Roth's Theorem

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Given (x_n) on $[0,1]$; it is u.d. iff $\frac{1}{N}\sum_{n=1}^{N}\delta_{x_n} \xrightarrow{\text{weak}^*} \text{Leb}$

$\forall f \in C[0,1]$ $\quad \underbrace{\frac{1}{N}\sum_{n=1}^{N}\int f\, d\delta_{x_n}}_{\frac{1}{N}\sum_{n=1}^{N} f(x_n)} \to \int_0^1 f(t)\,dt$

$f = 1_{[a,b]}$

Thm (Roth) 1953: Any $A \subset \mathbb{N}$ with $\bar{d}(A) > 0$ has a 3 term AP, i.e., $\exists\, x, y \in \mathbb{N}$ s.t. $\{x, x+y, x+2y\} \subset A$.

Thm (Furstenberg) Let $(X, \mathcal{B}, \mu, T)$ be a m.p.s., let $A \in \mathcal{B}$, $\mu(A) > 0$. Then $\exists n \in \mathbb{N}$ s.t. $\mu(A \cap T^{-n}A \cap T^{-2n}A) > 0$. $(**)$

Recall: PRT $\exists n\ \mu(A \cap T^{-n}A) > 0$

Klintchine $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(A\cap T^{-n}A) > 0$ $\longleftarrow$ $\int_X 1_A \cdot \underbrace{\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} T^n 1_A}_{\uparrow \text{ MET}}\, d\mu$

$\int 1_A \cdot T^n 1_A\, d\mu$

Polynomial $\quad \mu(A \cap T^{-n^2}A) > 0$ $\qquad \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} T^{n^2} f = \int f\, d\mu$

if $(X, \mathcal{B}, \mu, T)$ is tot. erg.

$T^n f := f \circ T^n$

Roth $\quad \underbrace{\mu(A \cap T^{-n}A \cap T^{-2n}A)}_{\int_X 1_A \cdot T^n 1_A \cdot T^{2n} 1_A\, d\mu} > 0$ $\qquad \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} T^n f \cdot T^{2n} f$

Notions of Mixing.

From the MET it follows that $(X,\mathcal{B},\mu,T)$ is ergodic iff

$$\forall A,B\in\mathcal{B}.\ \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(A\cap T^{-n}B)=\mu(A)\mu(B)$$

A natural strengthening is the following:

<u>Def</u>: $(X,\mathcal{B},\mu,T)$ is called (strong) mixing if $\forall A,B\in\mathcal{B}$

$$\lim_{n\to\infty}\mu(A\cap T^{-n}B)=\mu(A)\mu(B)$$

<u>Def</u>: $(X,\mathcal{B},\mu,T)$ is weak mixing if $\forall A,B\in\mathcal{B}$

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\left|\mu(A\cap T^{-n}B)-\mu(A)\mu(B)\right|=0$$

<u>Thm</u>: Let $(X,\mathcal{B},\mu,T)$ be a m.p.s Then TFAE:

(1) The system is WM.

(2) $X\times X$ is ergodic

(3) For any ergodic system Y, $X\times Y$ is ergodic

(4) $\forall A,B\in\mathcal{B}\ \exists E\subset\mathbb{N}$ with $\bar{d}(E)=0$ s.t. $\displaystyle\lim_{\substack{n\to\infty\\ n\notin E}}\mu(A\cap T^{-n}B)=\mu(A)\mu(B)$

(5) $\forall f,g\in L^2$ $\displaystyle\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\left|\int_X f\cdot T^n g\,d\mu-\int_X f\,d\mu\int_X g\,d\mu\right|=0.$

<u>Obs</u>: Combining (2) and (3) it follows that $(X\times X)\times(X\times X)$ is ergodic so $X\times X$ is WM.

<u>Thm</u>: Let $(X,\mathcal{B},\mu,T)$ be a weak mixing system. Then $\forall f,g\in L^\infty$. (*)

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}T^n f\cdot T^{2n}g=\int_X f\,d\mu\int_X g\,d\mu \quad \text{in } L^2 \text{ norm.}$$

<u>Cor</u>: Let $(X,\mathcal{B},\mu,T)$ be weak mixing. Then $\forall A,B\ C\in\mathcal{B}$

Cor: Let $(X,\mathcal{B},\mu,T)$ be weak mixing. Then $\forall\, A,B,C\in\mathcal{B}$

$$\frac{1}{N}\sum_{n=1}^{N}\mu(A\cap T^{-n}B\cap T^{-2n}C)\longrightarrow\mu(A)\mu(B)\mu(C)$$

Pf: Exercise.

