Ergodic Ramsey Theory - Class 1:
Ramsey Theory and Ergodic Theory

Ergodic Ramsey Theory–an Update

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0. Introduction.

This survey is an expanded version and elaboration of the material presented by the author at the *Workshop on Algebraic and Number Theoretic Aspects of Ergodic Theory* which was held in April 1994 as part of the 1993/1994 *Warwick Symposium on Dynamics of* $\mathbf{Z}^n$*-actions and their connections with Commutative Algebra, Number Theory and Statistical Mechanics.* The leitmotif of this paper is: Ramsey theory and ergodic theory of multiple recurrence are two beautiful, tightly intertwined and mutually perpetuating disciplines. The scope of the survey is mostly limited to Ramsey-theoretical and ergodic questions about $\mathbf{Z}^n$–partly because of the proclaimed

Thm (Schur, 1916): If $\mathbb{N} = C_1 \cup \dots \cup C_r$, then one of the C_i contains a triple $\{x, y, x+y\}$.

Thm (Schur, again): For any $r \in \mathbb{N}$ $\exists$ $N \in \mathbb{N}$ s.t. if $\{1, \dots, N\} = C_1 \cup \dots \cup C_r$, then one of the C_i contains $\{x, y, x+y\}$.

Thm (van der Waerden, 1927) If $\mathbb{N} = C_1 \cup \dots \cup C_r$ then one of the C_i contains arbitrarily long arithmetic progressions.

$\forall k$ $\exists$ $x, y \in \mathbb{N}$ s.t. $\{x, x+y, x+2y, \dots, x+ky\} \subset C_i$

Thm (Szemerédi's theorem, 1975) Let $A \subset \mathbb{N}$ such that $\bar{d}(A) = \limsup_{N\to\infty} \frac{1}{N}|A \cap \{1, \dots, N\}| > 0$, then A contains arbitrarily long arithmetic progressions.

upper density

Exercise: $\bar{d}(A \cup B) \leqslant \bar{d}(A) + \bar{d}(B)$

Thm (Roth) 1953: Every set $A \subset \mathbb{N}$ with $\bar{d}(A) > 0$ contains

$\{x, x+y, x+2y\}$ for some $x, y \in \mathbb{N}$.

Thm (Sárközi, 1978): If $A \subset \mathbb{N}$ has $\bar{d}(A) > 0$, then $\exists\, x, y \in A$ s.t. $x-y$ is a perfect square. In other words, $A \supset \{x, x+n^2\}$

Def: A set $R \subset \mathbb{Z}$ is intersective if $\forall A \subset \mathbb{N}$, $\bar{d}(A) > 0$, $A - A := \{x-y : x, y \in A\}$ satisfies $(A-A) \cap R \neq \emptyset$.

Example: If $0 \in R$ then R is intersective.

For $n \in \mathbb{N}$, the set $\{nx : x \in \mathbb{N}\}$ is intersective

The set $2\mathbb{Z}+1$ is not intersective.

In fact $\forall k \in \mathbb{N}$, if R does not contain a multiple of k, then R is not intersective.

Thm (Kamae - Mendès-France): Let $f \in \mathbb{Z}[x]$. Then $R = \{f(x) : x \in \mathbb{N}\}$ is intersective if and only if it contains a multiple of every $k \in \mathbb{N}$.

Eg: if $f(0) = 0$.

Thm (Bergelson - Leibman 1996): Let $f_1, f_2, \dots, f_k \in \mathbb{Z}[x]$ s.t. $f_i(0) = 0$. Then for any $A \subset \mathbb{N}$ with $\bar{d}(A) > 0$ $\exists\, x, y \in \mathbb{N}$ s.t.

$$A \supset \{x, x + f_1(y), x + f_2(y), \dots, x + f_k(y)\}.$$

If $f_i(y) = iy$, we recover Szemerédi's theorem.

If $f_i(y) = iy^2$, we obtain $A \supset \{x, x+y^2, x+2y^2, \dots, x+ky^2\}$

Thm (Green-Tao, 2008) The set of prime numbers contains arbitrarily long arithmetic progressions.

Problem: Prove that for any finite coloring on $\mathbb{N}$ there is

Problem: Prove that for any finite coloring on $\mathbb{N}$ there is a monochromatic solution to $x^2+y^2=z^2$.

