gives non-trivial solutions when $a = a' = a''$. The article ends with three computed examples.

2.2 Note on Mr. Lowri's solution

At a first glance, Somerville's choices in the paramtrizations of x, y and z might seem surprising. We notice that the next published solution uses roughly the same parameterization up to a permutation of x, y, z. It turns out that Mr. Lowri's solution gives us a deeper insight into Somerville's one, and perhaps into a common method of solving simultaneous quadratic equations of the nineteenth century in the British mathematical community.

Mr. Lowri starts by dividing the first two expressions by x^2 and the third one by z^2. Hence, the Prize Question becomes equivalent to showing that the following are squares:

$$\begin{cases} 1 + a\frac{y}{x} + \frac{y^2}{x^2} \\ 1 + a'\frac{z}{x} + \frac{z^2}{x^2} \\ 1 + a''\frac{z}{y} + \frac{z^2}{y^2} \end{cases} \tag{5}$$

Next, Lowry assumes that $1 \mp \frac{vy}{ux}$ is a root for the first expression and $1 \mp \frac{mz}{nx}$ is a root for the second. Then he gets

$$\frac{y}{x} = \frac{u(au \pm 2v)}{v^2 - u^2}, \quad \frac{z}{x} = \frac{n(a'n \pm 2m)}{m^2 - n^2}.$$

Therefore $\frac{z}{y} = \frac{n(a'n \pm 2m)(v^2 - u^2)}{u(au \pm 2v)(m^2 - n^2)}$. Mr. Lowri notices that this expression can be simplified if one assumes $v^2 - u^2 = m^2 - n^2$ and he writes down the most obvious way to do this: by letting $m = v$ and $n = u$. It turns out that this simplification gives exactly Somerville's first parameterization (1) up to a permutation of x, y, z. From here his method is very similar to Somerville's. This can also be seen by examining the common examples they both give: when $a = 1, a' = 3, a'' = 7$ they get the same result $x = 24, y = 11, z = 56$ and when $a = a' = a'' = -1$ they get the same result up to a sign $x = 832, y = 667, z = 520$.

Later on, Mr. Lowri considers a generalisation of this method, where he assumes $v^2 - u^2 = m^2 - n^2$ by letting $v = \frac{r^2+s^2}{2rs}m + \frac{r^2-s^2}{2rs}n$ and $u = \frac{r^2-s^2}{2rs}m + \frac{r^2+s^2}{2rs}n$. Perhaps not surprising, this leads to a more complicated formula for the solutions. Moreover, Mr. Lowri proposes an even more general solution which we will discuss in section 2.5.

He ends the discussion by giving solutions to the generalized problem that asks for simultaneous zeros of $x^2 + axy + by^2$, $x^2 + a'xz + b'z^2$, and $y^2 + a''yz + b''z^2$, but which are not of interest to us at the moment.

2.3 Prize Question Solution via Algebraic Geometry

A present-day mathematician would rewrite the statement in the following way. Find such integer values of x, y, z such that:

$$\begin{cases} x^2 + axy + y^2 = w^2 \\ x^2 + a'xz + z^2 = w'^2 \\ y^2 + a''yz + z^2 = w''^2. \end{cases} \tag{6}$$

As the above equations are homogeneous, a natural way to view their common solutions is as a projective variety in $\mathbb{P}^5$. More concretely, consider the following homogeneous polynomials of degree 2:

$$\begin{cases} f_1(x,y,z,w,w',w'') = x^2 + axy + y^2 - w^2 \\ f_2(x,y,z,w,w',w'') = x^2 + a'xz + z^2 - w'^2 \\ f_3(x,y,z,w,w',w'') = y^2 + a''yz + z^2 - w''^2. \end{cases} \tag{7}$$

Define the projective variety:

$$S_{a,a',a''} := \{\mathbf{p} = [x:y:z:w:w':w''] \in \mathbb{P}^5 : f_1(\mathbf{p}) = 0, f_2(\mathbf{p}) = 0, f_3(\mathbf{p}) = 0\}.$$

Consider the set of rational points $S_{a,a',a''}(\mathbb{Q}) = S_{a,a',a''} \cap \mathbb{P}^5(\mathbb{Q})$. Note that by the fact that our equations are homogeneous, each rational point has an integer representative in $\mathbb{P}^5$. Hence, for the rest of this section we would like to describe the geometry of $S_{a,a',a''}(\mathbb{Q})$.