Lemma: (Van der Corput trick) Let H be a Hilbert space, let $(u_n)_{n=1}^{\infty}$ be a bounded sequence in H.
If $\lim_{M\to\infty}\frac{1}{M}\sum_{h=1}^{M}\limsup_{N\to\infty}\frac{1}{N}\left|\sum_{n=1}^{N}\langle u_{n+h},u_n\rangle\right|=0$ Then $\frac{1}{N}\sum_{n=1}^{N}u_n\xrightarrow[N\to\infty]{}0$ (in $\|\cdot\|_H$)

Pf: Exercise.

Pf of $(*)$: Subtract from g its average $\int_X g\,d\mu$ We can assume that $\int_X g\,d\mu=0$.

Let $u_n=T^nf\,T^{2n}g$. $\quad\langle u_{n+h},u_n\rangle=\int_X T^{n+h}f\cdot T^{2n+2h}g\cdot T^{n}\bar{f}\cdot T^{2n}\bar{g}\,d\mu$

$$=\int_X T^hf\cdot T^{n+2h}g\cdot\bar{f}\cdot T^n\bar{g}\,d\mu=\int_X T^hf\cdot\bar{f}\cdot T^n\left(T^{2h}g\cdot\bar{g}\right)d\mu$$

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle u_{n+h},u_n\rangle=\int_X T^hf\cdot\bar{f}\cdot\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}T^n\left(T^{2h}g\cdot\bar{g}\right)d\mu$$

$$=\int_X T^hf\cdot\bar{f}\cdot\left(\int_X T^{2h}g\cdot\bar{g}\,d\mu\right)d\mu$$

$$\lim_{M\to\infty}\frac{1}{M}\sum_{h=1}^{M}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle u_{n+h},u_n\rangle=\lim_{M\to\infty}\frac{1}{M}\sum_{h=1}^{M}\overbrace{\int T^hf\cdot\bar{f}\,d\mu}^{\text{bdd}}\cdot\int T^{2h}g\cdot\bar{g}\,d\mu=0$$

We know that $\lim_{M\to\infty}\frac{1}{M}\sum_{h=1}^{M}\left|\int_X T^{2h}g\cdot\bar{g}\,d\mu\right|=0$, So $\lim_{M\to\infty}\frac{1}{M}\sum_{h=1}^{M}\left|\int_X T^{2h}g\cdot\bar{g}\,d\mu\right|\cdot\left|\int T^hf\cdot\bar{f}\,d\mu\right|=0$

Weak mixing (5)

By the vdC trick $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}T^nf\,T^{2n}g=0=\int_X f\,d\mu\int_X g\,d\mu$ ∎

Simplifications:

(1) For the proof of Roth's thm it suffices to assume that T is invertible, i.e. T is a.e. bijective and T^{-1} is measurable and measure preserving.

Ex: Go through the proof of the correspondence principle replacing $\mathbb{N}$ with $\mathbb{Z}$ to check that we can obtain an invertible system.

(2) We can assume that ($(X, \mathcal{B}, \mu)$ is a standard Borel space.) X is a compact metric space, $\mathcal{B}$ = Borel. Moreover we can assume T is continuous.

Ex: Show that $(**)$ is equivalent when we make assumptions (1) and (2).

(3) We can assume that $(X, \mathcal{B}, \mu, T)$ is ergodic.

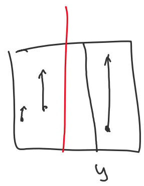
Thm (Ergodic Decomposition): Let X be a compact metric space, let $\mathcal{B}$ = Borel, let $T: X \to X$ be continuous. Let μ be a Borel T-invariant prob. measure. Then $\exists$ prob. space (Y, ν) and a map $y \mapsto \mu_y$ from Y into $\mathcal{M}(X, T)$ s.t.

(1) For ν-a.e. $y \in Y$, μ_y is ergodic [i.e. $(X, \mathcal{B}, \mu_y, T)$ is ergodic]

(2) $\mu = \int_Y \mu_y \, d\nu(y)$ $\left[\text{i.e. } \forall f \in C(X),\ \int_X f \, d\mu = \int_Y \left(\int_X f \, d\mu_y\right) d\nu(y)\right]$

Exercise: Show that we can assume ergodicity in $(**)$ using the EDT.

Example: Consider $X = \mathbb{T}^2 = [0,1]^2$, $T: (y, x) \mapsto (y, x+y)$



Take $Y = \mathbb{T}$, ν = Leb and $\mu_y = \delta_y \otimes$ Leb

Ex: Think why we can't assume that $(X, \mathcal{B}, \mu, T)$ is totally ergodic

Compact Vectors

Let $(X, \mathcal{B}, \mu, T)$ be a m.p.s. A function $f \in L^2(X)$ is called compact if its orbit closure $\overline{\{T^n f : n \in \mathbb{N}\}}$ is compact (in norm).

