The modular approach for solving

\[ x^r + y^r = z^p \]

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Generalized Fermat Equation:

\[ x^p + y^q = z^r, \quad p, q, r \in \mathbb{Z}_{\geq 2}. \] (1)
Introduction

Generalized Fermat Equation:

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Conjecture (Fermat-Catalan)

Over all choices of prime exponents \( p, q, r \) satisfying \( 1/p + 1/q + 1/r < 1 \) the equation (1) admits only finitely many integer solutions \((a, b, c)\) which are non-trivial (i.e. \( abc \neq 0 \)) coprime (i.e. \( \gcd(a, b, c) = 1 \)). (Here solutions like \( 2^3 + 1^q = 3^2 \) are counted only once.)
Introduction

Generalized Fermat Equation:

\[ x^p + y^q = z^r, \quad p, q, r \in \mathbb{Z}_{\geq 2}. \]

**Theorem (Darmon-Granville 1995)**

If we fix the prime exponents \( p, q, r \) such that \( 1/p + 1/q + 1/r < 1 \), then there are only finitely many coprime integers solutions to the above equation.
Introduction

Generalized Fermat Equation:

\[ x^p + y^q = z^r, \quad p, q, r \in \mathbb{Z}_{\geq 2}. \]

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Families of signatures that have been 'solved':

- \((n, n, n), n \geq 3\) Wiles, Taylor–Wiles 1995;
- \((n, n, 2), n \geq 4\) and \((n, n, 3), n \geq 3\) Darmon–Merel, Poonen 1998;
- \((3j, 3k, n), j, k \geq 2, n \geq 3\) Kraus 1998;
- \((2n, 2n, 5), n \geq 2\) Bennett 2006;
- \((5, 5, 7), (5, 5, 19),\) and \((7, 7, 5)\) Dahmen, Siksek 2014;
- \((5, 5, n)*\) Billerey, Chen, Dembélé, Dieulefait and Freitas 2022;
- \((11, 11, n)*,(13, 13, n)*\) Billerey, Chen, Dieulefait, Freitas and Najman 2022.
Asymptotic \((r, r, p)\)

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**Theorem (M.)**

Fix \(r \geq 5\) such that \(r \not\equiv 1 \mod 8\). Let \(\mathbb{Q}^+ := \mathbb{Q}(\zeta_r + \zeta_r^{-1})\), suppose that 2 is inert in \(\mathbb{Q}^+\) and \(2 \nmid h_{\mathbb{Q}^+}^+\). Then, there is a constant \(B_r\) (depending only on \(r\)) such that for each rational prime \(p > B_r\), the equation \(x^r + y^r = z^p\) has no integer solutions with \(2|z\).
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**Theorem (M.)**

Fix $r \geq 5$ such that $r \not\equiv 1 \mod 8$. Let $\mathbb{Q}^+ := \mathbb{Q}(\zeta_r + \zeta_r^{-1})$, suppose that 2 is inert in $\mathbb{Q}^+$ and $2 \nmid h_{\mathbb{Q}^+}$. Then, there is a constant $B_r$ (depending only on $r$) such that for each rational prime $p > B_r$, the equation $x^r + y^r = z^p$ has no integer solutions with $2|z$.

**Example**

This implies that there are no integer solutions $(x, y, z)$ with $2|z$ for $p$ large enough for signatures:

$$(5, 5, p), (7, 7, p), (11, 11, p), (13, 13, p), (19, 19, p), (23, 23, p), (37, 37, p), (43, 43, p).$$
Modular Method - Sketch

Step 1: Select a Frey curve. Suppose we have \( x, y, z \in \mathbb{Z} \), non-trivial, coprime with \( x \) and \( y \) s.t. \( x^p + y^p = z^p \). We construct the following Frey Curve:

\[
E/\mathbb{Q} : Y^2 = X(X - x^p)(X + y^p)
\]

which has Artin conductor \( N_p = 2 \).