Proposition 2.1 (Dimension)**.** *The projective variety $S_{a,a',a''}$ has dimension 2.*

Proof. The dimension of $S_{a,a',a''}$ can be computed as the degree of the Hilbert polynomial of the corresponding homogeneous ideal $I = I(S_{a,a',a''}) := \langle f_1, f_2, f_3 \rangle$. The Hilbert polynomial can be computed to be $4x^2 + 2$ (for example, via MAGMA [1]), hence the dimension is 2. □

Proposition 2.2 (Irreducibility)**.** *The number of irreducible components depend on the choices of (a, a', a'') as follows.*

1. *If $a \notin \{\pm 2\}$, $a' \notin \{\pm 2\}$ and $a'' \notin \{\pm 2\}$, then $S_{a,a',a''}$ is irreducible.*
2. *If exactly one of a, a' or a'' is ± 2, then $S_{a,a',a''}$ has 2 irreducible components, that intersect in a one dimensional variety.*
3. *If exactly two of a, a' or a'' are ± 2, then $S_{a,a',a''}$ has 4 irreducible components, regardless of what combination of signs we choose.*
4. *If all of a, a' and a'' are ± 2, then $S_{a,a',a''}$ has 8 irreducible components, regardless of what combination of signs we choose.*

Proof. We first note that $S_{a,a',a''} = V(f_1, f_2, f_3) = V(f_1) \cap V(f_2) \cap V(f_3)$. We will prove the following.

Claim 1. The projective variety $V(f_1)$ is

1. either reducible, with exactly two irreducible components when $a = \pm 2$;
2. or irreducible when $a \neq \pm 2$.

By symmetry, the claim is true for $V(f_2)$ and $V(f_3)$. Then, the proposition follows by counting the number of irreducible components in the intersection $V(f_1, f_2, f_3) = V(f_1) \cap V(f_2) \cap V(f_3)$.

Proof of Claim 1.1. If $a = \pm 2$, then

$$f_1 = x^2 \pm 2xy + y^2 - w^2 = (x \pm y - w)(x \pm y + w)$$

hence giving $V(f_1) = V(x \pm y - w) \cup V(x \pm y + w)$. Note that the two components are irreducible as they have degree 1.
If $a \neq \pm 2$ we consider f_1 as a polynomial in w by fixing the other variables. This factorizes if and only if $x^2 + axy + y^2$ is a square. Consider now the degree two polynomial

$$p(x, y) = x^2 + axy + y^2.$$

By fixing y, we get that $x^2+axy+y^2$ is a square if and only if its discriminant $\Delta_p = a^2y^2 - 4y^2$ is a square. This is equivalent to $a^2 - 4 = t^2$, for some integer t. This is equivalent to solving $(a - t)(a + t) = 4$ in the ring of integers. By considering the parities of the factorisation of 4 we get that the only possibilities are

$$\begin{cases} a - t = \pm 2 \\ a + t = \pm 2 \end{cases}.$$

These give $a = \pm 2$, a contradiction. Hence f_1 does not factorize if $a \neq \pm 2$.

□

Proposition 2.3 (Singularities)**.** *For any integer triple (a, a', a''), $S_{a,a',a''}$ is singular and the singularities depend on the choices of (a, a', a'') as follows.*

1. *If $a \notin \{\pm 2\}$, $a' \notin \{\pm 2\}$ and $a'' \notin \{\pm 2\}$, then the singularities of $S_{a,a',a''}$ consist in 12 isolated singularities given by:*

$$\begin{aligned} &[-1:0:0:-1:1:0], [-1:0:0:1:1:0], [0:-1:0:-1:0:1],\\ &[0:-1:0:1:0:1], [0:0:-1:0:-1:1], [0:0:-1:0:1:1],\\ &[0:0:1:0:-1:1], [0:0:1:0:1:1], [0:1:0:-1:0:1],\\ &[0:1:0:1:0:1], [1:0:0:-1:1:0], [1:0:0:1:1:0]. \end{aligned} \tag{8}$$

