Heart and Soul of Ergodic Ramsey Theory

Sarközy's theorem: $\forall\ A \subset \mathbb{N}$, $\bar{d}(A) > 0$

$\exists\ x, y \in A$ s.t. $x - y = n^2$ for some n.

$\exists\ x, n \in \mathbb{N}$ s.t. $\{x, x+n^2\} \subset A$

$\Leftrightarrow \exists n \in \mathbb{N}$ s.t. $A \cap (A - n^2) \neq \emptyset$

Let $T: \mathbb{N} \to \mathbb{N}$ $\quad Tx \mapsto x+1$

$T^{-1}A = A - 1 \qquad T^{-n}A = A - n \qquad \bar{d}(A) > 0$

$\Leftrightarrow \exists\ n \in \mathbb{N}$ s.t. $\boxed{A \cap T^{-n^2}A} \neq \emptyset$

$\bar{d}(T^{-1}A) = \bar{d}(A)$ *

Furstenberg Correspondence Principle

For many "arithmetic purposes" one can think of $\boxed{(\mathbb{N}, \mathcal{P}(\mathbb{N}), \bar{d}, T)}$ as a measure preserving system

Ex: $\exists$ sets $A, B \subset \mathbb{N}$, $A \cap B = \emptyset$

$\bar{d}(A) = \bar{d}(B) = 1$

Recall Szemerédi's theorem: $\forall\ A \subset \mathbb{N}$, $\bar{d}(A) > 0$, $\forall\ k \in \mathbb{N}\ \exists\ x, n$ s.t. $\{x, x+n, x+2n, \dots, x+kn\} \subset A$

Ex: Show that $\exists\ \mathbb{N} = A \cup B$ s.t. neither A nor B contains an infinite arithmetic progression.

$\exists\ x, n$ s.t. $\{x, x+n, x+2n, \dots, x+kn\} \subset A$

$\Leftrightarrow \exists n \in \mathbb{N}$ s.t. $A \cap (A-n) \cap (A-2n) \cap \dots \cap (A-kn) \neq \emptyset$

$\Leftrightarrow \exists\ n \in \mathbb{N}$ s.t. $A \cap T^{-n}A \cap T^{-2n}A \cap \dots \cap T^{-kn}A \neq \emptyset$

$\Leftarrow \exists\ n \in \mathbb{N}$ s.t. $\boxed{\bar{d}(A \cap T^{-n}A \cap \dots \cap T^{-kn}A) > 0}$

Thm (Furstenberg, 1977; "Multiple Recurrence Theorem".)

Thm (Furstenberg, 1977; "Multiple Recurrence Theorem".)
$\forall (X, \mathcal{B}, \mu, T)$ $\forall A \in \mathcal{B}$ $\mu(A) > 0$, $\forall k \in \mathbb{N}$ $\exists n \in \mathbb{N}$ s.t. $\boxed{\mu(A \cap T^{-n}A \cap T^{-2n}A \cap \cdots \cap T^{-kn}A) > 0}$

Prop: Let $E \subset \mathbb{N}$. Then $\exists (X, \mathcal{B}, \mu, T)$ m.p.s. and $A \in \mathcal{B}$ s.t. $\bar{d}(E) = \mu(A)$ and $\forall n_1, \ldots, n_k \in \mathbb{N}$

$$\bar{d}\left(E \cap (E - n_1) \cap \cdots \cap (E - n_k)\right) \geq \mu(A \cap T^{-n_1}A \cap \cdots \cap T^{-n_k}A)$$

Pf: X, $\mathcal{B}$, T and A don't depend on E.
$X = \{0,1\}^{\mathbb{N}_0}$ is a compact space. $\mathcal{B}$ = Borel σ-algebra.
$T : (x_n)_{n=0}^{\infty} \mapsto (x_{n+1})_{n=0}^{\infty}$ (it is continuous)
$A = \{(x_n)_{n=0}^{\infty} : x_0 = 1\}$. Let $x \in X$ s.t. $x_n = 1$ iff $n \in E$.
For each $N \in \mathbb{N}$, consider $\mu_N = \frac{1}{N}\sum_{n=1}^{N} \delta_{T^n x}$.
Choose (N_k) s.t. $\bar{d}(E) = \lim_{k\to\infty} \underbrace{\frac{1}{N_k} |E \cap \{1, \ldots, N_k\}|}_{\mu_{N_k}(A)}$
Recall that the space of Borel prob. measures on X is a compact metric space under the weak* topology.
Pass if needed to a subsequence of μ_{N_k} so that
$\mu = \lim_{k\to\infty} \mu_{N_k}$ exists in the weak* topology.
One can show that μ is T-invariant.
Recall: $\nu_n \longrightarrow \nu$ in the weak* topology
iff $\forall f \in C(X)$, $\int_X f\, d\nu_n \longrightarrow \int_X f\, d\nu$

