Szemeredi's theorem

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Def: Given two m.p.s. $\mathbb{X} = (X, \mathcal{B}, \mu, T)$ and $\mathbb{Y} = (Y, \mathcal{C}, \nu, S)$, a factor map is a map $\pi : X \to Y$ s.t. $\pi\mu = \nu$ (i.e. $\forall A \in \mathcal{C}$, $\mu(\pi^{-1} A) = \nu(A)$) and $\pi \circ T = S \circ \pi$ a.e.

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ Y \arrow[r, "S"] & Y \end{tikzcd}$$

If a factor map π is invertible and the inverse is also a factor map, we say that π is an isomorphism and $\mathbb{X}$ and $\mathbb{Y}$ are isomorphic.

Ex: The projection from a product $\mathbb{X} \times \mathbb{Y}$ to either $\mathbb{X}$ or $\mathbb{Y}$ is a factor map.

Ex: The Bernoulli system $(\{0,1\}^{\mathbb{N}}$, Borel, Product$(\frac{1}{2}, \frac{1}{2})$-measure, Shift$)$ is isomorphic to the doubling map $([0,1)$, Borel, Leb, $x \mapsto 2x \bmod 1)$.

Kronecker factor

Def: A m.p.s. $\mathbb{X}$ is a Kronecker system if it is isomorphic to $(K, \text{Borel}, \text{Haar}, T)$ where K is a compact (metrizable) abelian group and $T : x \mapsto x + \alpha$ for some $\alpha \in K$.

A Kronecker system is ergodic iff α generates a dense subgroup.

Recall: A function $f \in L^2(X)$ is compact if $\{T^n f : n \in \mathbb{N}\}$ is pre-compact. Denoted by H_c the set of all compact functions.

Thm: An ergodic m.p.s. $\mathbb{X}$ is a Kronecker system iff $H_c = L^2(X)$.

Obs. Given a factor map $\pi : \mathbb{X} \to \mathbb{Y}$ we can embed $L^2(Y)$ into $L^2(X)$ by taking $f \in L^2(Y)$ to $f \circ \pi \in L^2(X)$.

Thm: Every ergodic system $\mathbb{X}$ has a factor $\mathbb{Y}$ with the following properties:

(1) $L^2(Y) = H_c(X)$

(2) Y is a Kronecker system

(3) For any other Kronecker system Z that is a factor of X, Z is also a factor of Y.

Y is called the Kronecker factor of X.

Multiple recurrence in Kronecker systems:

Thm (MR): Let $X=(X,\mathcal{B},\mu,T)$ be an ergodic system and let $A\in\mathcal{B}$ with $\mu(A)>0$. Then $\forall\, k\in\mathbb{N}$

$$\liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(A\cap T^{-n}A\cap T^{-2n}A\cap\cdots\cap T^{-kn}A)>0$$

Pf of (MR) for Kronecker systems: We will use the following fact:

Fact: If f is compact, then $\forall\,\varepsilon>0$, $S=\{n:\|T^nf-f\|<\varepsilon\}$ is syndetic.

$\mu(A\cap T^{-n}A\cap\cdots\cap T^{-kn}A)=\int_X 1_A\cdot T^n1_A\cdot T^{2n}1_A\cdots T^{kn}1_A\,d\mu$. Fix $\varepsilon<\frac{\mu(A)}{k^2}$, let S be as above for $f=1_A$

For any $n\in S$ we have $\|T^{in}1_A-1_A\|\leq\|T^{in}1_A-T^{(i-1)n}1_A\|+\dots+\|T^n1_A-1_A\|$

$$=\|T^n1_A-1_A\|+\|T^n1_A-1_A\|+\dots+\|T^n1_A-1_A\|$$

$$\leq k\varepsilon$$

$$\mu(A\cap T^{-n}A\cap\cdots\cap T^{-kn}A)\geq\int_X 1_A\cdot1_A\cdots1_A\,d\mu-k^2\varepsilon=\mu(A)-k^2\varepsilon>0\quad(\forall n\in S)$$

Therefore $\liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\underbrace{\mu(A\cap T^{-n}A\cap\cdots\cap T^{-kn}A)}_{\geq(\mu(A)-k^2\varepsilon)1_S(n)}=(\mu(A)-k^2\varepsilon)\,\underline{d}(S)>0$ $\blacksquare$

$T \simeq Id$ in Sup of norm

$X \to X$

$C(X) \to C$

$\Longrightarrow$ $L^2(X) \to L^2(X)$, $f \mapsto f \circ T^n$ $\quad \simeq Id$

$L^\infty(X) \to L^\infty(X)$, $f \mapsto f \circ T^n$

<u>Pf of (MR) for weak mixing systems:</u>

If fact we will show that $\forall f_1, \dots, f_k \in L^\infty(X)$

$$\boxed{\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f_1 \cdot T^{2n} f_2 \cdot \dots \cdot T^{kn} f_k = \int_X f_1\, d\mu \cdot \int_X f_2\, d\mu \cdots \int_X f_k\, d\mu} \text{ (in } L^2)$$

Putting $f_i = 1_A$ we deduce $\displaystyle\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap \dots \cap T^{-kn} A) = \mu(A)^{k+1}$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int 1_A \cdot T^n 1_A \cdots \cancel{1} T^{nk} 1_A \, d\mu = \int 1_A \cdot \mu(A)^k \, d\mu$$

Pf: Subtracting $\int_X f_k \, d\mu$ from f_k and using induction, we can assume that $\int_X f_k \, d\mu = 0$.