2. *If $a = \pm 2$, $a' \notin \{\pm 2\}$ and $a'' \notin \{\pm 2\}$ with $a' \neq \mp a''$ (or any other permutation of (a, a', a'') which respects these conditions), then the singularities of $S_{a,a',a''}$ consist in:*
   (a) *points on an 1-dimensional variety (corresponding to the intersection of the 2 irreducible components);*
   (b) *8 isolated points given by:*

$$\begin{aligned} &[-1:0:0:-1:1:0], [-1:0:0:1:1:0], [0:-1:0:-1:0:1],\\ &[0:-1:0:1:0:1], [0:1:0:-1:0:1],\\ &[0:1:0:1:0:1], [1:0:0:-1:1:0], [1:0:0:1:1:0]. \end{aligned} \tag{9}$$

3. *If* $a = \pm 2$, $a' \notin \{\pm 2\}$ *and* $a'' \notin \{\pm 2\}$ *with* $a' = \mp a''$ *(or any other permutation of* (a, a', a'') *which respects these conditions), then the singularities of* $S_{a,a',a''}$ *consist in:*

   (a) *points on two distinct* 1*-dimensional varieties (corresponding to the intersection of the* 2 *irreducible components) which intersect in two points if* $(a')^2 - 4$ *is a square or are disjoint otherwise;*

   (b) *the* 8 *isolated points described in (9).*

4. *If* $a \in \{\pm 2\}$, $a' \in \{\pm 2\}$ *and* $a'' \notin \{\pm 2\}$ *with (or any other permutation of* (a, a', a'') *which respects these conditions), then the singularities of* $S_{a,a',a''}$ *consist in:*

   (a) *points on four distinct* 1*-dimensional varieties* V_1, V_2, V_3, V_4 *(corresponding to the intersection of the* 4 *irreducible components) which intersect as follows. For every* $i \neq j$, $V_i \cap V_j = \bigcap_i V_i$ *which intersect in two points if* $(a' + a'')$ *is a square or are dijoint otherwise.*

   (b) 4 *isolated points given by:*

$$\begin{aligned} &[-1:0:0:-1:1:0], [-1:0:0:1:1:0],\\ &[1:0:0:-1:1:0], [1:0:0:1:1:0]. \end{aligned} \tag{10}$$

5. *If* $a \in \{\pm 2\}$, $a' \in \{\pm 2\}$ *and* $a'' \in \{\pm 2\}$*, then the singularities of* $S_{a,a',a''}$ *consist in* 12 *lines* $L_1, \cdots L_{12}$ *(corresponding to the intersection of the irreducible components).*

Proof. We are going to use the Jacobian condition for smoothness to compute the singularities of the projective variety $S_{a,a',a''}$. In our case, this reduces to finding the points $\mathbf{p} \in S_{a,a',a''}$, such that the rank of the Jacobian matrix evaluated at $\mathbf{p}$ is less than 3, in particular all 3×3 minors of $J(\mathbf{p})$ vanish, where $J(\mathbf{p})$ is the Jacobian matrix evaluated at $\mathbf{p} = [x : y : z : w : w' : w'']$.

$$J(\mathbf{p}) = \begin{pmatrix} 2x + ay & ax + 2y & 0 & -2w & 0 & 0 \\ 2x + a'z & 0 & a'x + 2z & 0 & -2w' & 0 \\ 0 & 2y + a''z & a''y + 2z & 0 & 0 & -2w'' \end{pmatrix}$$

By computing the 3×3 minors of $J(\mathbf{p})$ and imposing that they are 0, we get different solutions according to whether a, a', a'' are ± 2, as described in the statement of the proposition. □

The next proposition tells us that the singularities described above are the "nicest" possible ones, when we restrict to irreducible components. We checked that this is indeed the case using MAGMA [1].

Proposition 2.4. *Each of the singularities described in Proposition 2.3 is at most a double rational point when viewed inside an irreducible component.*

Theorem 2.5. *Let k be a field of characteristic* 0. *Assume that X is a surface over k of one of the following three types:*

1. *a quartic surface in $\mathbb{P}^3_k$,*

2. *an intersection of a cubic and a quadric hypersurface in $\mathbb{P}^4_k$,*

3. *an intersection of three quadrics in $\mathbb{P}^5_k$.*

Furthermore, assume that all singularities of X are rational double points. Then the minimal regular model of X is a K3-surface.

