

# Mary Somerville's Diophantine Equations

Diana Mocanu



# Mary Somerville

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Mary Somerville (born Mary Fairfax 1780 - 1872)



Figure: Painting by Thomas Phillips, 1834

# Equation 1

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**I. QUESTION 311**, by Mr. John Hynes, Dublin.

To divide a given square number  $n^2$ , into two such parts that the sum of their squares and the sum of their cubes may both be rational squares.

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Solution published under the pseudonym "a Lady".

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Somerville begins her solution by denoting the first part by  $x$ . Hence, in her notation, the problem asks to find such  $x, n$  that make

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Expanding she gets that  $x^2 + (n^2 - x)^2 = n^4 - 2n^2x + 2x^2$  and  $x^3 + (n^2 - x)^3 = n^2(n^4 - 3n^2x + 3x^2)$  must be squares and notices that the later reduces to  $n^4 - 3n^2x + 3x^2$  being a square.

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Consequently, she assumes that

$$\begin{cases} -3n^2 + 3x = -2n^2px + p^2x^2 \\ -2n^2 + 2x = -2n^2q + q^2x \end{cases} \quad (1)$$

which will give  $n^4 - 2n^2x + 2x^2 = (n^2 - qx)^2$  and  $n^4 - 3n^2x + 3x^2 = (n^2 - px)^2$ .

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Here, she makes the strict assumptions that  $3 - 2p = 2 - 2q$  and  $3 - p^2 = 2 - q^2$ , which gives her  $q = \frac{3}{4}$  and  $p = \frac{5}{4}$ . Finally, these values give the desired values  $x = \frac{8n^2}{23}$  and  $n^2 - x = \frac{15n^2}{23}$ .

## Equation 2

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**XX. PRIZE QUESTION 310**, by Mr. W. Wallace.

Find such integer values of  $x, y, z$  as shall render the three expressions  $x^2 + axy + y^2$ ,  $x^2 + a'xz + z^2$ ,  $y^2 + a''yz + z^2$  squares,  $a, a', a''$  being given numbers.



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She starts her solution by assuming that  $x, y$  and  $z$  are parameterized by two new variables  $m, n$  in the following way:

$$\begin{cases} x = an^2 + 2mn, \\ y = m^2 - n^2, \\ z = a''n^2 + 2mn. \end{cases} \quad (2)$$

These make the first and third expression into a square as Somerville writes:

$$\begin{cases} x^2 + axy + y^2 = (m^2 + amn + n^2)^2, \\ y^2 + a''yz + z^2 = (m^2 + a''mn + n^2)^2. \end{cases} \quad (3)$$

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Next, in order to make the middle expression into a square, Somerville reparameterizes  $an + 2m = p^2 - q^2$ ,  $a''n + 2m = a'q^2 + 2pq$  in order to get

$$x^2 + a'xz + z^2 = n^2(p^2 + a'pq + q^2)^2.$$

The rest of her solution consists in writing  $m, n$  in terms of the parameters  $p, q$  which after several simplifications leads to the following formulae:

Recall  $x = an^2 + 2mn$

$$\begin{cases} m = a''p^2 - 2apq - (a'a + a'')q^2, \\ n = 2(a' + 1)q^2 + 4pq - 2p^2. \end{cases} \quad (4)$$

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## Examples

- Linear Diophantine equation

$$ax + by = c$$

- Pell's equations

$$x^2 - ny^2 = \pm 1$$

- Elliptic Curves

$$y^2 = x^3 + ax + b$$

- Generalized Fermat Equation

$$x^n + y^m = z^r$$

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- $n = 3, V(a_1x_1 + \dots + a_3x_3) =$  hyperplane,  $V(x^2 + y^2 - z^2) =$  cone .

