09 October 2021 16:42

Polynomial Recurrence

Sárközy's theorem: The set $\boxed{\{n^2 : n \in \mathbb{N}\}}$ is an intersective, i.e.

If $A \subset \mathbb{N}$, $\bar{d}(A) > 0$, then $A - A$ contains a perfect square

<u>Uniform distribution mod 1:</u> <u>Def</u>: A sequence (x_n) in $[0,1)$ is <u>uniformly distributed</u> if $\forall\, (a,b) \subset [0,1)$, $\quad d(\{n : x_n \in (a,b)\}) = b - a$, i.e.

$$\lim_{N\to\infty} \frac{1}{N} \left| \{ n \leq N : x_n \in (a,b) \} \right| = b - a.$$

<u>Proposition</u>: (Weyl) Let (x_n) be a sequence in $[0,1)$. TFAE

(1) (x_n) is u.d.

(2) The sequence $\mu_N = \frac{1}{N}\sum_{n=1}^{N} \delta_{x_n}$ converges in the weak* topology to Leb.

(3) $\forall f \in C[0,1]$, $\quad \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)\,dx \qquad (*)$

(4) $\forall k \in \mathbb{N}$ $\quad \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k x_n} = 0$

<u>Pf</u>: [Sketch]: (2) $\Leftrightarrow$ (3) ✓ $\quad$ (3) $\Rightarrow$ (4) ✓

(4) $\Rightarrow$ (3) Finite linear combinations of characters are dense $C([0,1])$.

(1) $\Leftrightarrow$ $(*)$ holds for $f = 1_{(a,b)}$.

<u>Example</u>: $x_n = n\alpha \mod 1$ is uniformly distributed $\forall \alpha \in \mathbb{R}\setminus\mathbb{Q}$.

$x_n = \sqrt{n} \mod 1$ ✓ $\qquad x_n = n^\alpha$ for $\alpha \notin \mathbb{N}$ $\alpha > 0$

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$x_n = \log n \mod 1$ ✗ $\qquad$ $x_n = n \cdot \log n \mod 1$ ✓

$x_n = \alpha^n$ is u.d. for a.e. $\alpha > 0$. (But no concrete value of α is known).

$\alpha = \frac{3}{2}$??

Thm (Weyl) If $P \in \mathbb{R}[x]$ has at least one (non-constant) coefficient irrational, then $P(n) \mod 1$ is u.d.

Thm (van der Corput): If (x_n) is a sequence and $\forall h \in \mathbb{N}$, $n \longmapsto x_{n+h} - x_n$ is u.d. then (x_n) is u.d.

Pf (of Weyl): Use induction on degree of P. Note that $n \mapsto P(n+h) - P(n)$ is a polynomial of degree less than P.

Thm: Let $u : \mathbb{N} \longrightarrow \mathbb{C}$ bounded s.t. $\forall h \in \mathbb{N}$, $\frac{1}{N} \sum_{n=1}^{N} u(n+h)\overline{u(n)} \longrightarrow 0$.
Then $\frac{1}{N} \sum_{n=1}^{N} u(n) \longrightarrow 0$

(4) $\forall k \in \mathbb{N}$ $\quad \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k x_n} = 0$

$u(n) = e^{2\pi i k x_n}$

$u(n+h)\overline{u(n)} = e^{2\pi i k (x_{n+h} - x_n)}$

Pf: Recall that $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \overset{\mu}{d}, \overset{T}{x \mapsto x+1})$ "is" a measure preserving system

$v \in L^2$ s.t. $\boxed{\int_{\mathbb{N}} T^h u \cdot \bar{u} \, d\mu = 0 \quad \forall h}$. $\qquad$ $\boxed{T^h u := u \circ T^h}$

Want $\int u \, d\mu = 0$. $\qquad$ $\langle T^{h_1 - h_2} u, u \rangle = \langle T^{h_1} u, T^{h_2} u \rangle = 0$

By the Bessel inequality, if (u_h) is a sequence of pairwise orthogonal vectors, then $\forall v$, $\sum \langle v, u_h \rangle^2 \le \|v\|^2$

vectors, then $\forall\, v$, $\sum \langle v, u_h \rangle^2 \leq \|v\|^2$ $\qquad ([N], \dots)$

$$\int u \, d\mu = \langle 1, u \rangle = \langle 1, T^h u \rangle = 0 \qquad \blacksquare$$

$v = 1$
$u_h = T^h u$

Let H be a Hilbert space.

Thm: Let $u : \mathbb{N} \to H$ bounded s.t. $\forall h \in \mathbb{N}$, $\frac{1}{N} \sum_{n=1}^{N} \langle u(n+h), u(n) \rangle \to 0$.
Then $\frac{1}{N} \sum_{n=1}^{N} u(n) \to 0$

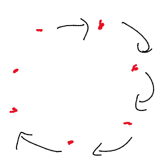
Thm: Let $(X, \mathcal{B}, \mu, T)$ be a m.p.s. Let $A \in \mathcal{B}$, $\mu(A) > 0$. Then

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{-n^2} A) > 0$$

Def: A m.p.s. $(X, \mathcal{B}, \mu, T)$ is totally ergodic if $\forall\, k \in \mathbb{N}$, the system $(X, \mathcal{B}, \mu, T^k)$ is ergodic (i.e. any $A \in \mathcal{B}$ s.t. $T^{-k} A = A$ for some k has $\mu(A) \in \{0,1\}$).