Proof. This is a well-known result. A proof can be found in the Appendix of [6]. □

Corollary 2.6. *The minimal regular model of $S_{a,a',a''}$ in the most general case (i.e. when $a \notin \{\pm 2\}$, $a' \notin \{\pm 2\}$ and $a'' \notin \{\pm 2\}$) is a K3-surface.*

Proof. By Proposition 2.2 we get that $S_{a,a',a''}$ is irreducible, and then by Proposition 2.4 it has only double rational points. Thus by Theorem 2.5 applied to an intersection of three quadrics in $\mathbb{P}^5$ we get the desired result. □

In the cases where $S_{a,a',a''}$ is reducible, we get that each irreducible component is a rational surface by MAGMA [1].

2.4 Back to Somerville's solution

Throughout this section we will work in the general case (i.e. when $a \notin \{\pm 2\}$, $a' \notin \{\pm 2\}$ and $a'' \notin \{\pm 2\}$) and moreover we assume that a, a' and a'' are all distinct. In the language of Algebraic Geometry, Somerville's solution proposes a rational map

$$F \colon \mathbb{P}^1 \dashrightarrow S_{a,a',a''}$$

defined over $\mathbb{Q}$, given as a composition of a couple explicit rational maps. More precisely, $F = f \circ g$, where f is given by (1) and g is given by (3) or (4).

$$\mathbb{P}^1 \overset{g}{\dashrightarrow} \mathbb{P}^1 \overset{f}{\dashrightarrow} S_{a,a',a''}, \qquad F = f \circ g \colon \mathbb{P}^1 \dashrightarrow S_{a,a',a''}$$

$$[p,q] \xrightarrow{\;g\;} [m,n] \xrightarrow{\;f\;} [x,y,z,w,w',w''].$$

Note that Somerville's approach comes with two symmetric solutions given by the following parameterizations:

$$\begin{cases} x = m^2 - n^2 \\ y = an^2 \pm 2mn \\ z = a'n^2 + 2mn \end{cases} \qquad \begin{cases} x = a'n^2 \pm 2mn \\ y = a''n^2 + 2mn \\ z = m^2 - n^2 \end{cases}.$$

These give two more choices for the map f, hence two more options for F. For the rest of this section we will work with Somerville's choices described in (1) but all the arguments work to the symmetric solutions too.

Note that Im(F) is a rational curve (i.e. isomorphic to $\mathbb{P}^1$). If Im(F) does not contain singularities of $S_{a,a',a''}$, it means that it remains a rational curve in the minimal resolution of $S_{a,a',a''}$ which we proved to be a K3-surface.

Proposition 2.7. *Suppose that we are in the general case and a, a', a'' are all distinct. Moreover, assume that $a' + 2$ is not a square. Then Im(F) does not contain any singularity of $S_{a,a',a''}$.*

Proof. By Proposition 2.3 1., the singular points in the general case must have two out of x, y, z equal to 0. If $x = y = 0$, by $x = an^2 \pm 2mn$ and the fact that $a \neq \pm 2$ it follows that $n = 0$. Moreover $y = m^2 - n^2$ implies $m = 0$, which will make $z = 0$, a contradiction. Same reasoning makes it impossible for a singularity with $y = z = 0$ to belong to Im(F). Lastly, if $x = z = 0$, then it follows that $n = 0$. Otherwise by 1 we would get $a = a'' = \frac{2m}{n}$, a contradiction. By Section 2, $n = 0$ implies $p^2 - q^2 = a'q^2 + 2pq$. Rearranging we get that $(p - q)^2 = (a' + 2)q^2$, which implies that $a' + 2$ ia a square, contradicting the hypothesis. ☐

In conclusion, for the general case when a, a', a'' are all distinct and moreover $a' + 2$ is not a square, a modern algebraic geometer would say that Somerville constructs a rational curve on the K3-surface which is the minimal resolution of $S_{a,a',a''}$.