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- **Chain of length  $m$**  is a chain of strict inclusions:  $\emptyset \neq X_0 \subsetneq X_1 \dots \subsetneq X_m$
- The **local dimension**  $\dim_p X$  of  $X$  at a point  $p \in X$  is the maximum over all lengths of chains starting with  $X_0 = \{p\}$ . The **dimension** of  $X$  is

$$\mathbf{dim} X := \max_{p \in X} \mathbf{dim}_p X.$$

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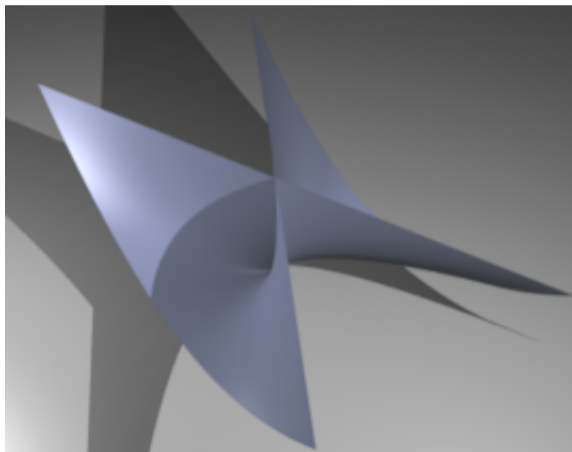
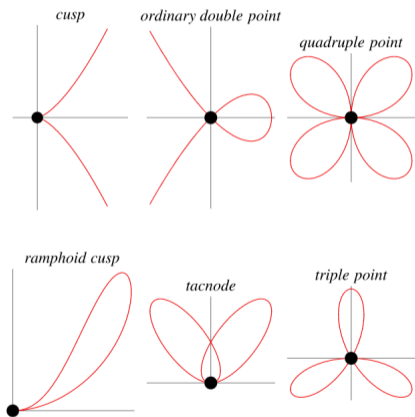
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## Theorem

Let  $X \subset \mathbb{A}^n$  be an irreducible aff. var. of dimension  $d$  with  $I(X) = \langle f_1, f_2, \dots, f_n \rangle$ .  
Then  $\text{Sing}(X) \subset X$  is a subvariety given by the vanishing in  $X$  of all  $(n-d) \times (n-d)$  minors of the Jacobian matrix

$$\text{Jac} = \left( \frac{\partial f_i}{\partial x_j} \right)$$

# Singularities



# Maps

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- $F: \mathbb{A}^n \rightarrow \mathbb{A}^m$  is a **morphism (or polynomial map)** if it is defined by polynomials:  
 $F(a) = (f_1(a), \dots, f_m(a))$  for some  $f_1, \dots, f_m \in R := k[x_1, \dots, x_n]$ .
- $F: X \rightarrow Y$  is a **morphism of affine varieties** if it is the restriction of a morphism  $\mathbb{A}^n \rightarrow \mathbb{A}^m$  (here  $X \subset \mathbb{A}^n$  and  $Y \subset \mathbb{A}^m$ ).  
 $F(a) = (f_1(a), \dots, f_m(a))$  for some  $f_1, \dots, f_m \in k[X] := R/I(X)$ .
- **isomorphism** = morphism which has an inverse which is a morphism

# Projective Space

---

- **Zariski topology** on  $\mathbb{A}^n$ : declare the basic open sets to be for any  $f \in R$ :

$$D(f) = \mathbb{A}^n \setminus V(f) := \{a \in k^n : f(a) \neq 0\}$$

$\implies$  induces a topology on an algebraic variety  $X \subset \mathbb{A}^n$

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- **Birational map** between two (affine/projective) varieties is “an isomorphism between open dense subsets”

# Resolution of Singularities

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Let  $k$  be an algebraically closed field.

- The problem of **resolution of singularities** asks whether every algebraic variety  $X$  has a resolution, a non-singular variety  $Y$  with a proper birational map  $Y \rightarrow X$ .
- Hironaka (1964): proved it for varieties over  $k$ , of characteristic 0
- Open for characteristic  $p$  and dimension  $> 4$ .

**TODAY:** characteristic 0: dimension 1 (curves), dimension 2 (surfaces).

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- Let  $C$  be a variety of dimension 1, then we call  $C$  an **algebraic curve**.
- Every algebraic curve has a unique nonsingular projective resolution.
- The **genus** of a smooth complete algebraic curve  $C$  is a numerical invariant under birational maps. If  $k = \mathbb{C}$ , then an algebraic curve  $C$  is the same as a Riemann surface. In this case, the smooth complex curve  $X$  of genus  $g$  is homeomorphic to the sphere with  $g$  handles.