Problem: Prove that for any finite coloring on $\mathbb{N}$ there exists a monochromatic configuration of the form $\{\underbrace{x, y, x+y}, xy\}$

Ergodic Theory

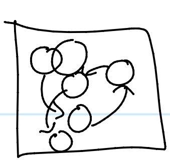
Def: (Measure preserving system): Let $(X, \mathcal{B}, \mu)$ be a probability space and let $T: X \to X$ be measurable.
If $\forall A \in \mathcal{B}$ $\mu(T^{-1}A) = \mu(A)$ we say that T preserves the measure and we call $(X, \mathcal{B}, \mu, T)$ a measure preserving system.

$T^{-1}A = \{x \in X : Tx \in A\}$.

Example (Circle rotation). Let $\alpha \in \mathbb{R}$ and consider $X = \mathbb{R}/\mathbb{Z} = \mathbb{T}$, let $\mathcal{B}$ be the Borel σ-algebra and μ be the Lebesgue/Haar measure on $\mathbb{T} = [0,1)$
Let $T: X \to X$, $Tx = x+\alpha \bmod 1$. Then $(X, \mathcal{B}, \mu, T)$ is a m.p.s.

Example: (Doubling map): Let $(X, \mathcal{B}, \mu)$ be as above. Let $T: x \mapsto 2x \bmod 1$.
Then $(X, \mathcal{B}, \mu, T)$ is a m.p.s.

Thm: (Poincaré Recurrence Theorem) Let $(X, \mathcal{B}, \mu, T)$ be a m.p.s. and let $A \in \mathcal{B}$ have $\mu(A) > 0$. Then $\exists\, n \in \mathbb{N}$ s.t. $\mu(A \cap T^{-n}A) > 0$.



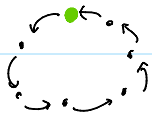
Pf: Consider $A, T^{-1}A, T^{-2}A, T^{-3}A, \dots$ All these sets have same measure. Since $1 = \mu(X) \geq \mu\left(\bigcup_{i=1}^{\infty} T^{-i}A\right)$, it follows that $\exists\, i < j$ s.t. $\mu(T^{-i}A \cap T^{-j}A) > 0$. $\boxed{n = j-i}$

$\underbrace{T^{-i}A \cap T^{-j}A} = T^{-i}(\underbrace{A \cap T^{-(j-i)}A}) \Rightarrow \mu(A \cap T^{-(j-i)}A) > 0$.

Def: A set $R \subset \mathbb{N}$ is called a <u>set of recurrence</u> if for any m.p.s. $(X, \mathcal{B}, \mu, T)$ and any $A \in \mathcal{B}$, $\mu(A) > 0$, $\exists\, n \in R$ s.t. $\mu(A \cap T^{-n}A) > 0$.

Examples: For any $k \in \mathbb{N}$, the set $\{kn : n \in \mathbb{N}\}$ is a set of recurrence.

For any $k \in \mathbb{N}$, $\mathbb{N} \setminus (k\mathbb{N})$ is not a set of recurrence.

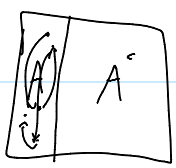


For any infinite set I, the set of differences $I - I$ is a set of recurrence.

The set of perfect squares is a set of recurrence.

Proposition: A set $R \subset \mathbb{N}$ is a set of recurrence if and only if it is an intersective set.

Def: A m.p.s. $(X, \mathcal{B}, \mu, T)$ is <u>ergodic</u> if any $A \in \mathcal{B}$ which is T-invariant (i.e. $T^{-1}A = A$) is trivial in the sense that $\mu(A) = 0$ or $\mu(A) = 1$.



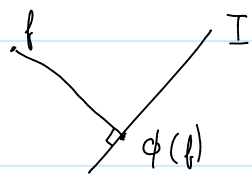
Given a m.p.s. $(X, \mathcal{B}, \mu, T)$ we can look at $L^2(X, \mathcal{B}, \mu)$.

Inside L^2 sits the set $I = \{f \in L^2 : f \circ T = f\}$.

I is a closed subspace of L^2, so $\exists\, \phi : L^2 \to I$ s.t.

- $\|\phi(f) - f\| = \inf_{g \in I} \|g - f\|$

- $\langle f - \phi(f), g \rangle = 0 \quad \forall g \in I$



Fact: $(X, \mathcal{B}, \mu, T)$ is ergodic if and only if $I = \{\text{constant functions}\}$.

