# Mary Somerville's Diophantine Equations

Diana Mocanu

THE UNIVERSITY OF
WARWICK

# Mary Somerville

Mary Somerville (born Mary Fairfax 1780 - 1872)

Figure: Painting by Thomas Phillips, 1834

# Equation 1

**I. QUESTION 311**, by Mr. John Hynes, Dublin.
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Solution published under the pseudonym "a Lady".

# Mary Somerville's Solution

Somerville begins her solution by denoting the first part by $x$. Hence, in her notation, the problem asks to find such $x, n$ that make

$$\begin{cases} x^2 + (n^2 - x)^2 = \text{square} \\ x^3 + (n^2 - x)^3 = \text{square} \end{cases}$$

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Expanding she gets that $x^2 + (n^2 - x)^2 = n^4 - 2n^2x + 2x^2$ and $x^3 + (n^2 - x)^3 = n^2(n^4 - 3n^2x + 3x^2)$ must be squares and notices that the later reduces to $n^4 - 3n^2x + 3x^2$ being a square.

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Consequently, she assumes that

$$\begin{cases} -3n^2 + 3x = -2n^2px + p^2x^2 \\ -2n^2 + 2x = -2n^2q + q^2x \end{cases} \tag{1}$$

which will give $n^4 - 2n^2x + 2x^2 = (n^2 - qx)^2$ and $n^4 - 3n^2x + 3x^2 = (n^2 - px)^2$.

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Here, she makes the strict assumptions that $3 - 2p = 2 - 2q$ and $3 - p^2 = 2 - q^2$, which gives her $q = \frac{3}{4}$ and $p = \frac{5}{4}$. Finally, these values give the desired values $x = \frac{8n^2}{23}$ and $n^2 - x = \frac{15n^2}{23}$.

# Equation 2

**XX. PRIZE QUESTION 310**, by Mr. W. Wallace.
Find such integer values of $x, y, z$ as shall render the three expressions $x^2 + axy + y^2$, $x^2 + a'xz + z^2$, $y^2 + a''yz + z^2$ squares, $a, a', a''$ being given numbers.

# Mary's solution

She starts her solution by assuming that $x, y$ and $z$ are parameterized by two new variables $m, n$ in the following way:

$$\begin{cases} x = an^2 + 2mn, \\ y = m^2 - n^2, \\ z = a''n^2 + 2mn. \end{cases} \tag{2}$$

These make the first and third expression into a square as Somerville writes:

$$\begin{cases} x^2 + axy + y^2 = (m^2 + amn + n^2)^2, \\ y^2 + a''yz + z^2 = (m^2 + a''mn + n^2)^2. \end{cases} \tag{3}$$

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Next, in order to make the middle expression into a square, Somerville reparameterizes $an + 2m = p^2 - q^2$, $a''n + 2m = a'q^2 + 2pq$ in order to get

$$x^2 + a'xz + z^2 = n^2(p^2 + a'pq + q^2)^2.$$

The rest of her solution consists in writing $m, n$ in terms of the parameters $p, q$ which after several simplifications leads to the following formulae:
Recall $x = an^2 + 2mn$

$$\begin{cases} m = a''p^2 - 2apq - (a'a + a'')q^2, \\ n = 2(a' + 1)q^2 + 4pq - 2p^2. \end{cases} \tag{4}$$

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## Examples

- Linear Diophantine equation
$$ax + by = c$$
- Pell's equations
$$x^2 - ny^2 = \pm 1$$
- Elliptic Curves
$$y^2 = x^3 + ax + b$$
- Generalized Fermat Equation
$$x^n + y^m = z^r$$

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- **Chain of length m** is a chain of strict inclusions: $\emptyset \neq X_0 \subsetneq X_1 ... \subsetneq X_m$
- The **local dimension** $\dim_p X$ of $X$ at a point $p \in X$ is the maximum over all lengths of chains starting with $X_0 = \{p\}$. The **dimension** of $X$ is

$$\textbf{dim} X := \max_{p \in X} \textbf{dim}_p X.$$

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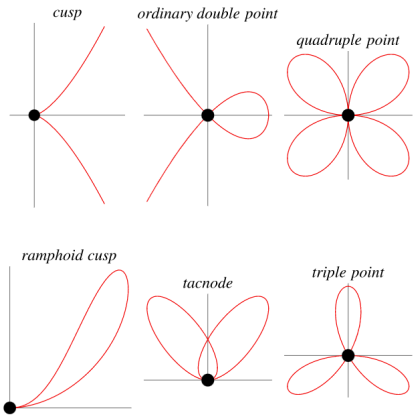
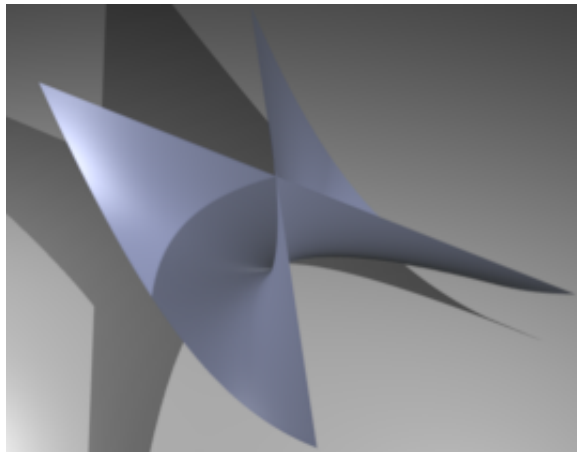
**Theorem**