2.5 Alternative Solution

Inspired by Mr. Lowri's comments, one might want to generalize Somerville's solution in (1) in the following way:

$$\begin{cases} x = (au^2 \pm 2uv)(m^2 - n^2), \\ y = (m^2 - n^2)(v^2 - u^2), \\ z = (a''n^2 + 2mn)(v^2 - u^2). \end{cases} \tag{11}$$

As in Somerville's solution, these choices make sure that the first and third expressions are squares, giving:

$$\begin{cases} x^2 + axy + y^2 = (v^2 \pm auv + u^2)^2(m^2 - n^2)^2, \\ y^2 + a''yz + z^2 = (m^2 + a''mn + n^2)^2(v^2 - u^2)^2. \end{cases} \tag{12}$$

Now, we have to impose that also $x^2 + a'xz + z^2$ is also a square. After writing this condition in terms of m, n, u, v and simplifying calculations we note that the following choice for u and v gives $x^2 + a'xz + z^2$ as a square:

$$\begin{cases} u = 4n(a''n + 2m)(an(a''n + 2m) - a'(m^2 - n^2)), \\ v = (4 - a'^2)(m^2 - n^2)^2 + 2aa'n(a''n + 2m)(m^2 - n^2) - (4 + a^2)n^2(a''n + 2m)^2. \end{cases} \tag{13}$$

We make the quick note that Somerville's solution takes $u = n, v = m$ and imposes constrains on m, n as discussed in Section 2.1. In the language of algebraic geometry this gives another rational curve on the surface $S_{a,a',a''}$.

However, for most choices of a, a', a'', this rational curve contains exactly one singularity, hence it does not stay rational when passing to the minimal resolution of $S_{a,a',a''}$.

2.6 Further Notes

Inspired by Somerville's solution, a present-day mathematician might be interested to find out how many rational curves one can construct on $S_{a,a',a''}(\mathbb{Q})$, or alternatively on its minimal resolution. Unfortunately, there is not much known about the rational curves on K3 surfaces over $\mathbb{Q}$. However, over algebraically closed fields, the following theorem holds.

Theorem 2.8. *Let X be a projective K3 surface over an algebraically closed field k of characteristic $p \geq 0$. Then X contains infinitely many rational curves, with the only possible exception if $k = \mathbb{F}_p$ with $p = 2, 3$ and X is isotrivially elliptic.*

It is conjectured that the above theorem holds for any algebraically closed field. Moreover, the analogous statement for genus 1 curves holds, with the only possible exception uniruled K3 surfaces in characteristic $p \leq 3$. See [2] for more details.

3 Question 311: an elliptic curve

We move our attention to a second Diophantine problem solved by Somerville in Volume 3 of *The New Series of the Mathematical Repository* [4]. It reads as follows:

I. QUESTION 311, by Mr. John Hynes, Dublin.
To divide a given square number n^2, into two such parts that the sum of their squares and the sum of their cubes may both be rational squares.

Three solutions were published: the first one by "a Lady" (Mary Somerville), the second one by by Mr. Cunliffe, (R. M. College) and the third one by "Mr. Lowry" (R. M. College).

3.1 Somerville's Solution

Somerville begins her solution by denoting the first part by x. Hence, in her notation, the problem asks to find such x, n that make $x^2 + (n^2 - x)^2$ and $x^3 + (n^2 - x)^3$ into squares. Expanding she gets that $x^2 + (n^2 - x)^2 = n^4 - 2n^2x + 2x^2$ and $x^3 + (n^2 - x)^3 = n^2(n^4 - 3n^2x + 3x^2)$ must be squares and notices that the later reduces to $n^4 - 3n^2x + 3x^2$ being a square.

Consequently, she assumes that

$$\begin{cases} -3n^2 + 3x = -2n^2px + p^2x^2 \\ -2n^2 + 2x = -2n^2q + q^2x \end{cases} \tag{14}$$

which will give $n^4-2n^2x+2x^2 = (n^2-qx)^2$ and $n^4-3n^2x+3x^2 = (n^2-px)^2$. Then she solves both equations in (14) for x and equalizes them, and hence getting

$$\frac{3n^2 - 2n^2p}{3 - p^2} = \frac{2n^2 - 2n^2q}{2 - q^2}.$$

Here, she makes the strict assumptions that $3-2p = 2-2q$ and $3-p^2 = 2-q^2$, which gives her $q = \frac{3}{4}$ and $p = \frac{5}{4}$. Finally, these values give the desired values $x = \frac{8n^2}{23}$ and $n^2 - x = \frac{15n^2}{23}$. The last part of her solution consists in a few examples for different n.