Note that $\delta_{T^n x}(A) = 1 \iff T^n x \in A \iff n \in E$
Therefore $\mu_N(A) = \frac{1}{N}\sum_{n=1}^{N} \delta_{T^n x}(A) = \frac{1}{N}|E \cap \{1, \ldots, N\}|$
Hence $\mu(A) = \bar{d}(E)$.
Finally, for $n_1, \ldots, n_k \in \mathbb{N}$, $\delta_{T^n x}(A \cap T^{-n_1}A \cap \cdots \cap T^{-n_k}A) = 1$
$\iff n \in E \cap (E - n_1) \cap \cdots \cap (E - n_k)$
which implies that $\bar{d}(E \cap \cdots \cap (E - n_k)) \boxed{\geq} \lim_{k\to\infty} \frac{1}{N_k} |E \cap \cdots \cap (E - n_k) \cap [N_k]|$
$= \mu(A \cap \cdots \cap T^{-n_k}A)$ ∎

Note that A is clopen, i.e. 1_A is continuous

Note that A is clopen, i.e. 1_A is continuous

Hence $\mu_N(A) = \int 1_A \, d\mu_N \longrightarrow \int 1_A \, d\mu = \mu(A)$

Recall that - $R \subset \mathbb{N}$ is a set of recurrence if $\forall (X, \mathcal{B}, \mu, T)$, $\forall A \in \mathcal{B}$ $\mu(A) > 0$ $\exists n \in R$ s.t. $\mu(A \cap T^{-n} A) > 0$

- $R \subset \mathbb{N}$ is intersective if $\forall E \subset \mathbb{N}$. $\bar{d}(E) > 0$, $(E - E) \cap R \neq \emptyset$.

Prop: Let $R \subset \mathbb{N}$. Then TFAE:

(1) R is a set of recurrence

(2) R is an intersective set

(3) $\forall E \subset \mathbb{N}$, $\bar{d}(E) > 0$, $\exists n \in R$ s.t. $\bar{d}(E \cap (E - n)) > 0$

Pf: (1) $\Rightarrow$ (3). Let R be a set of recurrence, let $E \subset \mathbb{N}$, $\bar{d}(E) > 0$. Apply the correspondence principle to get $(X, \mathcal{B}, \mu, T)$ and $A \in \mathcal{B}$ s.t. $\mu(A) > 0$. Then $\exists n \in R$ s.t. $\mu(A \cap T^{-n} A) > 0$, so $\bar{d}(E \cap (E - n)) > 0$.

(3) $\Rightarrow$ (2) If $E \cap (E - n) \neq \emptyset$, then $n \in E - E$.

(2) $\Rightarrow$ (1) Let R be an intersective set. Let $(X, \mathcal{B}, \mu, T)$ be a m.p.s., let $A \in \mathcal{B}$ $\mu(A) > 0$. For each $x \in X$, let $E_x = \{ n : T^n x \in A \}$.

$$\bar{d}(E_x) = \limsup_{N \to \infty} \frac{1}{N} |E_x \cap \{1, \dots, N\}| = \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} 1_A(T^n x).$$

$$\int_X \bar{d}(E_x) \, d\mu = \int_X \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} 1_A(T^n x) \, d\mu \geq \text{(Fatou's lemma)}$$

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \overbrace{\int_X 1_A \circ T^n \, d\mu}^{\mu(A)} = \mu(A) > 0.$$

Hence $B = \{ x \in X : \bar{d}(E_x) > \mu(A)/2 \}$ has $\mu(B) > 0$.

For each $x \in B$ $\exists$ $a_x, b_x \in E_x$ and $n_x \in R$ s.t. $a_x - b_x = n_x$

There are only countably many choices for a_x, b_x, n_x so $\exists$ a, b, n and $C \subset B$ with $\mu(C) > 0$ s.t. $\forall x \in C$, $a, b \in E_x$ and $a - b = n$.