Thm (Birkhoff's pointwise ergodic thm): Let $(X, \mathcal{B}, \mu, T)$ be a m.p.s, let $f \in L^2(X)$. Then

$$\boxed{\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f \circ T^n} = \phi(f) \quad \text{a.e.}$$

In particular, if the system is ergodic $\quad \lim \frac{1}{N} \sum^{N} f \circ T^n = \int f \, d\mu$ a.e.

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \ldots$$

In particular, if the system is ergodic, $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f\circ T^n = \int_X f\,d\mu$ a.e.

<u>Notation:</u> Given a sequence (a_n) we write

$$\lim_{N-M\to\infty} \frac{1}{N-M}\sum_{n=M}^{N} a_n = L \quad \text{to mean that}$$

$$\forall \varepsilon>0 \ \exists\ K \text{ s.t. } \forall\, N,M\in\mathbb{N},\ N-M>K,\ \left|\frac{1}{N-M}\sum_{n=M}^{N} a_n - L\right|<\varepsilon.$$

<u>Exercise:</u> $\lim_{N-M\to\infty} \frac{1}{N-M}\sum_{n=M}^{N} a_n = L$ iff for every sequence $(F_N)_{N=1}^{\infty}$ of intervals $F_N=\{b_N, b_N+1, \ldots, b_N+N\}$,

$$\lim_{N\to\infty} \frac{1}{|F_N|}\sum_{n\in F_N} a_n = L.$$

Thm (von Neumann ergodic thm) Let $(X,\mathcal{B},\mu,T)$ be a m.p.s., let $f\in L^2$. Then

$$\lim_{N-M\to\infty} \frac{1}{N-M}\sum_{n=M}^{N} f\circ T^n = \phi(f) \quad \text{in } L^2 \text{ norm.}$$

<u>Thm (Khintchine):</u> Let $(X,\mathcal{B},\mu,T)$ be a m.p.s. and let $A\in\mathcal{B}$, $\mu(A)>0$. Then

$$\lim_{N-M\to\infty} \frac{1}{N-M}\sum_{n=M}^{N} \mu(A\cap T^{-n}A) \geq \mu^2(A)$$

In particular, $\forall \varepsilon>0\ \exists\, n\in\mathbb{N}$ s.t. $\mu(A\cap T^{-n}A)>\mu^2(A)-\varepsilon$.

This is not necessarily true. $\rightarrow$ $\boxed{\mu(A\cap T^{-n}A)\geq\mu^2(A)}$

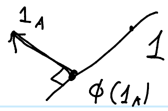
<u>Pf:</u> $$\lim_{N-M\to\infty} \frac{1}{N-M}\sum_{n=M}^{N} \mu(A\cap T^{-n}A) = \lim_{N-M\to\infty} \frac{1}{N-M}\sum_{n=M}^{N} \int 1_{A\cap T^{-n}A}\,d\mu$$

$$\int \lim_{N-M\to\infty} \frac{1}{N-M}\sum_{n=M}^{N} 1_A\cdot 1_A\circ T^n\,d\mu = \int 1_A\cdot \lim_{N-M\to\infty} \frac{1}{N-M}\sum_{n=M}^{N} 1_A\circ T^n\,d\mu$$

$$= \int 1_A\cdot\phi(1_A)\,d\mu = \langle 1_A, \phi(1_A)\rangle = \|\phi(1_A)\|^2\cdot\|1\|^2$$

$$= \int_X 1_A \cdot \phi(1_A)\, d\mu = \langle 1_A, \phi(1_A) \rangle = \|\phi(1_A)\|^2 \cdot \|1\|^2$$

$$\geq \langle \phi(1_A), 1 \rangle^2 \underset{\uparrow}{=} \langle 1_A, 1 \rangle^2 = \mu(A)^2$$



$\langle 1_A - \phi(1_A), 1 \rangle = 0$

<u>Corollary:</u> Let $(X, \mathcal{B}, \mu, T)$ be a m.p.s., let $A \in \mathcal{B}$ have $\mu(A) > 0$.

Then $R = \{ n : \mu(A \cap T^{-n} A) > 0 \}$ has bounded gaps, i.e.

$\mathbb{N} \setminus R$ does not contain intervals of arbitrary length.

Such sets are called syndetic.