Note 3.1. From now on, we denote a solution by a pair (n, x). For example, Somerville found the pairs $(n, \frac{8n^2}{23})$ with $n \in \mathbb{Z}$. Note that it is enough to search for pairs of the form $(1, x)$ which is going to be the focus of the next sections.

3.2 Somerville's solution - the modern perspective

We will first make the substitution $y = n^2$. We can rewrite the problem as finding the common zeros of the following two equations:

$$\begin{cases} y^2 - 2yx + 2x^2 = w^2 \\ y^2 - 3yx + 3x^2 = w'^2. \end{cases} \tag{15}$$

Let's denote by C the vanishing set of these two equations in $\mathbb{P}^3$. It can be shown (for example in MAGMA [1]) that this is a singular curve of genus 1 which has an obvious rational point $[0 : 1 : 1 : 1]$. Thus C is birational to an elliptic curve. Note that we don't have to worry about $y = n^2$ as we can just consider the points with $y = 1$ in projective coordinates and compute solutions of the form $(1, x)$ (see Note 3.1).

Somerville's solution computes a rational point on C, namely $S = [8 : 23 : 17 : 13] = [\frac{8}{23}, 1, \frac{17}{23}, \frac{13}{23}]$ which corresponds to a point on the birational elliptic curve associated to C.

A present-day algebraic geometer would perhaps want to investigate how other points on this curve look like and maybe find a larger set of solutions to the equation. We computed with MAGMA [1] an elliptic curve E in Weierstras form which is birationally equivalent to C and has $[0 : 1 : 1 : 1]$ as the point at infinity:

$$E : y^2 = x^3 + 8x^2 + 12x.$$

Somerville's point $S = [8 : 23 : 17 : 13]$ corresponds to the affine point $S_E = (48, 360)$ on E. We can now make use of the group law on E. We compute using MAGMA that the order of S is infinite. Thus, it is a source of infinite points on our curve, namely $[k]S_E$ for k an integer. Next, we might wonder how often does it happen that different points on the elliptic curve E give rise to the same solution to our equation.
We define the equivalence relation $P \sim P'$ if the points P and P' of the curve C give rise to the same solution $(1, x)$. Note that each equivalence class has at most 4 elements because if $[x, 1, w, w']$ is a solution then $[x, 1, -w, w']$, $[x, 1, w, -w']$, $[x, 1, -w, -w']$ are the rest of the elements in its class.
By the fact that E and C are birational, all but finitely many points of E can be viewed as points of C. And as each equivalence class is finite, it follows that there is an infinite number of equivalence classes and hence an infinite number of solutions.

As an example $[2]S_E = (36481/3600, 9620479/216000)$ gives rise to a solution. It corresponds to the point $[10130640 : 18240049 : 12976609 : 9286489]$ on the initial curve C. Now, since we assumed that $y = 1$, we read this projective point as

$$[\frac{10130640}{18240049}, 1, \frac{12976609}{18240049}, \frac{9286489}{18240049}].$$

A simple check shows that and $(1, \frac{10130640}{18240049})$ is a solution to the initial problem. Moreover, any pair $(n, \frac{10130640}{18240049}n^2)$ gives rise to a solution.

3.3 Further Notes

The curious modern mathematician might wonder if $\{[k]S_E : k \text{ integer}\}$ are the only rational points on the curve E. The set of all rational points on an elliptic curve denoted by $E(\mathbb{Q})$ forms a group named the Mordell Weil group of E. In this case we can compute it in MAGMA [1] to be

$$E(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}.$$

Hence, we deduce that Somerville's point S_E belongs to the free part of the abstract group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$.
We can compute the generators of this group in MAGMA [1] and get

$$S_{1,E} = (-6, 0), \quad S_{2,E} = (-2, 0), \quad S_{3,E} = (6, -24).$$

They correspond to the points $S_1 = [1 : 1 : 1 : -1]$, $S_2 = [1 : 1 : -1 : 1]$, $S_3 = [1 : 1 : -1 : -1]$ on the initial curve C, which all lead to the rather uninteresting pair of solutions $(1, 1)$. However, Mordel Weil group provides us with an infinite number of distinct solutions as described in Section 3.2. To see how, note that Somerville's point $S_E = S_{3,E} + S_{3,E}$.
In conclusion, a modern algebraic geometer would say that Somerville found a point of infinite order on an elliptic curve.

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