$\Downarrow$ $T^a x, T^b x \in A$

$$C \subseteq T^{-a} A \cap T^{-b} A = T^{-b}(T^{-n} A \cap A) \Rightarrow \mu(A \cap T^{-n} A) > 0 \quad \blacksquare$$

Thm: Let $E_1, \dots, E_k \subset \mathbb{N}$ all with positive upper density

Then $(E_1 - E_1) \cap (E_2 - E_2) \cap \dots \cap (E_k - E_k)$ is syndetic.

$E_1 \times \dots \times E_k \subset \mathbb{N}^k$

Def: A set $S \subset \mathbb{N}$ is called syndetic if it has bounded gaps (i.e. if $\exists M$ s.t. $S \cup S - 1 \cup \dots \cup S - M = \mathbb{N}$).

Thm (Khintchine's Recurrence): $\forall (X, \mathcal{B}, \mu, T)$ $\forall A \in \mathcal{B}$ $\mu(A) > 0$

gaps (i.e. if $\exists M$ s.t. $S \cup S-1 \cup \cdots \cup S-M = \mathbb{N}$).

Thm (Khintchine's Recurrence): $\forall (X, \mathcal{B}, \mu, T)$ $\forall A \in \mathcal{B}$ $\mu(A)>0$,
$\{n : \mu(A \cap T^{-n}A) > 0\}$ is syndetic.

Pf of (*) Apply the Correspondence principle for each E_i to get a m.p.s. $(X_i, \mathcal{B}_i, \mu_i, T_i)$ and $A_i \in \mathcal{B}_i$ s.t. $\mu_i(A_i) > 0$.
Let $(X, \mathcal{B}, \mu, T)$ be the product of these k systems, i.e.
$X = X_1 \times \cdots \times X_k$, $\mathcal{B} = \mathcal{B}_1 \otimes \cdots \otimes \mathcal{B}_k$, $\mu = \mu_1 \otimes \cdots \otimes \mu_k$,
$T(x_1, \ldots, x_k) := (T_1 x_1, \ldots, T_k x_k)$. Let $A = A_1 \times \cdots \times A_k$.
Note that $\mu(A) = \prod \mu_i(A_i) > 0$, so $\underbrace{\{n : \mu(A \cap T^{-n}A) > 0\}}^{=: R}$ is syndetic. Note $A \cap T^{-n} A = (A_1 \cap T_1^{-n} A_1) \times \cdots \times (A_k \cap T_k^{-n} A_k)$
so $\mu_i(A_i \cap T_i^{-n} A_i) > 0$ $\forall i \in \{1, \ldots, k\}$ and $\forall n \in R$
Therefore $\bar{d}(E_i \cap E_i - n) > 0$ $\forall i \in \{1, \ldots, k\}$ and $\forall n \in R$
Hence $n \in E_i - E_i$ $\forall i \in \{1, \ldots, k\}$ and $\forall n \in R$ ∎

Thm (Schur): If $\mathbb{N} = C_1 \cup \cdots \cup C_r$ $\exists$ $x, y \in \mathbb{N}$ $\exists i$ s.t.
$\{x, y, x+y\} \subset C_i$.

Ex: Show that, under the same conditions we can choose x, y s.t. $\{x, y, xy\} \subset C_i$.

Pf of Schur's thm: (Bergelson, 1980's)
Find $s \in \{1, \ldots, r\}$ s.t. $C_1, \ldots, C_s$ have positive upper density and $\bar{d}(C_i) = 0$ $\forall i > s$.
$(C_1 - C_1) \cap (C_2 - C_2) \cap \cdots \cap (C_s - C_s)$ is syndetic, and in particular, it is not contained in $\underbrace{C_{s+1} \cup \cdots \cup C_r}_{\bar{d}(\ ) = 0}$

Therefore $\exists i \in \{1, \ldots, s\}$ s.t. $(C_1 - C_1) \cap \cdots \cap (C_s - C_s) \cap C_i \neq \emptyset$
Take x in that intersection, then $x \in C_i \cap (C_i - C_i)$, so $x = z - y$ for some $z, y \in C_i$. Hence $\{x, y, x+y\} \subset C_i$ ∎

$\exists x : \{x, x+n^2\} \subset E \iff E \cap (E - n^2) \neq \emptyset$

$\mu(A \cap T^{-n^2} A) > 0$

$\exists n: \{x, x+n^2\} \subset E \iff E \cap (E - n^2) \neq \emptyset$

$\mu(A \cap T^{-n^2} A) > 0$.