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\[
E_{x,y,z} : Y^2 = X(X - A)(X + B)(2)
\]

defined over the totally real number field \( \mathbb{Q}^+ = \mathbb{Q}(\zeta_r + \zeta_r^{-1}) \). The Artin conductor of \( E \) is \( N_p = 2 e_2 P e_r r \).
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$$E_{x,y,z} : Y^2 = X(X - A)(X + B) \quad (2)$$

defined over the totally real number field $\mathbb{Q}^+ := \mathbb{Q}(\zeta_r + \zeta_r^{-1})$. The Artin conductor of $E$ is

$$N_p = 2^{e_2} \mathfrak{p}_r^{e_r}. $$

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Step 2: Modularity.
Freitas, Hung and Siksek (2013) proved Modularity of elliptic curves over totally real fields (up to a finite number of exceptions).

$$E \sim \mathfrak{f}$$

where $\mathfrak{f}$ is a Hilbert newform of parallel weight 2 and level $N$, where $N$ is the conductor of $E$. 
Step 3: Irreducibility.
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Freitas and Siksek (2015) proved irreducibility of $\bar{\rho}_{E,p}$ for elliptic curves $E$ over totally real number fields under a few technical assumptions, if $p$ is large enough.
Modular Method - Sketch

Step 4: Level Lowering
Ribet’s Level Lowering Theorem (1986) implies that there exists a rational newform $f$ of level $N_p = 2$ and weight 2 such that $E \sim_p f$. 
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Step 4: Level lowering. Use level lowering theorems, which require irreducibility of $\bar{\rho}_{E,p}$, to conclude that

$$\bar{\rho}_{E,p} \simeq \bar{\rho}_{\mathfrak{f},p}$$

where $\mathfrak{f}$ is a Hilbert newform over $\mathbb{Q}^+$ of parallel weight 2, of trivial character, and rational Hecke eigenvalues, with level equal to the Artin conductor $N_p$ of $E$. 

Level Lowering

Step 5: Eliminate
Simply there are no newforms at level $N_p = 2$, hence giving the desired contradiction.
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Prove that among the finitely many Hilbert newforms predicted above, none of them corresponds to $\bar{\rho}_{E,p}$ and get the desired contradiction.
Modular Method - Step 5

Step 5 is challenging in general.

Example

The approach we used:

1. an 'Eichler-Shimura'-type result;
2. image of inertia comparison arguments;
3. the study of certain $S$-unit equations;

to get a contradiction.
Modular Method - Recap

Recap steps 1, 2, 3, 4:
Assuming we have a (non-trivial, primitive) solution

\[(x, y, z) \text{ to } x^r + y^r = z^p \text{ with } 2 | z + \text{ some class field theoretic assumptions}\]
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Step 1 \((x, y, z) \rightsquigarrow E_{x, y, z}, \text{ Frey elliptic curve}\)
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**Step 1** \((x, y, z) \leadsto E_{x, y, z}, \text{ Frey elliptic curve}\)

**Steps 2, 3, 4** Get a Hilbert newform \(f\), of level \(\mathcal{N}_p\), such that

\[\bar{\rho}_{E, p} \simeq \bar{\rho}_{f, p}\]
1. **Eichler-Shimura** gives an elliptic curve $E'$ with conductor $\mathcal{N}_p$ such that

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1. **Eichler-Shimura** gives an elliptic curve $E'$ with conductor $\mathcal{N}_p$ such that

$$\bar{\rho}_{E,p} \simeq \bar{\rho}_{f,p} \simeq \bar{\rho}_{E',p}$$

2. **Image of inertia comparison** at the bad prime 2 gives $v_2(j_{E'}) < 0$.

3. Elliptic curves $E'$ with such properties are parametrized by $S$-unit equations. By **finiteness of solutions** to $S$-unit equations + the class field theoretic assumptions we get the contradiction.
Modular Method

Select a Frey Curve - Modularity - Irreducibility - Level lowering - Eliminate
Thank you!