# Examples

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Every smooth curve of genus 1 is birationally isomorphic to smooth cubic curves in  $\mathbb{P}^2$ . In characteristic 0 they have an affine model:

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- **genus  $\geq 2$** : Every smooth curve of genus  $\geq 2$  is birationally isomorphic to hyper-elliptic curves and non-hyper-elliptic curves.

Example: hyperelliptic curves:

$$C/k : y^2 = f(x), \deg(f) \geq 5$$



## Dimension 2: Surfaces

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Let  $k$  be an algebraically closed field.

- Let  $S$  be a variety of dimension 2, then we call  $S$  an **algebraic surface**.
- Surfaces have many different nonsingular projective resolutions (unlike the case of curves). However, a surface still has a unique minimal resolution.
- $\kappa$  is the **Kodaira dimension**, an arithmetic invariant which classifies algebraic surfaces.

# Examples

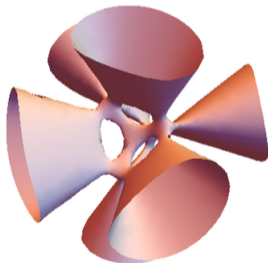
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- $\kappa = -\infty$ : rational surfaces, ruled surface

Any rational surface is birational to  $\mathbb{P}^2$ . A ruled surface: every point lies on a line.

- $\kappa = 0$ : abelian surfaces, hyperelliptic surface, K3 surface, Enriques surface

For example the K3 surface given by  $x^4 + y^4 + z^4 + w^4 = 0$ :



- $\kappa = 1$ : elliptic surface

For example,  $E \times C$  where  $E$  is an elliptic curve and  $C$  is any curve.

- $\kappa = 2$ : surface of general type

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**I. QUESTION 311**, by Mr. John Hynes, Dublin.

To divide a given square number  $n^2$ , into two such parts that the sum of their squares and the sum of their cubes may both be rational squares.

Somerville rewrites the problem such that one needs to find such  $x, n$  that make  $x^2 + (n^2 - x)^2$  and  $x^3 + (n^2 - x)^3$  into squares. Expanding she gets that  $x^2 + (n^2 - x)^2 = n^4 - 2n^2x + 2x^2$  and  $x^3 + (n^2 - x)^3 = n^2(n^4 - 3n^2x + 3x^2)$  must be squares and notices that the later reduces to  $n^4 - 3n^2x + 3x^2$  being a square.

We denote by  $y = n^2$ .

# Diophantine Equation 1

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- We can rewrite the problem as finding the common zeros of the following two equations:

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- Let's denote by  $C$  the vanishing set of these two equations in  $\mathbb{P}^3$  of coordinates  $[x : y : w : w']$ . We note that Somerville's solution corresponds to the point  $S = [8 : 23 : 17 : 13]$ .  
Let's examine  $C(\mathbb{Q})$  which are the points in  $\mathbb{P}^3$  with rational coordinates.

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Let's examine  $C(\mathbb{Q})$  which are the points in  $\mathbb{P}^3$  with rational coordinates.
- Dimension of  $C$  is 1 and genus of  $C$  is 1 and has an obvious rational point  $[0 : 1 : 1 : 1] \implies C$  is birational to an elliptic curve.



# Diophantine Equation 1: An Elliptic Curve

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- Since  $C$  is birational to an elliptic curve, it has an model which can be computed to be:

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- Somerville's point  $S = [8 : 23 : 17 : 13]$  corresponds to the affine point  $S_E = (48, 360)$  on  $E$ .

Hence, a modern algebraic geometer would say that Mary Somerville constructed a point on an elliptic curve.

# Other rational points?

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Recall that elliptic curves come with a group law.