Example: $([0,1), \text{Borel}, \text{Leb}, T : x \mapsto x + \alpha)$. If $\alpha \in \mathbb{Q}$ the system is not ergodic. If $\alpha \notin \mathbb{Q}$, then the system is totally ergodic.

Example: Let T be a rotation on finitely many points. Then T is ergodic, but not totally ergodic.



Thm: Let $(X, \mathcal{B}, \mu, T)$ be totally ergodic and let $f \in L^2$. Then for any $q, r \in \mathbb{N}$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} T^{qn+r} f = \int_X f \, d\mu.$$

Pf: By the ergodic thm applied to the ergodic system $(X, \mathcal{B}, \mu, T^q)$,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \underbrace{T^{qn+r} f}_{(T^q)^n \cdot T^r f} = \int_X T^r f \, d\mu = \int_X f \, d\mu.$$

Thm: Let $(X,\mathcal{B},\mu,T)$ be totally ergodic and let $f\in L^2$. Then for any $P\in\mathbb{Z}[x]$ with positive leading coefficient.

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} T^{P(n)} f = \int_X f \, d\mu.$$

Let H be a Hilbert space.
Thm: Let $u:\mathbb{N}\to H$ bounded s.t. $\forall h\in\mathbb{N}$, $\frac{1}{N}\sum_{n=1}^{N}\langle u(n+h),u(n)\rangle\to 0$. Then $\frac{1}{N}\sum_{n=1}^{N} u(n)\to 0$

Pf: Proceed by induction on the degree of P. If $\deg P=1$, follows from previous thm. Assume $\deg P>1$. If the conclusion holds when $\int_X f\,d\mu=0$, then it holds for any f, so assume $\int_X f\,d\mu=0$. Let $u(n)=T^{P(n)}f$. By the vdC trick, need to show that $\forall h\in\mathbb{N}$,

$$\frac{1}{N}\sum_{n=1}^{N}\underbrace{\int T^{P(n+h)} f\cdot T^{P(n)}\bar{f}\, d\mu}\to 0.$$

WTS
$$\frac{1}{N}\sum_{n=1}^{N}\int_X T^{P(n+h)-P(n)} f\cdot\bar{f}\,d\mu\to 0$$

$n\mapsto P(n+h)-P(n)$ is a polynomial of degree smaller that $\deg P$.

By induction
$$\frac{1}{N}\sum_{n=1}^{N} T^{P(n+h)-P(n)} f\to 0 \text{ (in norm)},$$ so

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\int_X T^{P(n+h)-P(n)} f\cdot\bar{f}\,d\mu = 0 \qquad \blacksquare$$

Corollary: If $(X,\mathcal{B},\mu,T)$ is totally ergodic, then $\forall A\in\mathcal{B}$, $\mu(A)>0$,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(A\cap T^{-n^2}A) = \mu^2(A).$$

Let $(X,\mathcal{B},\mu,T)$ be a m.p.s. Consider the subspaces of L^2

$$H_{per} := \overline{\{ f \in L^2 : T^k f = f \text{ for some } k \}}$$

$$H_{te} := \overline{\{ f \in L^2 : \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^{kn} f = 0 \ \forall k \}}.$$

<u>Lemma</u>: $L^2 = H_{per} \oplus H_{te}$.

<u>Thm</u>: Let $(X,\mathcal{B},\mu,T)$ be a m.p.s. Let $A \in \mathcal{B}$, $\mu(A) > 0$. Then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{-n^2} A) > 0$$

<u>Pf</u>: Let $1_A = f + g$ where $f \in H_{per}$ and $g \in H_{te}$.

Since $1 \in H_{per}$, $\mu(A) = \langle 1, 1_A \rangle = \langle 1, f \rangle + \langle 1, g \rangle = \langle 1, f \rangle \leq \|1\| \cdot \|f\| \Rightarrow \|f\| \geq \mu(A) > 0$.

Find $\varepsilon > 0$ s.t. $\varepsilon < \|f\|^2$, then find h s.t. $\|f - h\| < \varepsilon$ and $\exists k \in \mathbb{N}$ s.t. $T^k h = h$.

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{-n^2} A) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int_X 1_A \cdot T^{n^2} 1_A \, d\mu$$

$$= \int_X 1_A \cdot \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^{n^2} f + T^{n^2} g \, d\mu$$

$$\geq \frac{1}{K} \int_X 1_A \cdot \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^{(kn)^2} f + T^{(kn)^2} g \, d\mu$$

$$\geq \frac{1}{k} \int_X 1_A \cdot \lim_{N\to\infty} \frac{1}{N} \sum_{n=1} T^{(kn)} f + T^{(kn)} g \, d\mu$$

$$\geq \frac{1}{k} \int_X 1_A \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^{(kn)^2} h + \cancel{T^{(kn)^2} g} \, d\mu \; - \varepsilon$$

Using a vdC-type argument as before, it follows that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^{(kn)^2} g = 0 .$$

So

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu\left(A \cap T^{-n^2} A\right) \geq \frac{1}{k} \int_X 1_A \cdot h \, d\mu - \varepsilon$$

$$= \frac{1}{k} \langle 1_A , h \rangle - \varepsilon = \frac{1}{k} \langle f + g , h \rangle - \varepsilon = \frac{1}{k} \langle f, h \rangle - \varepsilon \geq \frac{1}{k}\left[\|f\|^2 - 2\varepsilon \right] > 0 \qquad \blacksquare$$