Let $u_n = \prod_{i=1}^{k} T^{in} f_i$. $\quad \langle u_{n+d}, u_n \rangle = \displaystyle\int_X \prod_{i=1}^{k} T^{i(n+d)} f_i \cdot \prod_{i=1}^{k} T^{in} \bar{f_i} \, d\mu$

$$= \int_X \prod_{i=2}^{k} T^{(i-1)n} \left(T^{id} f_i \cdot \bar{f_i} \right) \cdot \left(T^d f_1 \cdot \bar{f_1} \right) d\mu$$

$$\boxed{\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle u_{n+d}, u_n \rangle} = \int_X \left[\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \prod_{i=2}^{k} T^{(i-1)n} \left(T^{id} f_i \cdot \bar{f_i} \right) \right] \cdot \left(T^d f_1 \cdot \bar{f_1} \right) d\mu$$

$$= \prod_{i=1}^{k} \int_X T^{id} f_i \cdot \bar{f_i} \, d\mu$$

$$\lim_{D\to\infty} \frac{1}{D} \sum_{d=1}^{D} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle u_{n+d}, u_n \rangle = \lim_{D\to\infty} \frac{1}{D} \sum_{d=1}^{D} \prod_{i=1}^{k} \int_X T^{id} f_i \cdot \bar{f_i} \, d\mu$$

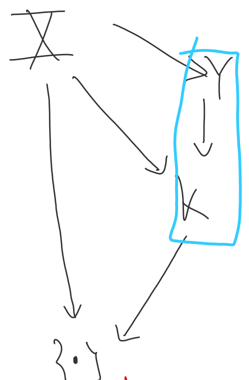
$$\le \lim_{D\to\infty} \frac{1}{D} \sum_{d=1}^{D} \boxed{\prod_{i=1}^{k-1} \|f_i\|^2} \left| \int_X T^{kd} f_k \cdot \bar{f}_k \, d\mu \right|$$

By the vdC lemma $\frac{1}{N}\sum_{n=1}^{N} u_n \longrightarrow 0$ as $N \to \infty$. $\qquad \overset{\nearrow}{\text{WM}} = 0 \qquad$ ∎

$$L^2(X) = H_w \oplus H_c$$

If $\mathbb{X}$ is not weak mixing, then it has a non-trivial Kronecker factor.

Def: We say that an ergodic system is SZ if it satisfies the MR theorem.

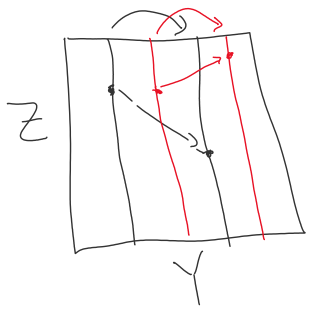


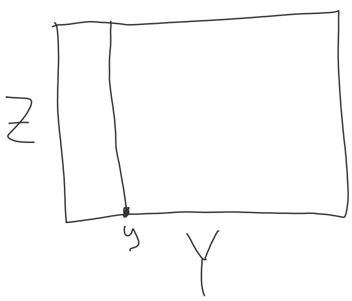
skew-product

$(X, \mathcal{B}, \mu, T) \qquad // \; (Y, \mathcal{C}, \nu, S)$

Lemma (Rokhlin): Let $\pi : \mathbb{X} \to \mathbb{Y}$ be a factor map. If $\mathbb{X}$ is ergodic, there exist $(Z, \mathcal{D}, \lambda)$ and a measurable function $\rho : Y \to \operatorname{Aut}(Z)$ s.t. $\mathbb{X}$ is iso. to

$$(Y \times Z, \mathcal{C} \otimes \mathcal{D}, \nu \otimes \lambda, R), \text{ where } R(y, z) = (Sy, \rho(y) z).$$





$$\int_Y \underbrace{\delta_y \otimes \lambda}_{\mu_y} \, d\nu(y) = \mu$$

Recall $\mathbb{X}$ is weak mixing iff $\mathbb{X} \times \mathbb{X}$ is ergodic

Def: A system X is a weak mixing extension of Y if (using the notation from Rokhlin's lemma) the relative product

$$(Z \times Y \times Z,\ D \otimes C \otimes D,\ \lambda \otimes \nu \otimes \lambda,\ R) \quad \text{is ergodic}$$

where $R(z_1, y, z_2) = (\rho(y) z_1,\ Sy,\ \rho(y) z_2)$

Thm: If X is a weak mixing extension of Y and Y is SZ, then X is SZ.

Recall: $f \in H_c$ iff $\forall \varepsilon > 0\ \exists g_1, \dots, g_r$ s.t. $\{T^n f : n \in \mathbb{N}\} \subset \bigcup_{i=1}^{r} B(g_i, \varepsilon)$

$$\Longleftrightarrow \forall n \in \mathbb{N} \quad \min_{1 \le i \le r} \|T^n f - g_i\| < \varepsilon$$

Def: An ergodic system X is a compact extension of a system Y if (using the notation of Rokhlin's lemma) there is a dense set of relatively compact functions.

$f \in L^2(X)$ is rel. compact if $\forall \varepsilon > 0\ \exists g_1, \dots, g_r \in L^2(X)$ s.t.

$$\forall n \in \mathbb{N} \quad \nu\left(\left\{ y : \min_{1 \le i \le r} \|T^n f - g_i\|_{L^2(\mu_y)} < \varepsilon \right\}\right) = 1$$

Thm: If X is a compact extension of a SZ system, then is SZ.

Thm: Let $\pi : X \to Y$ be a factor map between ergodic systems. If X is not a weak mixing extension of Y, then there exists an intermediate extension Z_1 such that Z_1 is a compact extension of Y.

$$\begin{tikzcd} X \arrow[d] \arrow[r] & Z_1 \arrow[dl] \\ Y & \end{tikzcd}$$

X

$$Z_3 \to Z_2 \to$$

Z_1