## Theorem (Mordell-Weil)

*Let  $E/\mathbb{Q}$  be an elliptic curve. Then,*

$$E(\mathbb{Q}) \cong \text{Tors} + \mathbb{Z}^r,$$

*where  $r$  is called the rank of  $E$ .*

## Other rational points?

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- In our case,  $E : y^2 = x^3 + 8x^2 + 12x$  and

$$E(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$$

with generators  $S_1 = (-6, 0)$ ,  $S_2 = (-2, 0)$ ,  $S_3 = (6, -24)$ .

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- They correspond to the points  $S_1 = [1 : 1 : 1 : -1]$ ,  $S_2 = [1 : 1 : -1 : 1]$ ,  $S_3 = [1 : 1 : -1 : -1]$  on the initial curve  $C$ , which all lead to the rather uninteresting pair of solutions. Somerville's point turns out to be  $S = S_3 + S_3 = [2]S_3$ , which has infinite order.

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- In conclusion, Somerville's point gives rise to infinitely many solutions. For example,

$$S + S = (36481/3600, 9620479/216000)$$

which gives rise to the solution:  $[10130640, 18240049, 12976609, 9286489]$ .

## Diophantine equation 2

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A present-day mathematician would rewrite the statement in the following way. Find such integer values of  $x, y, z$  such that:

$$\begin{cases} x^2 + axy + y^2 = w^2 \\ x^2 + a'xz + z^2 = w'^2 \\ y^2 + a''yz + z^2 = w''^2. \end{cases} \quad (6)$$

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As the above equations are homogeneous, a natural way to view their common solutions is as a projective variety in  $\mathbb{P}^5$ . More concretely, consider the following homogeneous polynomials of degree 2:

$$\begin{cases} f_1(x, y, z, w, w', w'') = x^2 + axy + y^2 - w^2 \\ f_2(x, y, z, w, w', w'') = x^2 + a'xz + z^2 - w'^2 \\ f_3(x, y, z, w, w', w'') = y^2 + a''yz + z^2 - w''^2. \end{cases} \quad (7)$$



## Diophantine equation 2

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- Define the projective variety:

$$S_{a,a',a''} := \{\mathbf{p} = [x : y : z : w : w' : w''] \in \mathbb{P}^5 : f_1(\mathbf{p}) = 0, f_2(\mathbf{p}) = 0, f_3(\mathbf{p}) = 0\}.$$

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- If  $a \notin \{\pm 2\}$ ,  $a' \notin \{\pm 2\}$  and  $a'' \notin \{\pm 2\}$ , then the singularities of  $S_{a,a',a''}$  consist in 12 isolated singularities, and  $S_{a,a',a''}$  is irreducible. These singularities are double points ("not too bad").

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- The resolution of singularities theorem tells us that  $S_{a,a',a''}$  has a minimal resolution belonging to the classification of surfaces. Which one?

## Diophantine equation 2: a K3 surface

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### Theorem

Let  $k$  be a field of characteristic 0. Assume that  $X$  is a surface over  $k$  of one of the following three types:

1. a quartic surface in  $\mathbb{P}_k^3$ ,
2. an intersection of a cubic and a quadric hypersurface in  $\mathbb{P}_k^4$ ,
3. an intersection of three quadrics in  $\mathbb{P}_k^5$ .

Furthermore, assume that all singularities of  $X$  are rational double points. Then the minimal regular model of  $X$  is a K3-surface.

# Back to Somerville's Solution

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- We will work in the general case (i.e. when  $a \notin \{\pm 2\}$ ,  $a' \notin \{\pm 2\}$  and  $a'' \notin \{\pm 2\}$ ) and moreover we assume that  $a, a'$  and  $a''$  are all distinct. In the language of Algebraic Geometry, Somerville's solution proposes a birational map

$$F: \mathbb{P}^1 \dashrightarrow S_{a,a',a''}$$

defined over  $\mathbb{Q}$ , given as a composition of a couple of explicit rational maps. More precisely,  $F = f \circ g$ , where  $f$  is given by (2) and  $g$  is given by (4).

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- Hence, Somerville's solution  $Im(F) \subset S_{a,a',a''}$  is birational to  $\mathbb{P}^1$  and it can be shown that it misses the singularities.
- Hence, in modern language, Somerville constructs a rational curve on a  $K3$  surface.



**Thank you!**