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<u>Exercise:</u> If f is compact, then for every $\varepsilon > 0$, $\{n : \|T^n f - f\| < \varepsilon\}$ is syndetic (has bounded gaps)

<u>Def:</u> A function $f \in L^2(X)$ is called weak-mixing if $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left| \int_X T^n f \cdot \bar{f} \, d\mu \right| = 0$

Note: $(X, \mathcal{B}, \mu, T)$ is weak mixing iff any $f \in L^2$ with $\int_X f \, d\mu = 0$ is weak-mixing.

<u>Thm</u> (Jacob - de Leeuw - Glicksberg decomposition): Let $(X, \mathcal{B}, \mu, T)$ be a m.p.s., let $H_c = \{f \in L^2 : f \text{ is compact}\}$ and let $H_{WM} = \{f \in L^2 : f \text{ is weak-mixing}\}$. Then

$$L^2 = \underline{H_c} \oplus \underline{H_{WM}}.$$

$$L^2 = \underline{H_{Per}} \oplus \underline{H_{te}} \qquad H_{WM} \subset H_{te} \subset H_{cob}$$

$$L^2 = H_{Inv} \oplus H_{cob} \qquad H_{Inv} \subset H_{Per} \subset H_c$$

$$H_{cob} = \overline{\operatorname{span}\{f - Tf\}}$$

<u>Pf of</u> (**) Let $1_A = f + g$ where f is compact and g is weak-mixing.

Then $\mu(A \cap T^{-n} A \cap T^{-2n} A) = \int_X 1_A \cdot T^n 1_A \cdot T^{2n} 1_A \, d\mu = \int_X 1_A \cdot T^n f \cdot T^{2n} f \, d\mu +$

$$\cancel{\int_X 1_A \cdot T^n f \cdot T^{2n} g \, d\mu} + \cancel{\int_X 1_A \cdot T^n g \cdot T^{2n} f \, d\mu} + \cancel{\int_X 1_A \cdot T^n g \cdot T^{2n} g \, d\mu}.$$

For each expression involving g, the Cesàro average is 0.

E.g. $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int_X 1_A \cdot T^n f \cdot T^{2n} g \, d\mu = \int_X 1_A \cdot \underbrace{\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f \cdot T^{2n} g}_{=0} \, d\mu$

This proved using the vdC trick and the fact that g is WM.

Since f is compact, using the exercise (let $\varepsilon = \mu^3(A)/100$) $\exists\, S \subset \mathbb{N}$ syndetic s.t. $\forall n \in S$

$\|T^n f - f\| < \varepsilon$. But $\|T^{2n} f - f\| \leq \|T^{2n} f$ …

... compact, using the exercise. (let $\varepsilon = \mu^3(A)/100$) $\exists$ $S \subset \mathbb{N}$ syndetic s.t. $\forall$ $n \in S$

$\|T^n f - f\| < \varepsilon$. But $\|T^{2n} f - f\| \leq \underbrace{\|T^{2n} f - T^n f\|}_{T^n(T^n f - f)} + \|T^n f - f\| \leq 2\varepsilon$

$$\boxed{\begin{array}{l} \|1_A \cdot f\|_{L^2} \leq 1 \\ \|1_A \cdot T^n f\|_{L^2} \leq 1 \end{array}}$$

For any $n \in S$, $\displaystyle\int_X 1_A \cdot T^n f \cdot T^{2n} f \, d\mu \geq \int_X 1_A \cdot f \cdot f \, d\mu - 3\varepsilon = \int_X 1_A \cdot f^2 \, d\mu - 3\varepsilon$

$1_A = f + g$ $\quad$ $g \perp f^2$

$\displaystyle = \int_X f^3 \, d\mu - 3\varepsilon$. Therefore $\displaystyle\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{-n}A \cap T^{-2n}A) \geq \bar{d}(S)\left(\int_X f^3 \, d\mu - 3\varepsilon\right)$

$\int_X g \, d\mu = 0$

$\displaystyle\int_X f \, d\mu = \int_X 1_A - g \, d\mu = \int_X 1_A \, d\mu = \mu(A) > 0$. $\quad \underbrace{f(x) \geq 0 \text{ for } \mu\text{-a.e. } x.}$

Hence $f^3 \geq 0$ and $f^3 \neq 0$, so $\displaystyle\int_X f^3 \, d\mu > 0$. $\blacksquare$