*Let $X \subset \mathbb{A}^n$ be an irreducible aff. var. of dimension $d$ with $I(X) =< f_1, f_2, ..., f_n >$. Then Sing($X$) $\subset X$ is a subvariety given by the vanishing in $X$ of all $(n-d) \times (n-d)$ minors of the Jacobian matrix*

$$Jac = \left( \frac{\partial f_i}{\partial x_j} \right)$$

# Singularities



# Maps

- $F\colon \mathbb{A}^n \to \mathbb{A}^m$ is **a morphism (or polynomial map)** if it is defined by polynomials: $F(a) = (f_1(a), ..., f_m(a))$ for some $f_1, ..., f_m \in R := k[x_1, ..., x_n]$.
- $F\colon X \to Y$ is a **morphism of affine varieties** if it is the restriction of a morphism $\mathbb{A}^n \to \mathbb{A}^m$ (here $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$).
  $F(a) = (f_1(a), ..., f_m(a))$ for some $f_1, ..., f_m \in k[X] := R/I(X)$.
- **isomorphism** = morphism which has an inverse which is a morphism

# Projective Space

- **Zariski topology** on $\mathbb{A}^n$: declare the basic open sets to be for any $f \in R$:

$$D(f) = \mathbb{A}^n \setminus V(f) := \{a \in k^n : f(a) \neq 0\}$$

$\implies$ induces a topology on an algebraic variety $X \subset \mathbb{A}^n$
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- **Birational map** between two (affine/projective) varieties is "an isomorphism between open dense subsets"

# Resolution of Singularities

Let $k$ be an algebraically closed field.

- The problem of **resolution of singularities** asks whether every algebraic variety $X$ has a resolution, a non-singular variety $Y$ with a proper birational map $Y \to X$.
- Hironaka (1964): proved it for varieties over $k$, of characteristic $0$
- Open for characteristic $p$ and dimension $> 4$.

**TODAY:** characteristic $0$: dimension $1$ (curves), dimension $2$ (surfaces).

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- Let $C$ be a variety of dimension $1$, then we call $C$ an **algebraic curve**.
- Every algebraic curve has a unique nonsingular projective resolution.
- The **genus** of a smooth complete algebraic curve $C$ is a numerical invariant under birational maps. If $k = \mathbb{C}$, then an algebraic curve $C$ is the same as a Riemann surface. In this case, the smooth complex curve $X$ of genus $g$ is homeomorphic to the sphere with $g$ handles.

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- **genus $\geq 2$**: Every smooth curve of genus $\geq 2$ is birationally isomorphic to hyper-elliptic curves and non-hyper-elliptic curves.
  Example: hyperelliptic curves:

$$C/k : y^2 = f(x), \deg(f) \geq 5$$

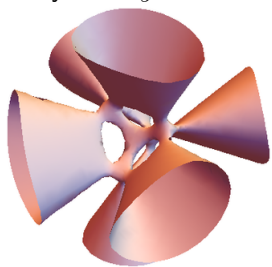
# Dimension 2: Surfaces

Let $k$ be an algebraically closed field.

- Let $S$ be a variety of dimension $2$, then we call $S$ an **algebraic surface**.
- Surfaces have many different nonsingular projective resolutions (unlike the case of curves). However, a surface still has a unique minimal resolution.
- $\kappa$ is the **Kodaira dimension**, an arithmetic invariant which classifies algebraic surfaces.

# Examples

- $\kappa = -\infty$: rational surfaces, ruled surface
  Any rational surface is birational to $\mathbb{P}^2$. A ruled surface: every point lies on a line.
- $\kappa = 0$: abelian surfaces, hyperelliptic surface, K3 surface, Enriques surface
  For example the K3 surface given by $x^4 + y^4 + z^4 + w^4 = 0$:



- $\kappa = 1$: elliptic surface
  For example, $E \times C$ where $E$ is an elliptic curve and $C$ is any curve.
- $\kappa = 2$: surface of general type

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We denote by $y = n^2$.

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  Let's examine $C(\mathbb{Q})$ which are the points in $\mathbb{P}^3$ with rational coordinates.

- Dimension of $C$ is $1$ and genus of $C$ is $1$ and has an obvious rational point $[0 : 1 : 1 : 1] \Longrightarrow C$ is birational to an elliptic curve.

# Diophantine Equation 1: An Elliptic Curve

- Since $C$ is birational to an elliptic curve, it has an model which can be computed to be:

$$E : y^2 = x^3 + 8x^2 + 12x$$

where $[0 : 1 : 1 : 1]$ is the point at infinity.

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where $[0 : 1 : 1 : 1]$ is the point at infinity.
- Somerville's point $S = [8 : 23 : 17 : 13]$ corresponds to the affine point $S_E = (48, 360)$ on $E$.

Hence, a modern algebraic geometer would say that Mary Somerville constructed a point on an elliptic curve.

# Other rational points?

Recall that elliptic curves come with a group law.

**Theorem (Mordell-Weil)**

*Let $E/\mathbb{Q}$ be an elliptic curve. Then,*

$$E(\mathbb{Q}) \cong \mathit{Tors} + \mathbb{Z}^r,$$

*where $r$ is called the rank of $E$.*

# Other rational points?

- In our case, $E : y^2 = x^3 + 8x^2 + 12x$ and

$$E(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$$

with generators $S_1 = (-6, 0), \quad S_2 = (-2, 0), \quad S_3 = (6, -24).$

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- They correspond to the points $S_1 = [1 : 1 : 1 : -1]$, $S_2 = [1 : 1 : -1 : 1]$, $S_3 = [1 : 1 : -1 : -1]$ on the initial curve $C$, which all lead to the rather uninteresting pair of solutions. Somerville's point turns out to be $S = S_3 + S_3 = [2]S_3$, which has infinite order.

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- In conclusion, Somerville's point gives rise to infinitely many solutions. For example,

$$S + S = (36481/3600, 9620479/216000)$$

  which gives rise to the solution: $[10130640, 18240049, 12976609, 9286489]$.

# Diophantine equation 2

A present-day mathematician would rewrite the statement in the following way. Find such integer values of $x, y, z$ such that:

$$\begin{cases} x^2 + axy + y^2 = w^2 \\ x^2 + a'xz + z^2 = w'^2 \\ y^2 + a''yz + z^2 = w''^2. \end{cases} \tag{6}$$

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As the above equations are homogeneous, a natural way to view their common solutions is as a projective variety in $\mathbb{P}^5$. More concretely, consider the following homogeneous polynomials of degree 2:

$$\begin{cases} f_1(x, y, z, w, w', w'') = x^2 + axy + y^2 - w^2 \\ f_2(x, y, z, w, w', w'') = x^2 + a'xz + z^2 - w'^2 \\ f_3(x, y, z, w, w', w'') = y^2 + a''yz + z^2 - w''^2. \end{cases} \tag{7}$$

# Diophantine equation 2

- Define the projective variety:

$$S_{a,a',a''} := \{\mathbf{p} = [x : y : z : w : w' : w''] \in \mathbb{P}^5 : f_1(\mathbf{p}) = 0, f_2(\mathbf{p}) = 0, f_3(\mathbf{p}) = 0\}.$$

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- The resolution of singularities theorem tells us that $S_{a,a',a''}$ has a minimal resolution belonging to the classification of surfaces.Which one?

# Diophantine equation 2: a K3 surface

**Theorem**

*Let $k$ be a field of characteristic $0$. Assume that $X$ is a surface over $k$ of one of the following three types:*

1. *a quartic surface in $\mathbb{P}^3_k$,*
2. *an intersection of a cubic and a quadric hypersurface in $\mathbb{P}^4_k$,*
3. *an intersection of three quadrics in $\mathbb{P}^5_k$.*

*Furthermore, assume that all singularities of $X$ are rational double points. Then the minimal regular model of $X$ is a K3-surface.*

# Back to Somerville's Solution

- We will work in the general case (i.e. when $a \notin \{\pm 2\}$, $a' \notin \{\pm 2\}$ and $a'' \notin \{\pm 2\}$) and moreover we assume that $a, a'$ and $a''$ are all distinct. In the language of Algebraic Geometry, Somerville's solution proposes a birational map

$$F \colon \mathbb{P}^1 \dashrightarrow S_{a,a',a''}$$

defined over $\mathbb{Q}$, given as a composition of a couple of explicit rational maps. More precisely, $F = f \circ g$, where $f$ is given by (2) and $g$ is given by (4).

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- Hence, Somerville's solution $Im(F) \subset S_{a,a',a''}$ is birational to $\mathbb{P}^1$ and it can be shown that it misses the singularities.
- Hence, in modern language, Somerville constructs a rational curve on a $K3$ surface.

# Thank you!